Complex Langevin Lattice QCD for Color-Superconductivity at High Density

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Collaboration

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[2] T. Yokota, Y. Ito, H. Matsufuru, Y. Namekawa, J. Nishimura, A. Tsuchiya and S. Tsutsui, JHEP **06** (2023), 061 [arXiv:2302.11273 [hep-lat]].

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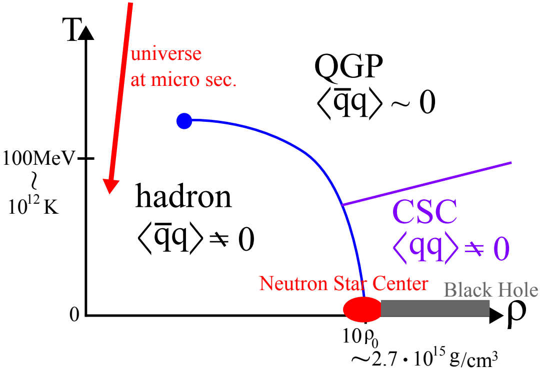
QCD Phase Diagram



- Complex Langevin Lattice QCD (CL-LQCD) overcoming Sign Problem,
- for Color-Superconductiviy (CSC).

c.f. Lefschetz Thimble, TRG, Quantum Computer...

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Complex Langevin in Lattice QCD

Complexification of Gluon Fields:

$$U_{\mu x} \in SU(3) \Longrightarrow \mathcal{U}_{\mu x}(\tau) \in SL(3, \mathbb{C})\,. \qquad (1)$$

Parisi '83; Klauder '84;
Aarts, Seiler, Stamatescu '09;
Aarts, James,Seiler, Stamatescu '11;
Seiler, Sexty, Stamatescu '13; Sexty '14;
Fodor, Katz, Sexty, Torok '15;
Nishimura, Shimasaki '15;
Nagata, Nishimura, Shimasaki '15;
Sinclair, Kogut '16

Complex Langevin Equation:

$$\mathcal{U}_{\mu x}(\tau+\epsilon) = \exp\Big[i\lambda^a\Big(-\epsilon\, \underbrace{v_{a\mu x}[\mathcal{U}(\tau)]}_{\text{drift term}} + \sqrt{\epsilon}\ \underbrace{\eta_{a\mu x}(\tau)}_{\text{gaussian noise}}\ \Big)\Big]\mathcal{U}_{\mu x}(\tau)\,. \qquad (2)$$

c.f. For $z \in \mathbb{C}$, $\quad \partial_\tau z(\tau) = \underbrace{-\partial_z S(z)}_{\text{drift term}} + \underbrace{\eta(\tau)}_{\text{gaussian noise}}\ .$

We expect (with no sign problem in the l.h.s.):

$$\lim_{\tau\to\infty} \langle O[\mathcal{U}(\tau)]\rangle_\eta \stackrel{!}{=} \int DU\, P[U] O[U]\,, \qquad (3)$$

which is justified only when No Singular Drift and No Excursion. To insure, we monitor the drift term histogram (Nagata et al.'16) & unitarity norm (Seiler et al'13).

Source-Term Method

To investigate diquark effects with manifest gauge invariance, we introduce a diquark source term: $S_{QCD} \to S_{QCD} + S_\phi + S_J$,

$$S_\phi = \sum_n m_\phi^2 \phi_n^{a,\dagger} \phi_{n,a} \ , \quad S_J = J \sum_n \left[\epsilon^{abc} (\chi_a \chi_b \phi_c)_n + h.c. \right] \ . \tag{4}$$

CSC Signal: Anomalous responses to the source coupling J.

The Langevin equation for the source field ϕ (color/site indices omitted):

$$\frac{\partial \phi_{1,2}(\tau)}{\partial \tau} = -\frac{\delta(S_\phi + S_J)}{\delta \phi_{1,2}(\tau)} + \eta_{1,2}(\tau), \quad \phi(\tau) = \frac{(\phi_1 + i\phi_2)(\tau)}{\sqrt{2}}, \quad \phi_{1,2}(0) \in \mathbb{R}. \tag{5}$$

Diquark Condensates (Observables):

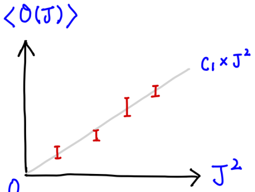
$$\langle \bar{\phi} \phi \rangle \sim \mathcal{O}(J^2) \ , \quad \frac{m_\phi^2}{2J} \langle \epsilon \chi \chi \phi + \epsilon \bar{\chi} \bar{\chi} \bar{\phi} \rangle \sim \mathcal{O}(J) \ , \quad \langle (\epsilon \bar{\chi} \bar{\chi})(\epsilon \chi \chi) \rangle \ . \tag{6}$$

We focus on the second one.

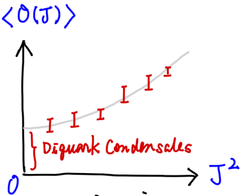
Source-Term Method

$$\langle \mathcal{O}(J) \rangle := \langle (\varepsilon \bar{\chi}\bar{\chi})(\varepsilon \chi\chi) \rangle (J) = \frac{-m_\phi^2}{2J} \langle \varepsilon \chi\chi\phi + \varepsilon \bar{\chi}\bar{\chi}\phi \rangle (J)$$

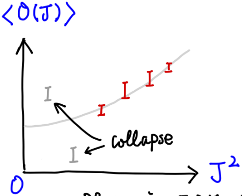
$$= \underbrace{C_0}_{\to \text{ zero or very small}} + C_1 J^2 + \mathcal{O}(J^4)$$



Normal Phase



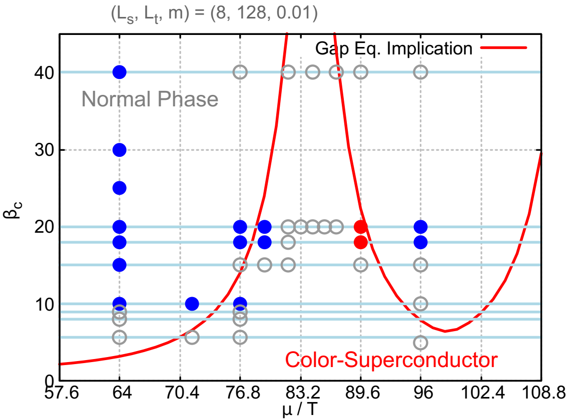
CSC Phase in Infinit Vol.-limit



CSC Phase in Finite Vol.

※ Discussions w. Prof. Imada (Tokyo-U)
Prof. Yamaji (NIMS)

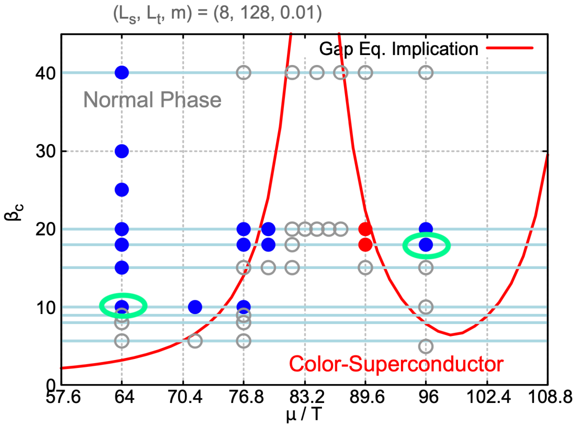
Guide



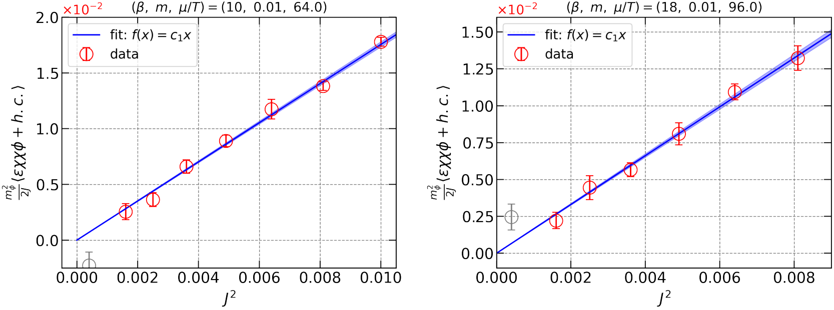
- Red Curves: A Phase Boundary suggested by Lattice Gap Eq (Yokota et al. [2]) at one-loop.
- Red/Blue Circles: CL-LQCD Simulation Points with Thermalization.

Guide

Normal Phase Expected Region.



$\epsilon\chi\chi\phi$ vs. J^2

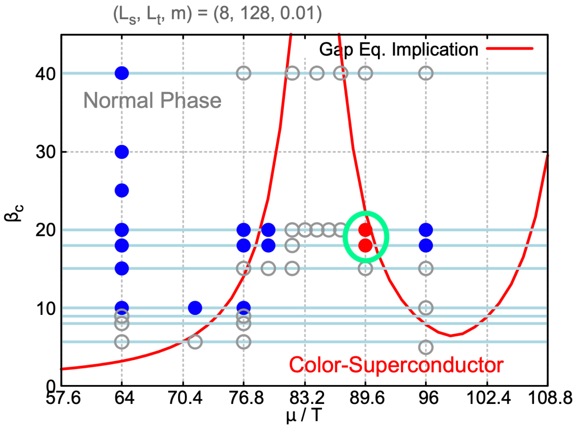


- Figure: Diquark condensates vs source term couplings J^2 in normal phase expected region. Two examples: $(\beta,\ \mu/T) = (10,\ 64)$ (left) & $(18,\ 96)$ (right).
- Only linear fit with a zero intercept works: Normal Phase.

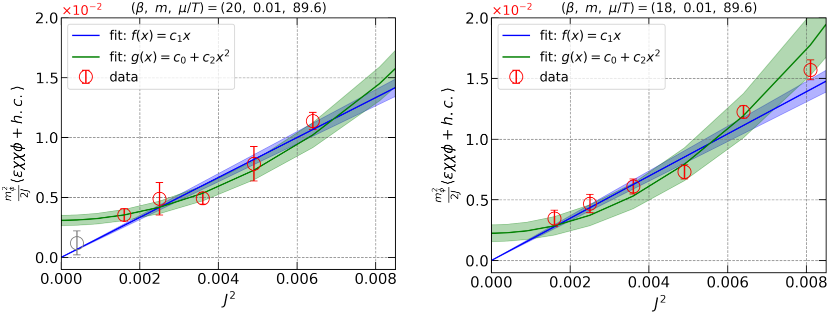
$$f(J^2) = c_1 \cdot J^2\ , \quad \chi^2/6 = 0.7\ \text{(left)}\ , \quad \chi^2/5 = 0.3\ \text{(right)}\ . \tag{7}$$

Guide

CSC Phase Right-Boundary: $\mu/T = 89.6$



$\epsilon\chi\chi\phi$ vs. J^2



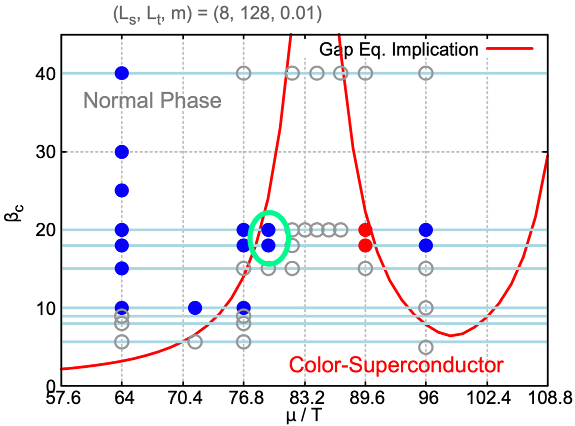
- Figure: Diquark condensates vs source term couplings J^2 with $\mu/T = 89.6$ for $\beta = 20$ (left) and 18 (right).

- The quadratic fit works better: CSC?

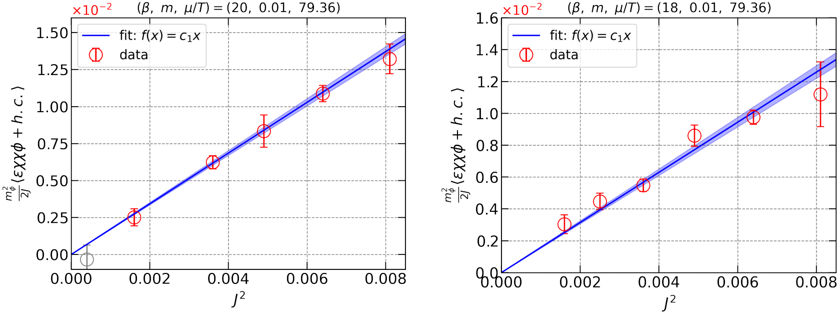
$$f(J^2) = c_0 + c_2 \cdot (J^2)^2 \,, \quad \chi^2/4 = 1.3(1.2) \text{ (left(right))} \,. \tag{8}$$

Guide

CSC/Normal Phase Left-Boundary: $\mu/T = 79.36$



$\epsilon\chi\chi\phi$ vs. J^2

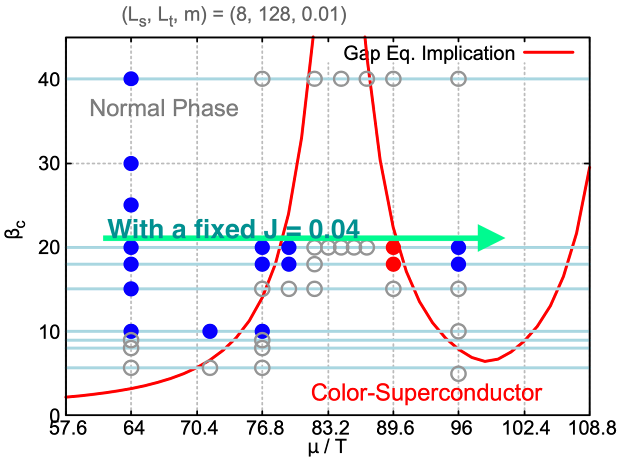


- Figure: Diquark condensates vs a source term coupling J^2 with $\mu/T = 79.36$ for $\beta = 20$ (left) and 18 (right).

- Only linear fit with a zero intercept works: Normal Phase.

$$f(J^2) = c_1 \cdot J^2\,, \quad \chi^2/5 = 0.4(1.0)\ \text{(left(right))}\,. \tag{9}$$

Guide



(L_s, L_t, m) = (8, 128, 0.01)
Gap Eq. Implication
Normal Phase
With a fixed J = 0.04
Color-Superconductor
β_c
40
30
20
10
0
57.6
64
70.4
76.8
83.2
89.6
96
102.4
108.8
μ / T

Diquark vs. μ

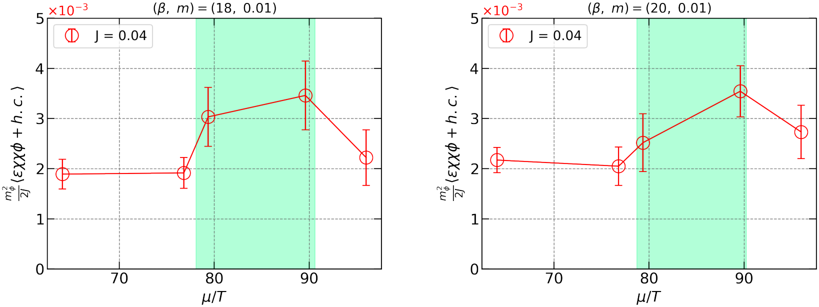


Figure:

- Diquark Condenstes vs. μ for a small-fixed $J = 0.04$. Left: $\beta = 18$. Right: $\beta = 20$. The green band is the CSC-expected region by the gap equation.
- The diquark condensates becomes larger for the CSC-expected μ/T, which suggests condensates remaining in the $J \to 0$ limit, i.e. the CSC signal.

Summary and Future Perspective

- We have investigated a small-box lattice QCD at cold and extremely high density by using the complex Langevin method.
- We have invented the diquark source-term method and investigated the color-superconductivity in a gauge-invariant way.
- Around the right-edge of the CSC-expected region, we have observed anomalous responses of the diquark source term, which may be the first indication of the CSC in the LQCD.
- For the CSC-expected μ/T, the diquark condensates tend to remain with decreasing source-term coupling, which would also be the signal of the CSC.
- As a future prospect, we will take (1) a infinite volume limit and then (2) $J \to 0$ limit to see whether the diquark condensates remain finite (color-superconductivity emerges).

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