Modified entangled states for perfect quantum teleportation

Sora Kobayashi

Graduate School of Science and Engineering, Chiba University

Sora Kobayashi, Kei-Ichi Kondo arXiv:2504.19747[quant-ph]

[1]C. H. Bennett et al. Phys. Rev. Lett. 70, 1895 (1993).
[2]D. Bouwmeester et al. Nature 390, 575–579 (1997).

Introduction

Quantum entangled states, whose applications are advancing in quantum mechanics particularly in the field of quantum information are states containing correlations unique to quantum systems that are not found in classical systems.

The theory of quantum teleportation using quantum entangled states was first formulated by Bennett et al[1]. using a 2-qubit entangled state (Bell state). Its experimental demonstration was first achieved by Zeilinger's group using a two-state system of photons[2].

Introduction

Similarly, quantum teleportation using 3 or more qubit entangled states, extended from 2-qubit entangled states, could also be considered.

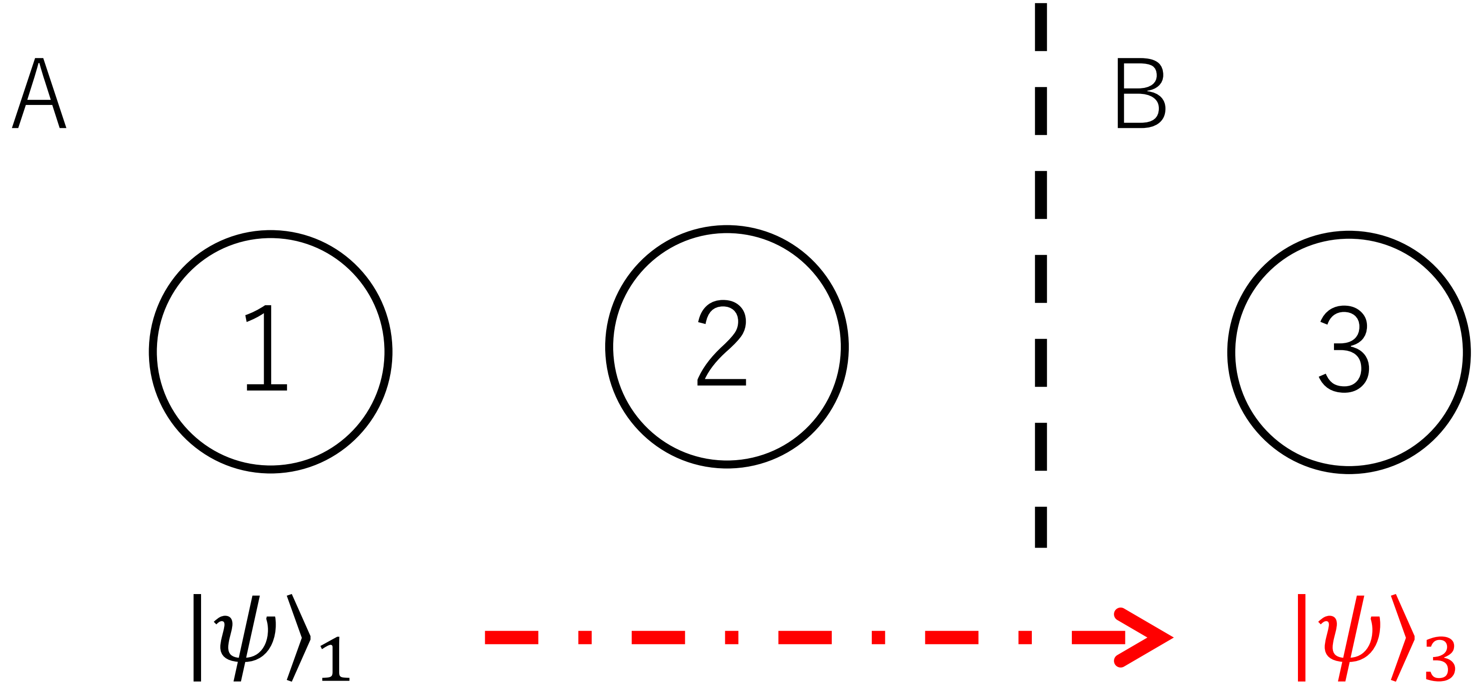
However, it is known that not all of these entangled states are suitable for quantum teleportation, particularly for perfect quantum teleportation.

Introduction

In this talk, we prove that the W state, an example of a 3-qubit entangled state, cannot be used for perfect teleportation. We then describe the construction of a modified W state (W-like state) for perfect quantum teleportation.

Quantum teleportation

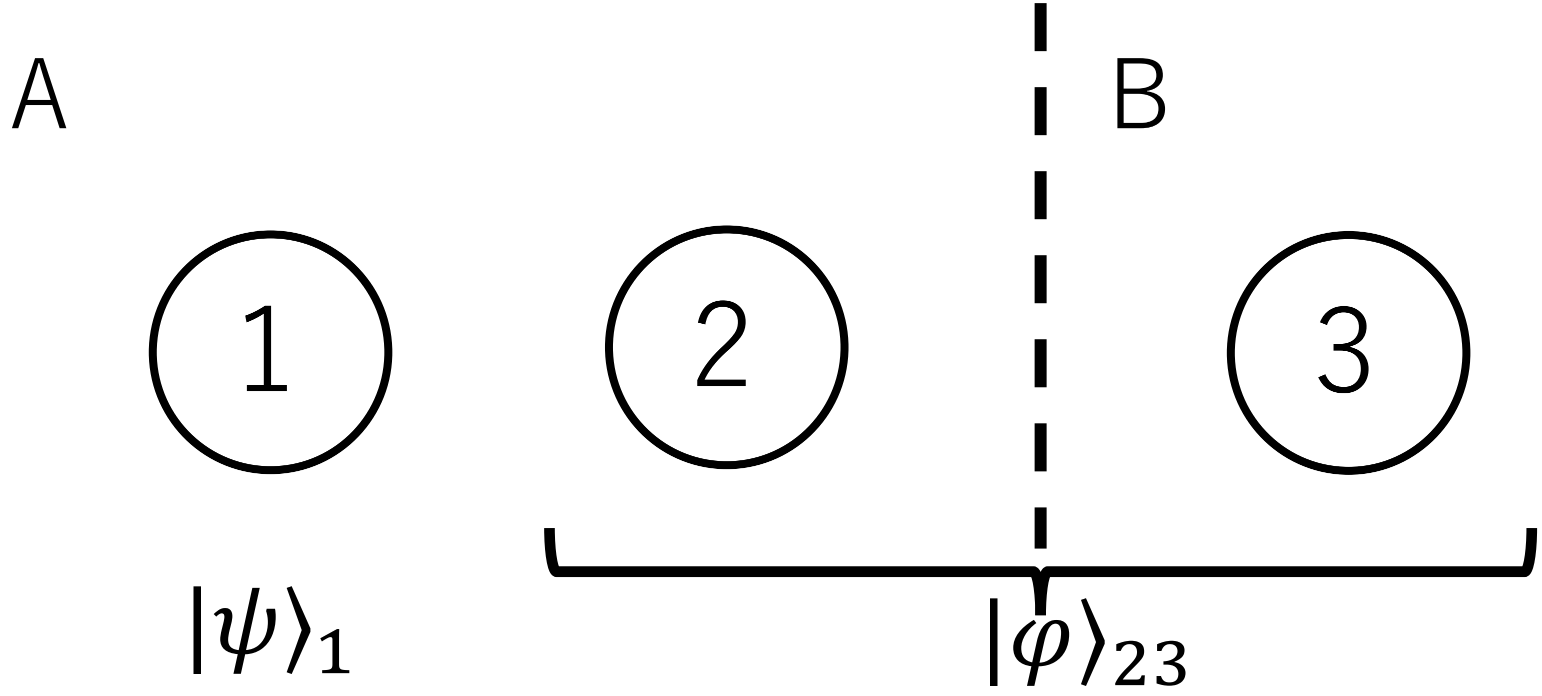
Quantum teleportation provides an example of transmitting and receiving an unknown 1-qubit state between two parties (A and B).



Quantum teleportation

(STEP1)

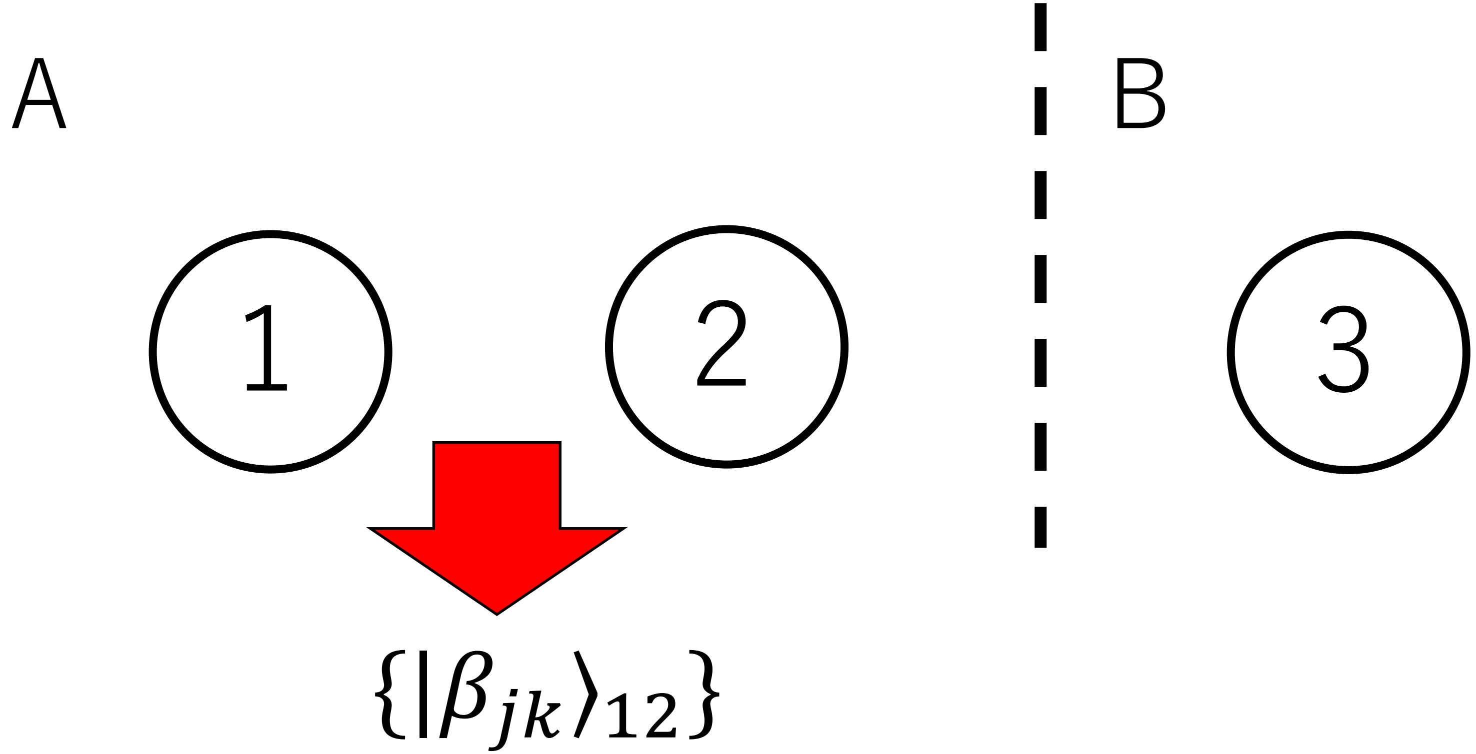
A and B share an entangled state $|\varphi\rangle_{23}$.



Quantum teleportation

(STEP2)

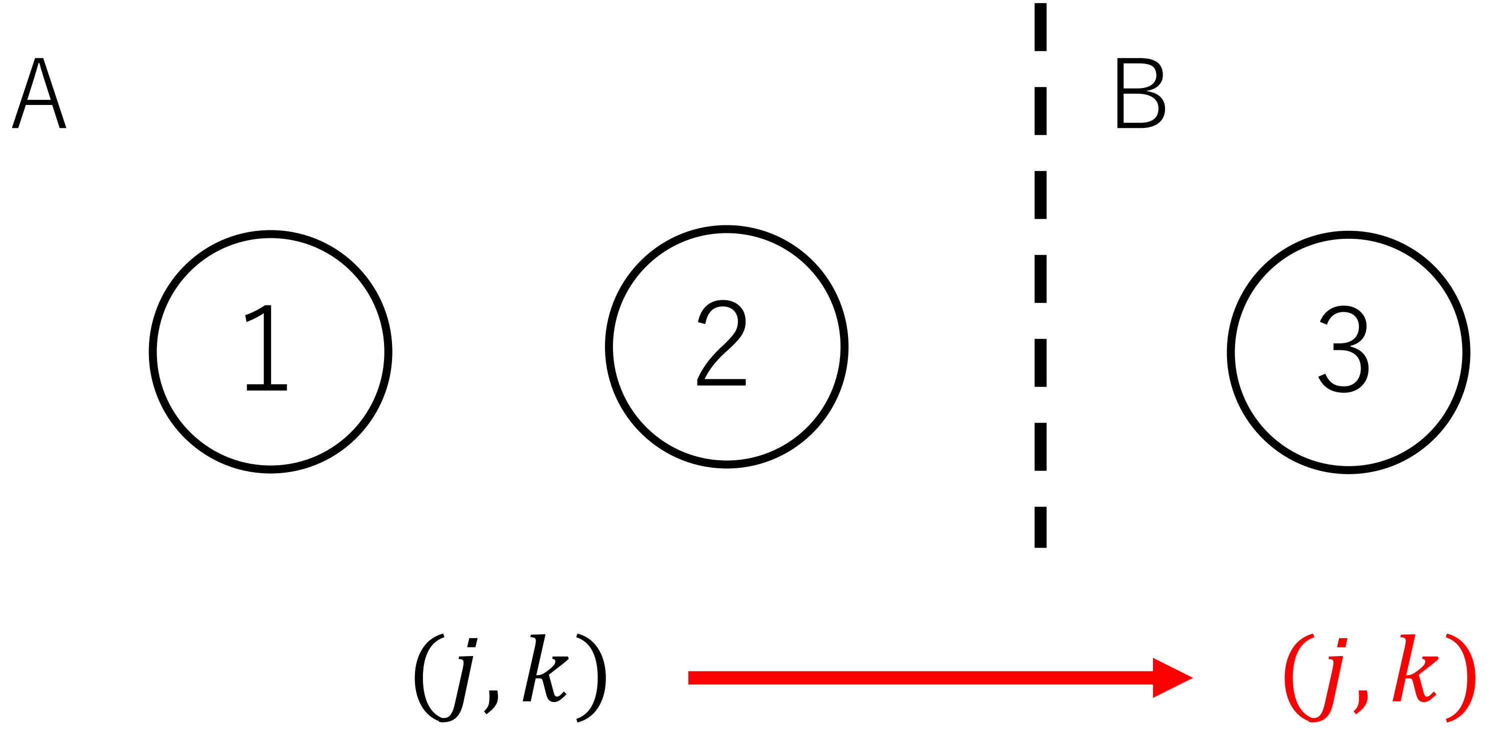
A performs a basis measurement $\{|\beta_{jk}\rangle_{12}\}(j,k \in \{0,1\})$ on the state of the composite system (12) possessed by A.



Quantum teleportation

(STEP3)

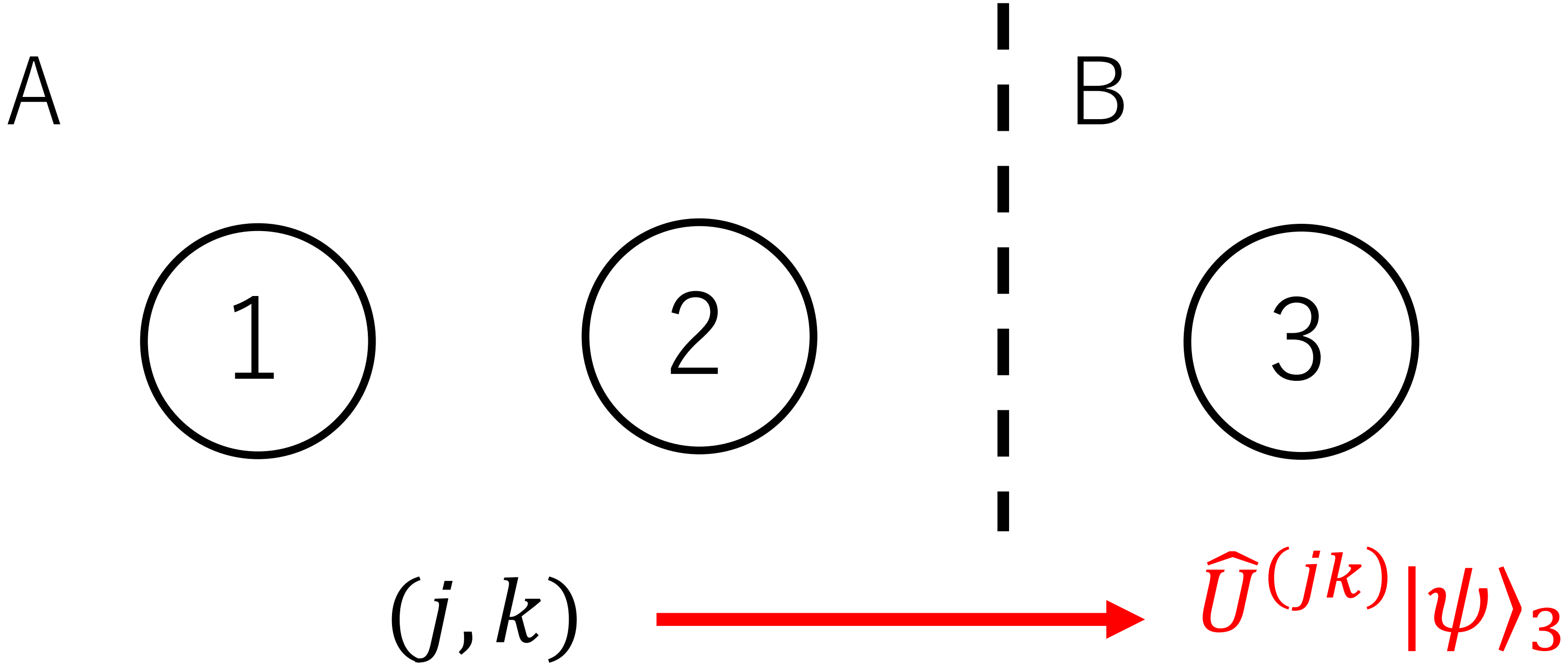
After measuring A, A communicates the label (j, k) identifying the measurement result to B through classical communication.



Quantum teleportation

(STEP4)

After receiving the measurement results(j,k), B performs the corresponding unitary operation $\hat{U}^{(jk)-1}$on the state of their own system 3 to reproduce the unknown quantum state $|\psi\rangle_3$.



Quantum teleportation

(STEP1 to STEP4)

These processes are expressed as follows:

$$|\psi\rangle_1 \otimes |\varphi\rangle_{23} = \frac{1}{2}|\beta_{00}\rangle_{12} \otimes \hat{U}^{(00)}|\psi\rangle_3 + \frac{1}{2}|\beta_{01}\rangle_{12} \otimes \hat{U}^{(01)}|\psi\rangle_3$$

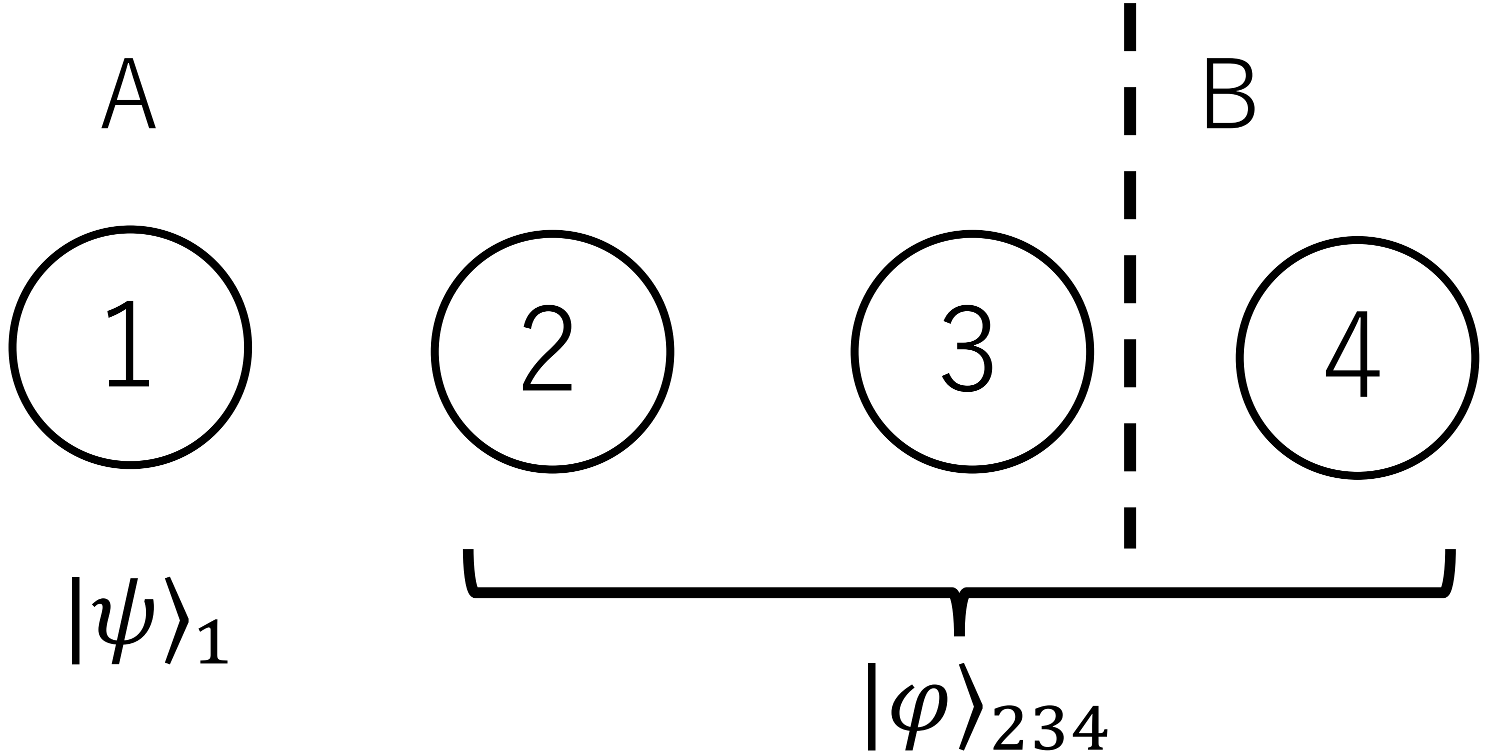
$$+\frac{1}{2}|\beta_{10}\rangle_{12} \otimes \hat{U}^{(10)}|\psi\rangle_3 + \frac{1}{2}|\beta_{11}\rangle_{12} \otimes \hat{U}^{(11)}|\psi\rangle_3.$$

B can always reproduce the unknown state $|\psi\rangle_3$ for any measurement result $(j,k) = (0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ of A

(Perfect quantum teleportation).

Extension to the 3-qubit entangled state

Next, extend the entangled state $|\varphi\rangle_{234}$ shared between A and B to a 3-qubit system, where A has 2-qubit and B has 1-qubit.



Extension to the 3-qubit entangled state

Just like in the case of a 2-qubit entangled state, the form of a perfect quantum teleportation when sharing a 3-qubit entangled state $|\varphi\rangle_{234}$ is as follows:

$$|\psi\rangle_1 \otimes |\varphi\rangle_{234} = \sum_{j,k=0}^{1} c_{jk} \left|\beta_{jk}\right\rangle_{123} \otimes \hat{U}^{(jk)} |\psi\rangle_4 \qquad (\forall c_{jk} \in \mathbb{C})$$

$\{|\beta_{jk}\rangle\}$: Measurement basis performed by A,

$\hat{U}^{(jk)}$: Unitary operator acting on state B.

Extension to the 3-qubit entangled state

In fact, when the GHZ state

$$|\varphi\rangle_{234} = |\mathrm{GHZ}\rangle_{234} = \frac{1}{\sqrt{2}}(|000\rangle_{234} + |111\rangle_{234})$$

is used as the shared state, A(system 23) and B(system 4) holding this state are maximally entangled.

The equation for perfect quantum teleportation can be specifically constructed:

$$|\psi\rangle_1 \otimes |\mathrm{GHZ}\rangle_{234} = \frac{1}{2}|\beta_{00}\rangle_{123} \otimes \hat{I}|\psi\rangle_4 + \frac{1}{2}|\beta_{01}\rangle_{123} \otimes \hat{\sigma}_x|\psi\rangle_4$$

$$+\frac{1}{2}|\beta_{10}\rangle_{12} \otimes -i\hat{\sigma}_y|\psi\rangle_4 + \frac{1}{2}|\beta_{11}\rangle_{12} \otimes \hat{\sigma}_z|\psi\rangle_4.$$

$\hat{I}$: Identity operator

$\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$: Pauli operator

Extension to the 3-qubit entangled state

The measurement basis for A is the following entangled state:

$$|\beta_{00}\rangle_{123} = \frac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123}),$$

$$|\beta_{01}\rangle_{123} = \frac{1}{\sqrt{2}}(|100\rangle_{123} + |011\rangle_{123}),$$

$$|\beta_{10}\rangle_{123} = \frac{1}{\sqrt{2}}(|100\rangle_{123} - |011\rangle_{123}),$$

$$|\beta_{11}\rangle_{123} = \frac{1}{\sqrt{2}}(|000\rangle_{123} - |111\rangle_{123}).$$

Impossibility of W states.

The W state is adopted as the shared state between A and B.

$$|\varphi\rangle_{234} = |\mathrm{W}\rangle_{234} = \frac{1}{\sqrt{3}}(|001\rangle_{234} + |010\rangle_{234} + |100\rangle_{234})$$

At this time, We prove that perfect quantum teleportation does not succeed.

Suppose A performs a measurement $\{|\beta_{jk}\rangle\}$ on the initial state of quantum teleportation $|\psi\rangle_1 \otimes |\mathrm{W}\rangle_{234}$ before sending an unknown quantum state $|\psi\rangle_1$ from A to B.

$$|\psi\rangle_1 \otimes |\mathrm{W}\rangle_{234} \longrightarrow {}_{123}\langle\beta_{jk}| \otimes |\psi\rangle_1 \otimes |\mathrm{W}\rangle_{234}$$

Impossibility of W states.

Perfect quantum teleportation succeeds when the state of B after measurement can always be written as follows:

$$ {}_{123}\langle\beta_{jk}| \otimes |\psi\rangle_1 \otimes |\mathrm{W}\rangle_{234} = \hat{T}^{(jk)}|\psi\rangle_4 $$

⇨ $\hat{T}^{(jk)} \propto$ Unitary operator

Impossibility of W states.

We have derived one conditional expression that operator $\hat{T}^{(jk)}$ must satisfy at this time.

$$\sum_{j,k=0}^{1}\left(\left|\langle 0|\hat{T}^{(jk)}|0\rangle\right|^2+\left|\langle 0|\hat{T}^{(jk)}|1\rangle\right|^2\right)=\sum_{j,k=0}^{1}\left(\left|\langle 1|\hat{T}^{(jk)}|0\rangle\right|^2+\left|\langle 1|\hat{T}^{(jk)}|1\rangle\right|^2\right)$$

Each term $\sum\left|\langle \ell|\hat{T}^{(jk)}|m\rangle\right|^2$ are determined from the shared state $|\mathrm{W}\rangle_{234}$.

Contradiction

$$(\text{left hand side})=\frac{4}{3}\neq\frac{2}{3}=(\text{right hand side})$$

Modified W state(W-like state)

We modify the coefficients of the basis constituting the W state:

$$|\mathrm{W}\rangle_{234} = \frac{1}{\sqrt{3}}(|001\rangle_{234} + |010\rangle_{234} + |100\rangle_{234})$$

$$\longrightarrow \quad |\mathrm{W\text{-}like}\rangle_{234} = a_0|001\rangle_{234} + a_1|010\rangle_{234} + a_2|100\rangle_{234}$$

We impose the requirement that these coefficients satisfy our conditions:

$$a_0 = \frac{1}{\sqrt{2}}, \qquad a_1 = \frac{1}{\sqrt{2}}e^{i\delta}\cos\gamma, \qquad a_2 = \frac{1}{\sqrt{2}}e^{i\omega}\sin\gamma \qquad \delta, \omega, \gamma \in \mathbb{R}$$

Modified W state(W-like state)

It is specifically confirmed that the W-like state, unlike the W state, is a shared maximally entangled state that can be used for perfect quantum teleportation.

$$|\text{W-like}\rangle_{234} = \frac{1}{\sqrt{2}}(|001\rangle_{234} + e^{i\delta}\cos\gamma\,|010\rangle_{234} + e^{i\omega}\sin\gamma\,|100\rangle_{234})$$

$$|\psi\rangle_1 \otimes |\text{W-like}\rangle_{234} = \frac{1}{2}|\beta_{00}\rangle_{123} \otimes \hat{I}|\psi\rangle_4 + \frac{1}{2}|\beta_{01}\rangle_{123} \otimes \hat{\sigma}_x|\psi\rangle_4$$

$$+\frac{1}{2}|\beta_{10}\rangle_{12} \otimes \hat{\sigma}_y|\psi\rangle_4 + \frac{1}{2}|\beta_{11}\rangle_{12} \otimes \hat{\sigma}_z|\psi\rangle_4.$$

Modified W state(W-like state)

The measurement basis for A is the following entangled state:

$$|\beta_{00}\rangle_{123} = \frac{1}{\sqrt{2}}\left(e^{i\delta}\cos\gamma\,|001\rangle_{123} + e^{i\omega}\sin\gamma\,|010\rangle_{123} + |100\rangle_{123}\right),$$

$$|\beta_{10}\rangle_{123} = \frac{1}{\sqrt{2}}\left(e^{i\delta}\cos\gamma\,|101\rangle_{123} + e^{i\omega}\sin\gamma\,|110\rangle_{123} - |000\rangle_{123}\right),$$

$$|\beta_{01}\rangle_{123} = \frac{1}{\sqrt{2}}\left(e^{i\delta}\cos\gamma\,|101\rangle_{123} + e^{i\omega}\sin\gamma\,|110\rangle_{123} + |000\rangle_{123}\right),$$

$$|\beta_{11}\rangle_{123} = \frac{1}{\sqrt{2}}\left(e^{i\delta}\cos\gamma\,|001\rangle_{123} + e^{i\omega}\sin\gamma\,|010\rangle_{123} - |100\rangle_{123}\right).$$

Conclusion

In this talk, we demonstrated that it is impossible to perfectly teleport an unknown 1-qubit quantum state between two parties(A and B) sharing a W state through basis measurements.

We succeeded in deriving an entangled state (W-like state) suitable for perfect quantum teleportation by modifying the coefficients of the basis constituting the W state.