Vacuum structures of QCD_2 from non-invertible anyon condensation

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Introduction

In this talk, we focus on 2d theory, so 0-form symmetries are described by topological defect lines (TDLs).

TDLs possess structure of a fusion category $\mathcal{C}$, rather than a group. [Bhardwaj–Tachikawa '17, Chang–Lin–Shao–Wang–Yin 18,...].

The fusion rule is described by

$$a \otimes b = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} N_{ab}^{c}\, c, \quad N_{ab}^{c} \in \mathbb{Z}_{\geq 0},$$

where the TDLs are denoted by $a, b, \ldots$.

In particular, they generally don't have an inverse, i.e., given a, no another TDL a^{-1} s.t. $a \otimes a^{-1} = a^{-1} \otimes a = \mathbf{1}$.

For example, the Ising CFT possesses the TDLs, so-called Verlinde lines $\{\mathbf{1}, \eta, \mathcal{N}\}$, satisfying the fusion rule:

$$\eta \otimes \eta = \mathbf{1}, \quad \eta \otimes \mathcal{N} = \mathcal{N} \otimes \eta = \mathcal{N}, \quad \mathcal{N} \otimes \mathcal{N} = \mathbf{1} \oplus \eta.$$

We see NO $\mathcal{N}^{-1}$.

In above description, we assume systems are bosonic.

To describe fermionic systems, we extend the fusion category to

fermionic fusion supercategory

[Bhardwaj–Inamura–Tiwari '24]

Fermionic fusion supercategories

In a fusion category, topological point-like defect (TPD) b/w a and b form a finite dim $\mathbb{C}$-vector space, $\mathrm{Hom}(a, b)$.

In a fusion supercategory, $\mathrm{Hom}(a, b)$ is equipped with a $\mathbb{Z}_2$-grading that represents the fermion parity of TPDs.

A simple line a is called an m-type if $\mathrm{Hom}(a, a) \cong \mathbb{C}^{1|0}$. Physically, a CANNOT have a fermionic TPD on it.

A simple line a is called a q-type if $\mathrm{Hom}(a, a) \cong \mathbb{C}^{1|1}$. Physically, a CAN have a fermionic TPD on it.

Any fermionic system has "$\mathbb{Z}_2$-symmetry", which is generated by a 1d fermionic invertible TQFT denoted by a π line.

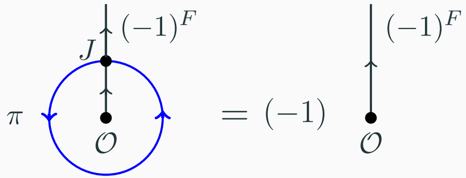
The π line is fermionically isomorphic to $\mathbf{1}$; the fermionic endpoint $\mathcal{O}_\pi \in \mathrm{Hom}(\mathbf{1}, \pi)$ exists.

When a is m-type, πa $(:= \pi \otimes a)$ is isomorphic to a via a fermionic isomorphism $\mathcal{O}_\pi \otimes 1_a$, where 1_a is a trivial TPD on a.

When a is q-type, there is also a bosonic isomorphism $\mathcal{O}_\pi \otimes f_a$ b/w a and πa, where $f_a \in \mathrm{Hom}(a, a)$ is a fermionic isomorphism from a to itself.

Any fermionic system also has a fermion parity symmetry $\mathbb{Z}_2^f$, which is non-anomalous.

The π and $(-1)^F$ lines have a canonical junction J



This shows that the π line acts as -1 on operators in the $(-1)^F$-twisted sector.

Namely, the π acts as -1 on the Ramond sector; it acts as $+1$ on the Neveu-Schwarz sector.

Vacua and TDLs in QCD_2

Let us consider massless QCD_2 w/ a gauge group G, a fermion in representation R.

The IR eff theory is described by the gauged WZW model with the coset

$$\frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)}},$$

where $I(R)$ is the Dynkin index.

A theory is gapped if and only if

$$c_{\mathrm{SO}(\dim(R_\pm))/G_{I(R_\pm)}} = c_{\mathrm{SO}(\dim(R_\pm))} - c_{G_{I(R_\pm)}} = 0,$$

equivalently, $G_{I(R)} \subset \mathrm{SO}(\dim(R))_1$ is the conformal embedding.

[Delmastro–Gomis–Yu '21]

Since $c_{\mathrm{SO}(\dim(R_\pm))/G_{I(R_\pm)}} = 0$ for the gapped theories, there are no dynamical d.o.f. in the IR.

However, there can exist topological d.o.f.

- topological local ops $\implies$ degeneracy of vacua
- topological line ops $\implies$ (non-invertible) symmetry

We focus on $\mathrm{SU}(N)$ adjoint QCD.

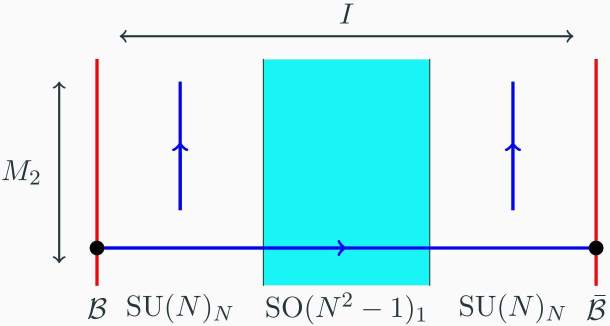
The vacuum structure and TDLs in the bosonized adjoint QCD is studied by [Komargodski–Ohmori–Roumpedakis–Seifnashri '20] in detail.

We study the (fermionic) adjoint QCD itself by applying the fermionic fusion supercategory.

Topological operators in QCD_2

To find the vacua and TDLs, we note that $\mathrm{SO}(N^2-1)_1$ WZW model can be obtained by gauging the (non-)invertible symmetry A of $\mathrm{SU}(N)_N$ WZW model.

One convenient way to understand is to realize the WZW model as a CS theory on $M_2 \times I$:



An example: $N = 3$

Vacua:

The vacua $v_i^{(p)}$ with 1-form charge $p \in \mathbb{Z}_3$ are

$$\alpha(L_{00}) = v_0^{(0)}, \quad \alpha(L_{10}) = v_1^{(1)}, \quad \alpha(L_{01}) = v_2^{(2)},$$
$$\alpha(L_{11}) = v_0^{(0)} + 2v_3^{(0)},$$

We find four vacua, and $v_{0,3}^{(0)}$ have the same 1-form charge (i.e., in the same universe).

TDLs:

The TDLs $\mathcal{L}^m_\mu$ and $\mathcal{L}^q_\mu$ are

$$\begin{aligned}
&\alpha^{\pm}_{00} = \mathbf{1},\ \alpha^{+}_{10} = \mathcal{L}^q_1,\ \alpha^{+}_{01} = \mathcal{L}^q_2,\ \alpha^{+}_{11} = \pi + \mathcal{L}^m_3 + \pi\mathcal{L}^m_3,\\
&\alpha^{-}_{10} = \mathcal{L}^q_4,\ \alpha^{-}_{01} = \mathcal{L}^q_5,\ \alpha_{11} = \pi + \mathcal{L}^m_6 + \pi\mathcal{L}^m_6,\\
&\alpha^{+}_{10}\alpha^{-}_{10} = \mathcal{L}^m_7,\ \alpha^{+}_{10}\alpha^{-}_{01} = \mathcal{L}^m_8,\ \alpha^{+}_{01}\alpha^{-}_{10} = \mathcal{L}^m_9,\ \alpha^{+}_{01}\alpha^{-}_{01} = \mathcal{L}^m_{10},\\
&\alpha^{+}_{10}\alpha^{-}_{11} = \mathcal{L}^m_1 + 2\mathcal{L}^q_{11},\ \alpha^{+}_{01}\alpha^{-}_{11} = \mathcal{L}^m_2 + 2\mathcal{L}^q_{12},\\
&\alpha^{+}_{11}\alpha^{-}_{10} = \mathcal{L}^m_4 + 2\mathcal{L}^q_{13},\ \alpha^{+}_{11}\alpha^{-}_{01} = \mathcal{L}^m_5 + 2\mathcal{L}^q_{14},\\
&\alpha^{+}_{11}\alpha^{-}_{11} = \mathbf{1} + \mathcal{L}^m_3 + \pi\mathcal{L}^m_3 + \mathcal{L}^m_6 + \pi\mathcal{L}^m_6 + 2(-1)^F + \pi(-1)^F,
\end{aligned}$$

where $\alpha^{\pm}_\mu := \alpha^{\pm}(L_\mu)$.

We find 8 m-type lines up to π line, and 8 q-type lines.

An example: $N=4$

Vacua:

The vacua $v_i^{(p)}$ with 1-form charge $p \in \mathbb{Z}_4$ are

$$\alpha(L_{000}) = v_0^{(0)},\ \alpha(L_{100}) = v_1^{(1)},\ \alpha(L_{010}) = v_2^{(2)},\ \alpha(L_{001}) = v_3^{(3)},$$
$$\alpha(L_{011}) = v_1^{(1)} + 2v_4^{(2)},\ \alpha(L_{020}) = 2v_5^{(0)},\ \alpha(L_{110}) = v_3^{(3)} + 2v_6^{(3)},$$
$$\alpha(L_{111}) = 2v_2^{(2)} + 2v_7^{(2)}.$$

We find totally eight vacua, and each universe has a two-fold degenerate vacuum.

TDLs:

The TDLs $\mathcal{L}^m_\mu$ and $\mathcal{L}^q_\mu$ are

$$\begin{aligned}
&\alpha^{\pm}_{000} = \mathbf{1},\ \alpha^{\pm}_{400} = \pi,\ \alpha^{+}_{100} = \mathcal{L}^q_1,\ \alpha^{+}_{010} = \mathcal{L}^q_2,\ \alpha^{+}_{001} = \mathcal{L}^q_3,\\
&\alpha^{+}_{011} = \mathcal{L}^q_1 + \mathcal{L}^m_4 + \pi\mathcal{L}^m_4,\ \alpha^{+}_{020} = \mathcal{L}^m_5 + \pi\mathcal{L}^m_5,\\
&\alpha^{+}_{110} = \mathcal{L}^q_3 + \mathcal{L}^m_6 + \pi\mathcal{L}^m_6,\ \alpha^{+}_{111} = 2\mathcal{L}^q_2 + 2\mathcal{L}^q_7,\\
&\alpha^{-}_{100} = \mathcal{L}^q_8,\ \alpha^{-}_{010} = \mathcal{L}^q_9,\ \alpha^{-}_{001} = \mathcal{L}^q_{10},\ \alpha^{-}_{011} = \mathcal{L}^q_8 + \mathcal{L}^m_{11} + \pi\mathcal{L}^m_{11},\\
&\alpha^{-}_{020} = \mathcal{L}^m_{12} + \pi\mathcal{L}^m_{12},\ \alpha^{-}_{110} = \mathcal{L}^q_{10} + \mathcal{L}^m_{13} + \pi\mathcal{L}^m_{13},\\
&\alpha^{-}_{111} = 2\mathcal{L}^q_9 + 2\mathcal{L}^q_{14}.
\end{aligned}$$

In addition, these fusions give rise to 25 m-type lines and 24 q-type lines. Consequently, in total there are 32 m-type lines (up to the π line) and 32 q-type lines.

Comparison with compactified QCD_2

The above results agree with those obtained from the small circle expansion of adjoint QCD_2 [Dempsey–Klebanov–Pufu–Søgaard '24].

We compactify the $\mathrm{SU}(N)$ adjoint QCD_2 on a circle S^1 with radius R.

For $gR \ll 1$, the leading order of the effective action is given by

$$S = \int \mathrm{d}t \left(\dot{\vec{q}} \cdot \dot{\vec{q}} - \frac{Ng^2}{2\pi} \vec{q} \cdot \vec{q} + \mathrm{i} \vec{\chi} \cdot \dot{\vec{\chi}} \right),$$

where $\vec{q}(t) = (q_1(t), \ldots, q_N(t))$ and $\vec{\chi}(t) = (\chi_1(t), \ldots, \chi_N(t))$ are bosons and fermions originating from the $A_\mu(t,x)$ and $\psi(t,x)$ in the adjoint representation, respectively.

They satisfy $\sum_i q_i = 0$ and $\sum_i \chi_i = 0$.

The action is nothing but $N-1$ harmonic oscillators w/ $N-1$ decoupled fermions.

We denote the creation and annihilation operators built from $\vec{q}$ by $a_j^\dagger, a_j$, and $\vec{\chi}$ by $c_j^\dagger, c_j$ $(j=1,\dots N-1)$, respectively.

By acting $c_j^\dagger$'s on $|0\rangle$, we find the degenerate vacua:

$\boldsymbol{N=3}$:

$\mathbb{Z}_3^{[1]}$:	0	1	2
	$\lvert 0\rangle$, $c_1^\dagger c_2^\dagger \lvert 0\rangle$	$c_1^\dagger \lvert 0\rangle$	$c_2^\dagger \lvert 0\rangle$

$\boldsymbol{N=4}$:

$\mathbb{Z}_4^{[1]}$:	0	1	2	3
	$\lvert 0\rangle$, $c_1^\dagger c_3^\dagger \lvert 0\rangle$	$c_1^\dagger \lvert 0\rangle$, $c_2^\dagger c_3^\dagger \lvert 0\rangle$	$c_2^\dagger \lvert 0\rangle$, $c_1^\dagger c_2^\dagger c_3^\dagger \lvert 0\rangle$	$c_3^\dagger \lvert 0\rangle$, $c_1^\dagger c_2^\dagger \lvert 0\rangle$

Summary

We discussed **vacua** and **non-invertible symmetry** in $\mathrm{SU}(N)$ adjoint massless QCD_2 based on formulation of fermionic fusion supercategory.

The gapped QCDs may have degenerate vacua with different 1-form symmetry charges.

We explicitly compute the vacuum degeneracy and TDLs in $\mathrm{SU}(N)$ adjoint QCD for $N = 3$ and $N = 4$.

The results agree with the analysis of the small circle expansion.

Back-up

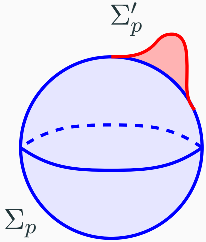
Symmetries are [Gaiotto–Kapustin–Seiberg–Willett '14]

Topological defects

A topological defect is an operator $\mathcal{D}(\Sigma_p)$ supported on a closed mfd Σ_p s.t.

$$\langle \cdots \mathcal{D}(\Sigma_p) \cdots \rangle = \left\langle \cdots \mathcal{D}(\Sigma'_p) \cdots \right\rangle ,$$

if no charged objects b/w the region Σ_p and Σ'_p.



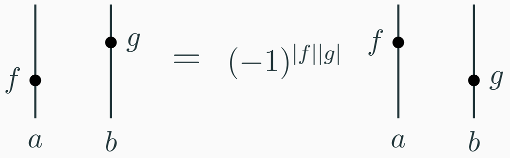
Fermionic fusion supercategories

In a fusion category, topological point-like defect (TPD) b/w a and b form a finite dim $\mathbb{C}$-vector space, $\mathrm{Hom}(a, b)$.

In a fusion supercategory, $\mathrm{Hom}(a, b)$ is equipped with a $\mathbb{Z}_2$-grading that represents the fermion parity of TPDs.

The $\mathbb{Z}_2$-grading of $f \in \mathrm{Hom}(a, b)$ is denoted by $|f|$, which is 0 if f is bosonic and 1 if f is fermionic.

The TPDs satisfy the (anti-)commutation relation depicted as



Gapped theories

The complete list of the gapped theories is

$\mathfrak{g}$	R
$\forall\mathfrak{g}$	adjoint
$\mathfrak{so}(N)$	$\square$
$\mathfrak{u}(N)$	$\square_q$
$\mathfrak{so}(N)$	$\square\square$
$\mathfrak{sp}(N)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\mathfrak{u}(N)$	$\square\square_q$
$\mathfrak{u}(N)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}_q$
$\mathfrak{su}(M)+\mathfrak{su}(N)+\mathfrak{u}(1)$	$(\square,\square)_q$
$\mathfrak{so}(M)+\mathfrak{so}(N)$	$(\square,\square)$
$\mathfrak{sp}(M)+\mathfrak{sp}(N)$	$(\square,\square)$

$\mathfrak{g}$	R
$\mathfrak{su}(2)$	$\mathbf{5}$
$\mathfrak{so}(9)$	$\mathbf{16}$
$\mathfrak{f}_4$	$\mathbf{26}$
$\mathfrak{sp}(4)$	$\mathbf{42}$
$\mathfrak{su}(8)$	$\mathbf{70}$
$\mathfrak{so}(16)$	$\mathbf{128}$
$\mathfrak{so}(10)+\mathfrak{u}(1)$	$\mathbf{16}_q$
$\mathfrak{e}_6+\mathfrak{u}(1)$	$\mathbf{27}_q$
$\mathfrak{su}(2)+\mathfrak{su}(2)$	$(\mathbf{2},\mathbf{4})$
$\mathfrak{su}(2)+\mathfrak{sp}(3)$	$(\mathbf{2},\mathbf{14})$
$\mathfrak{su}(2)+\mathfrak{su}(6)$	$(\mathbf{2},\mathbf{20})$
$\mathfrak{su}(2)+\mathfrak{so}(12)$	$(\mathbf{2},\mathbf{32})$
$\mathfrak{su}(2)+\mathfrak{e}_7$	$(\mathbf{2},\mathbf{56})$

α-induction

Mathematically, it is given as follows. Let $\mathcal{C}$ be the fusion category of the diagonal $\mathrm{SU}(N)_N$ WZW model. Then the vacua of the IR theory are given by A-module $\mathcal{C}_A$; the TDLs are given by A-A-bimodule ${}_A\mathcal{C}_A$.

To identify the vacua and TDLs in the QCD_2, we need to find maps $\mathcal{C} \to \mathcal{C}_A$ and $\mathcal{C} \to {}_A\mathcal{C}_A$. The latter is called α-induction.