# Semi-classical correspondence between Petz map and Bayes' rule: from Lindbladian to reverse diffusion

Ryota Nasu[1]
in collaboration with: Gota Tanaka[2] and Asato Tsuchiya[1]

[1]Shizuoka Univ. , [2]Meiji Gakuin Univ.

arXiv: 2510.18512

KEK Theory Workshop 2025
17 - 19 December 2025

# 1.Background

# Background1: Petz map and Bayes' rule

- Petz map[Petz, 1988]

$$\mathcal{P}_{\mathcal{A},\hat{\gamma}}(\cdot) = \hat{\gamma}^{1/2}\mathcal{A}^{\dagger}(\mathcal{A}(\hat{\gamma})^{-1/2}(\cdot)\mathcal{A}(\hat{\sigma})^{-1/2})\hat{\gamma}^{1/2}$$

  - $\mathcal{A}$: TPCP map($\mathcal{A}^{\dagger}$: adjoint with respect to the Hilbert-Schmidt inner product)
  - $\hat{\gamma}$: reference state (density matrix)
  - By definition, $\mathcal{P}_{\mathcal{A},\hat{\gamma}} \circ \mathcal{A}(\hat{\gamma}) = \hat{\gamma}$.

    $\Rightarrow$ Petz map is a "good" reverse process for quantum dynamics.

- Petz map is nowadays used in
  - Quantum error correction (cf. [Barnum and Knill, 2002])
  - Black hole information paradox (cf.[Penington et al., 2022])
  - Bulk reconstruction in the AdS/CFT correspondence (cf. [Chen et al., 2020])

$\Rightarrow$ It is important to understand the Petz map better.

- Petz map is a quantum analog of Bayes' rule.

# Schematic correspondence between Petz map and Bayes' rule

- Bayes' rule

$$P(x_t,t|x_s,s)=\frac{P(x_s,s|x_t,t)P(x_t,t)}{P(x_s,s)}\quad (s>t)$$

$\to$ transition probability that traces time backward

$$Q(x_t,t):=\int_{x_s}P(x_t,t|x_s,s)Q(x_s,s)$$

$$=P(x_t,t)^{1/2}\left[\int_{x_s}P(x_s,s|x_t,t)\left(P(x_s,s)^{-1/2}Q(x_s,s)P(x_s,s)^{-1/2}\right)\right]P(x_t,t)^{1/2}$$

- Note: $P(x_s,s)=\int_{x_t}P(x_s,s|x_t,t)P(x_t,t)$
- Petz map

$$\mathcal{P}_{\mathcal{A},\hat{\gamma}}(\rho)=\hat{\gamma}^{1/2}\mathcal{A}^{\dagger}(\mathcal{A}(\hat{\gamma})^{-1/2}(\rho)\mathcal{A}(\hat{\gamma})^{-1/2})\hat{\gamma}^{1/2}$$

$\to$ We see that the Petz map is a quantum analog of Bayes' rule.

# Backgrounds 2: A reverse process for quantum and classical processes

- In this talk, we focus on the Markov process.
- Quantum Markov process (semigroup TPCP) ⇔ Lindblad equation, a master equation for open quantum system

$$\frac{\partial \hat{\gamma}_t}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_t, \hat{\gamma}_t\right] + \sum_\alpha \left(\hat{L}_{\alpha,t}\hat{\gamma}_t\hat{L}^\dagger_{\alpha,t} - \frac{1}{2}\left\{\hat{L}^\dagger_{\alpha,t}\hat{L}_{\alpha,t}, \hat{\gamma}_t\right\}\right)$$

  - A reverse process for the Lindblad dynamics based on the Petz map was constructed in [Kwon and Kim, 2019, Kwon et al., 2022].
- Classical Markov process ⇔ Fokker-Planck equation

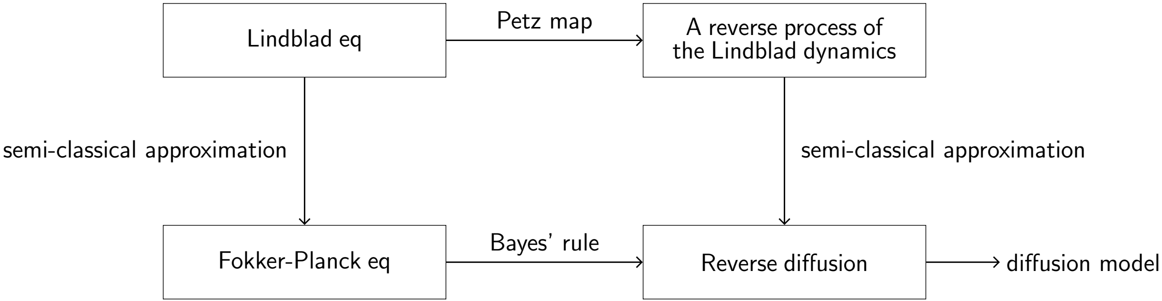
$$\frac{\partial P(x_t, t)}{\partial t} = -\sum_i \frac{\partial}{\partial x^i_t}\left[f^i(x_t, t)P(x_t, t)\right] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x^i_t \partial x^j_t}\left[D^{ij}(x_t, t)P(x_t, t)\right]$$

  - A reverse process of the Fokker-Planck dynamics based on Bayes' rule is known as reverse diffusion [Anderson, 1982].
  - This provides a mathematical foundation for diffusion models [Song et al., 2020], which have recently gained attention in the context of image-generating AI.

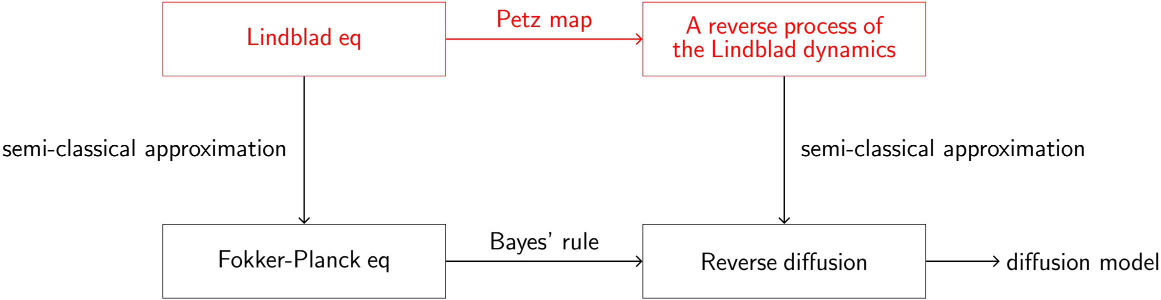
⇒ Are there any relationships between quantum and classical reverse processess?

# The goal of this talk.

- In this talk, we show that there is a concrete correspondence between the reverse process of the Lindblad equation and the reverse diffusion process, by using a semi-classical approximation based on the Wigner transform and an $\hbar$ expansion.



# 2.Reverse process of the Lindblad dynamics (review)



# Reverse process of the Lindblad dynamics [Kwon and Kim, 2019, Kwon et al., 2022]

- Lindblad equation:

$$\frac{\partial \hat{\gamma}_t}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_t, \hat{\gamma}_t\right] + \sum_\alpha \left(\hat{L}_{\alpha,t}\hat{\gamma}_t\hat{L}^\dagger_{\alpha,t} - \frac{1}{2}\left\{\hat{L}^\dagger_{\alpha,t}\hat{L}_{\alpha,t}, \hat{\gamma}_t\right\}\right)$$

$\Downarrow$ Petz map which uses $\hat{\gamma}_t$ as reference state.

- A reverse process of the Lindblad dynamics

$$-\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{\tilde{H}}_t, \hat{\rho}\right] + \sum_\alpha \left(\hat{\tilde{L}}_{\alpha,t}\hat{\rho}\hat{\tilde{L}}^\dagger_{\alpha,t} - \frac{1}{2}\left\{\hat{\tilde{L}}^\dagger_{\alpha,t}\hat{\tilde{L}}_{\alpha,t}, \hat{\rho}\right\}\right)$$

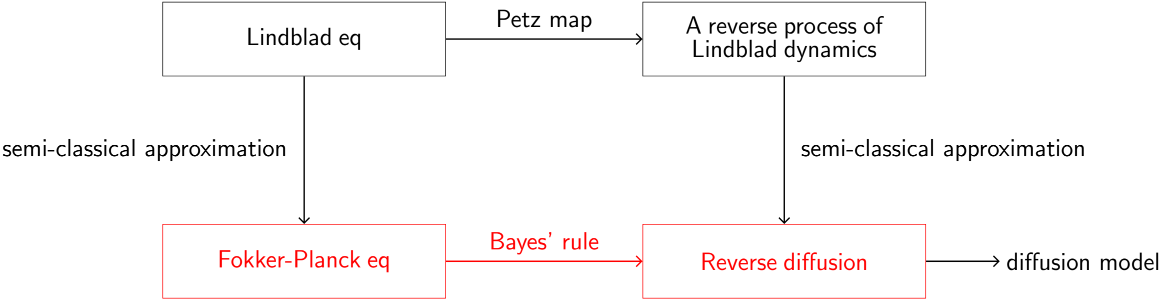
where

$$\hat{G}_t = \hat{\gamma}_t^{1/2}, \quad \dot{\hat{G}}_t = d\hat{G}_t/dt \ ,$$

$$\hat{\tilde{H}}_t = -\frac{1}{2}\left(\hat{G}_t\hat{H}_t\hat{G}_t + i\hbar\dot{\hat{G}}_t\hat{G}_t^{-1} + \frac{i\hbar}{2}\sum_\alpha \hat{G}_t\hat{L}^\dagger_{\alpha,t}\hat{L}_{\alpha,t}\hat{G}_t^{-1}\right) + h.c. \ ,$$

$$\hat{\tilde{L}}_{\alpha,t} = \hat{G}_t\hat{L}^\dagger_{\alpha,t}\hat{G}_t^{-1} \ .$$

# 3.Reverse diffusion (review)



# Reverse diffusion[Anderson, 1982]

- Fokker – Planck eq:

$$\frac{\partial P(x_t,t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_t^i}\left[f^i(x_t,t)P(x_t,t)\right] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_t^i \partial x_t^j}\left[D^{ij}(x_t,t)P(x_t,t)\right]$$

$\Downarrow$ Bayes' rule: $Q(x_t,t) = \int_{x_s} P(x_t,t|x_s,s)Q(x_s,s) = \int_{x_s} Q(x_s,s)\frac{P(x_s,s|x_t,t)P(x_t,t)}{P(x_s,s)} (s > t)$

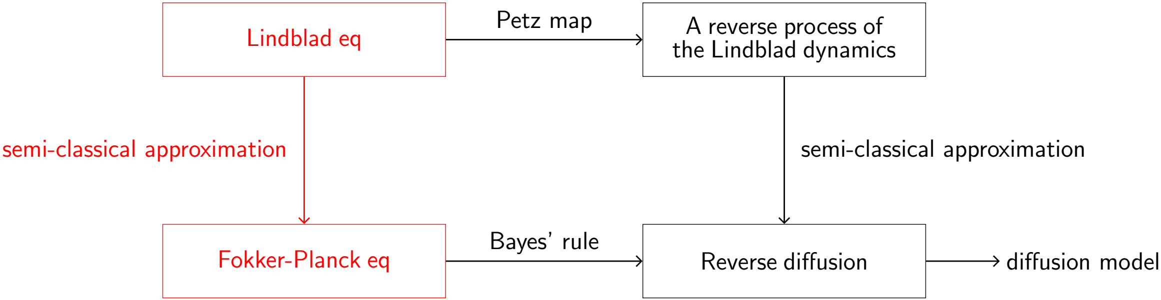
- Reverse diffusion :

$$-\frac{\partial Q(x_t,t)}{\partial t} = \sum_i \frac{\partial}{\partial x_t^i}\left[\bar{f}^i(x_t,t)Q(x_t,t)\right] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_t^i \partial x_t^j}\left[D^{ij}(x_t,t)Q(x_t,t)\right]$$

where $\bar{f}^i(x_t,t) = f^i(x_t,t) - \frac{1}{P(x_t,t)}\sum_j \frac{\partial}{\partial x_t^j}\left[D^{ij}(x_t,t)P(x_t,t)\right]$

$\rightarrow$the drift term adjusts the distribution to the initial distribution $P(x,0)$.
(cf. diffusion model$\rightarrow$ approximate the diffusion correction term by neural networks.)

# 4.Fokker–Planck eq from Lindblad eq



# Wigner transformation

- Wigner transformation: A transformation between operators in the Hilbert space and functions on the phase space.

$$O(\boldsymbol{Q},\boldsymbol{P}) = \int d^N\sigma \left\langle \boldsymbol{Q}+\frac{\boldsymbol{\sigma}}{2}\right| \hat{O}(\hat{q},\hat{p}) \left|\boldsymbol{Q}-\frac{\boldsymbol{\sigma}}{2}\right\rangle e^{-\frac{i}{\hbar}\boldsymbol{P}\cdot\boldsymbol{\sigma}}$$

$$\text{where} \quad \boldsymbol{Q} = \frac{\boldsymbol{q}_1+\boldsymbol{q}_2}{2},\ \boldsymbol{\sigma} = \boldsymbol{q}_1 - \boldsymbol{q}_2$$

- Wigner function: Wigner transform of the density operator.

$$W^{(\rho)}(\boldsymbol{Q},\boldsymbol{P}) = \int d^N\sigma \left\langle \boldsymbol{Q}+\frac{\boldsymbol{\sigma}}{2}\right| \hat{\rho} \left|\boldsymbol{Q}-\frac{\boldsymbol{\sigma}}{2}\right\rangle e^{-\frac{i}{\hbar}\boldsymbol{P}\cdot\boldsymbol{\sigma}}$$

  - Properties:

$$\begin{cases} \displaystyle\int \frac{d^N Q d^N P}{(2\pi\hbar)^N} W^{(\rho)}(\boldsymbol{Q},\boldsymbol{P}) = 1 \\ \displaystyle\left\langle \hat{O} \right\rangle = \mathrm{Tr}\left(\hat{O}\hat{\rho}\right) = \int \frac{d^N Q d^N P}{(2\pi\hbar)^N} O(\boldsymbol{Q},\boldsymbol{P}) W^{(\rho)}(\boldsymbol{Q},\boldsymbol{P}) \end{cases}$$

$\Rightarrow$ (quasi) probability on phase space.

# Moyal product

- Wigner transform of the product of operators:

$$\hat{f}\hat{g} \xrightarrow{\text{Wigner}} f(\boldsymbol{Q},\boldsymbol{P}) \star g(\boldsymbol{Q},\boldsymbol{P})$$

Here

$$f(\boldsymbol{Q},\boldsymbol{P}) \star g(\boldsymbol{Q},\boldsymbol{P}) := \exp\left[\frac{i\hbar}{2}\sum_{i=1}^{N}\left(\frac{\partial}{\partial Q_i}\frac{\partial}{\partial P_i'} - \frac{\partial l}{\partial Q_i'}\frac{\partial}{\partial P_i}\right)\right] f(\boldsymbol{Q},\boldsymbol{P}) g(\boldsymbol{Q'},\boldsymbol{P'})\Bigg|_{\substack{\boldsymbol{Q'}=\boldsymbol{Q}\\ \boldsymbol{P'}=\boldsymbol{P}}}$$

$$= fg + \frac{i\hbar}{2}\{f,g\}_p - \frac{\hbar^2}{8}\sum_i\left[\left\{\frac{\partial f}{\partial Q_i},\frac{\partial g}{\partial P_i}\right\}_p - \left\{\frac{\partial f}{\partial P_i},\frac{\partial g}{\partial Q_i}\right\}_p\right] + \mathcal{O}\left(\hbar^3\right) ,$$

where $\{,\}_p$ is the Poisson bracket.

# Lindblad eq→Fokker-Planck eq

- We assume $\hat{L}_{\alpha,t} = \hbar^{-1/2}\hat{\ell}_{\alpha,t}$.
- Apply the Wigner transformation to the Lindblad equation and expand it up to the first order in $\hbar$.
  ⇒ we obtain the Fokker-Planck type equation as follows:

$$\frac{\partial W_t^{(\gamma)}}{\partial t} = -\sum_{\mu} \frac{\partial}{\partial X^\mu}\left(f^\mu(X,t)W_t^{(\gamma)}\right) + \frac{1}{2}\sum_{\mu,\nu}\frac{\partial^2}{\partial X^\mu \partial X^\nu}\left(G_R^{\mu\nu}(X,t)W_t^{(\gamma)}\right) .$$

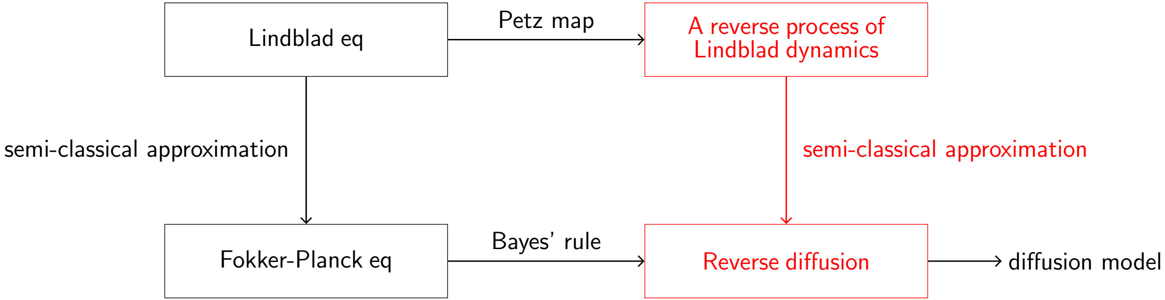
- Here, $X^\mu = (\boldsymbol{Q}, \boldsymbol{P})$ is a position on the phase space and

$$f^\mu(X,t) := \sum_{\nu}\omega^{\mu\nu}\left[\frac{\partial H_t}{\partial X^\nu} + \sum_{\alpha}\left(\mathrm{Im}\left(\ell_{\alpha,t}\frac{\partial \ell^*_{\alpha,t}}{\partial X^\nu}\right) - \frac{\hbar}{2}\mathrm{Re}\left\{\ell_{\alpha,t}, \frac{\partial \ell^*_{\alpha,t}}{\partial X^\nu}\right\}_p\right)\right] , \tag{1}$$

$$G^{\mu\nu}(X,t) = \hbar\sum_{\lambda,\rho}\sum_{\alpha}\omega^{\mu\lambda}\omega^{\nu\rho}\,\mathrm{Re}\left(\frac{\partial \ell_{\alpha,t}}{\partial X^\lambda}\frac{\partial \ell^*_{\alpha,t}}{\partial X^\rho}\right) , \tag{2}$$

where $\omega^{\mu\nu}$ is a sympletic form.

# 4. Reverse diffusion from a reverse process for Lindblad dynamics



# A reverse process of Lindblad eq$\rightarrow$ reverse diffusion

- The reverse process of the Lindblad dynamics is also a Lindblad equation, so we can apply the result for the forward process:

$$-\frac{\partial W_t^{(\rho)}}{\partial t} = -\sum_{\mu} \frac{\partial}{\partial X^{\mu}} \left( \tilde{f}^{\mu}(X,t) W_t^{(\rho)} \right) + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial X^{\mu} \partial X^{\nu}} \left( \tilde{G}_R^{\mu\nu}(X,t) W_t^{(\rho)} \right)$$

Here,

$$\tilde{f}, \tilde{G}_R = \left( \begin{array}{l} \text{The one obtaind by replacing the } H_t \text{ and } \ell_{\alpha,t} \text{ appearing in } f \text{ and } G_R \\ \text{in the forward process with } \tilde{H}_t \text{ and } \tilde{\ell}_{\alpha,t} \end{array} \right)$$

- $\tilde{H}_t$ and $\tilde{\ell}_{\alpha,t}$ are still contains Moyal product$\rightarrow$ Expand them up to the first order in $\hbar$.
- Diffusion matrix: $\tilde{G}_R = G_R + \mathcal{O}\left(\hbar^2\right)$

# A reverse process of Lindblad eq→ reverse diffusion: correction for drift

$$\hat{\tilde{H}}_t = -\hat{H}_t + \frac{i}{4}\sum_k \left\{\left[\hat{\ell}_{k,t}\hat{\ell}_{k,t}, \hat{G}\right], \hat{G}^{-1}\right\} + \underbrace{\frac{1}{2}\left[\left[\hat{H}_t, \hat{G}\right], \hat{G}^{-1}\right] - \frac{i\hbar}{2}\left[\dot{\hat{G}}, \hat{G}^{-1}\right]}_{\sim\mathcal{O}(\hbar^2)}$$

$$\xrightarrow{\text{Wigner\&}\hbar\text{ expansion}} -H_t + \frac{i}{2}\sum_k \left(i\hbar\left\{|\ell_{k,t}|^2, G_W\right\}_p G_W^{-1}\right) + \mathcal{O}(\hbar^2)$$

$$\hat{\tilde{\ell}}_{k,t} = \hat{\ell}^\dagger_{k,t} - \hat{G}\left[\hat{G}^{-1}, \hat{\ell}^\dagger_{k,t}\right] \xrightarrow{\text{Wigner\&}\hbar\text{ expansion}} \ell^*_{k,t} - i\hbar G_W\left\{G_W^{-1}, \ell^*_{k,t}\right\}_p + \mathcal{O}(\hbar^2)$$

- On the zeroth-order term of $\hbar$,

$$\left\{|\ell_{k,t}|^2, G_W\right\}_p G_W^{-1} \to \frac{1}{2}\frac{1}{W_t^{(\gamma)}}\frac{\partial W_t^{(\gamma)}}{\partial X^\mu}, \quad G_W\left\{G_W^{-1}, \ell^*_{k,t}\right\}_p \to -\frac{1}{2}\frac{1}{W_t^{(\gamma)}}\frac{\partial W_t^{(\gamma)}}{\partial X^\mu}$$

⟶The structure of the correction term for the drift term appearing in the reverse diffusion.

# A reverse process of Lindblad eq→reverse diffusion: result

$$-\frac{\partial W_t^{(\rho)}}{\partial t} = \sum_{\mu} \frac{\partial}{\partial X^{\mu}} \left( \bar{f}^{\mu}(X,t) W_t^{(\rho)} \right) + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial X^{\mu'} \partial X^{\nu'}} \left( G_R^{\mu\nu}(X,t) W_t^{(\rho)} \right)$$

$$\text{where} \quad \bar{f}^{\mu} = f^{\mu} - \frac{1}{W_t^{(\gamma)}} \sum_{\nu} \frac{\partial}{\partial X^{\nu'}} \left( G_R^{\mu\nu} W_t^{(\gamma)} \right)$$

$$\Downarrow$$

We obtain the reverse diffusion equation from the reverse process of the Lindblad equation.

# 5. Application to the renormalization group (on going)

# A inverse of the renormalization group flow

- Motivation:
  - Predicting high-energy theory from low-energy data
  - A relationship between AdS/CFT and RG→Understanding the Emergence of Bulk Geometry [de Boer et al., 2000, Swingle, 2012, Nozaki et al., 2012]
  - application to critical slowing down[Bachtis et al., 2022]
- Exact renormalization group equation for density matrix (our work)

$$
\begin{aligned}
&-\Lambda\frac{\partial}{\partial\Lambda}\rho_\Lambda[\varphi^+,\varphi^-] \\
&= \int_{\boldsymbol{p}}\left[-\frac{1}{2}\frac{\dot{K}_{\Lambda,\boldsymbol{p}}}{2\omega_{\boldsymbol{p}}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)\left(\frac{\delta}{\delta\varphi^+}+\frac{\delta}{\delta\varphi^-}\right)^2-\frac{1}{2}\frac{\dot{K}_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}\left(\frac{\delta}{\delta\varphi^+}+\frac{\delta}{\delta\varphi^-}\right)(\varphi^++\varphi^-)\right. \\
&\qquad\left.+\frac{1}{4}\frac{\dot{K}_{\Lambda,\boldsymbol{p}}}{K^2_{\Lambda,\boldsymbol{p}}}\omega_{\boldsymbol{p}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)(\varphi^+-\varphi^-)^2\right]\rho_\Lambda[\varphi^+,\varphi^-]
\end{aligned}
$$

# The feature of ERG eq for density matrix

- The ERG eq for the density matrix is equivalent to the Lindblad eq, which has the following Hamiltonian and jump operators:
  - "Hamiltonian":

$$X_\Lambda = -\frac{1}{4}\int_{\boldsymbol{p}} \frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}\left(\hat{\pi}_{-\boldsymbol{p}}\hat{\phi}_{\boldsymbol{p}} + \hat{\phi}_{\boldsymbol{p}}\hat{\pi}_{-\boldsymbol{p}}\right) = -\frac{i}{4}\int_{\boldsymbol{p}}\left(\hat{a}^\dagger_{\Lambda,\boldsymbol{p}}\hat{a}^\dagger_{\Lambda,-\boldsymbol{p}} - \hat{a}_{\Lambda,\boldsymbol{p}}\hat{a}_{\Lambda,-\boldsymbol{p}}\right)$$

  - Jump operators:

$$L_1(\boldsymbol{p}) = \sqrt{\frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{2K_{\Lambda,\boldsymbol{p}}}\left(\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)+1\right)}\,\hat{a}_{\Lambda,\boldsymbol{p}}, \quad L_2(\boldsymbol{p}) = \sqrt{\frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{2K_{\Lambda,\boldsymbol{p}}}\left(\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)-1\right)}\,\hat{a}^\dagger_{\Lambda,\boldsymbol{p}}$$

$$\hat{a}_{\Lambda,\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}}\hat{\phi}_{\boldsymbol{p}} + i\sqrt{\frac{K_{\Lambda,\boldsymbol{p}}}{\omega_{\boldsymbol{p}}}}\hat{\pi}_{\boldsymbol{p}}\right)$$

- It becomes the Fokker-Planck equation on the phase space by Wigner transformation without $\hbar$ expansion.

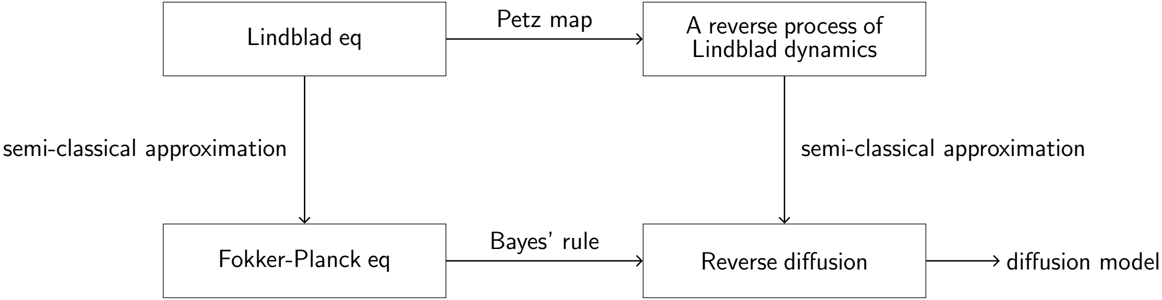
⇒We can analyze the inverse of the renormalization group in terms of the diffusion model by using the previous result.

# 6. Conclusion

# Conclusion

## Conclusion

Using the semi-classical approximation based on the Wigner transform and $\hbar$ expansion, we showed the relationship between the reverse process of the Lindblad dynamics via Petz map and the reverse diffusion process via Bayes' rule.



# Future prospects

- Numerical Verification
  - $\hbar$ dependence
  - Using the ERG eq for density matrix

$$\hat{H} \propto (\hat{a}^\dagger)^2 - (\hat{a})^2, \quad \hat{L}_1 \propto \hat{a}, \quad \hat{L}_2 \propto \hat{a}^\dagger$$

    Note: ERG eq for density matrix becomes a Fokker-Planck type equation without $\hbar$ expansion.

- Analysis of the inverse of the renormalization group based on the diffusion model.
- Construct a quantum version of the diffusion model based on Petz map (cf. [Liu et al., 2025, Hu et al., 2025]).

# References I


Anderson, B. D. (1982).
Reverse-time diffusion equation models.
*Stochastic Processes and their Applications*, 12(3):313–326.

Bachtis, D., Aarts, G., Di Renzo, F., and Lucini, B. (2022).
Inverse Renormalization Group in Quantum Field Theory.
*Phys. Rev. Lett.*, 128(8):081603.

Barnum, H. and Knill, E. (2002).
Reversing quantum dynamics with near-optimal quantum and classical fidelity.
*J. Math. Phys.*, 43(5):2097.

Chen, C.-F., Penington, G., and Salton, G. (2020).
Entanglement Wedge Reconstruction using the Petz Map.
*JHEP*, 01:168.

de Boer, J., Verlinde, E. P., and Verlinde, H. L. (2000).
On the holographic renormalization group.
*JHEP*, 08:003.

Hu, F., Liu, G., Zhang, Y., and Gao, X. (2025).
Local Diffusion Models and Phases of Data Distributions.

# References II


Kuwahara, T., Nasu, R., Tanaka, G., and Tsuchiya, A. (2024).
Quantum Error Correction Realized by the Renormalization Group in Scalar Field Theories.
*PTEP*, 2024(8):083B01.

Kwon, H. and Kim, M. (2019).
Fluctuation theorems for a quantum channel.
*Physical Review X*, 9(3):031029.

Kwon, H., Mukherjee, R., and Kim, M. S. (2022).
Reversing Lindblad Dynamics via Continuous Petz Recovery Map.
*Phys. Rev. Lett.*, 128(2):020403.

Liu, X., Zhuang, J., Hou, W., and You, Y.-Z. (2025).
Measurement-Based Quantum Diffusion Models.

Nozaki, M., Ryu, S., and Takayanagi, T. (2012).
Holographic Geometry of Entanglement Renormalization in Quantum Field Theories.
*JHEP*, 10:193.

Penington, G., Shenker, S. H., Stanford, D., and Yang, Z. (2022).
Replica wormholes and the black hole interior.
*JHEP*, 03:205.

# References III


Petz, D. (1988).
SUFFICIENCY OF CHANNELS OVER VON NEUMANN ALGEBRAS.
*Quart. J. Math. Oxford Ser.*, 39(1):97–108.

Song, Y., Sohl-Dickstein, J., Kingma, D. P., Kumar, A., Ermon, S., and Poole, B. (2020).
Score-based generative modeling through stochastic differential equations.
*arXiv preprint arXiv:2011.13456*.

Swingle, B. (2012).
Entanglement Renormalization and Holography.
*Phys. Rev. D*, 86:065007.

# Backup slides

# Reverse process of the Lindblad dynamics[Kwon and Kim, 2019, Kwon et al., 2022]

- Lindblad equation:

$$\frac{\partial \hat{\rho}_t}{\partial t} = \mathcal{L}(\hat{\rho}_t) = -\frac{i}{\hbar}\left[\hat{H}_t, \hat{\rho}_t\right] + \sum_k \left(\hat{L}_{k,t}\hat{\rho}_t\hat{L}_{k,t}^\dagger - \frac{1}{2}\left\{\hat{L}_{k,t}^\dagger\hat{L}_{k,t}, \hat{\rho}_t\right\}\right)$$

- Infinitesimal time evolution:

$$\hat{\rho}_{t+\Delta t} = \mathcal{T}_{\Delta t}(\hat{\rho}_t) = (1 + \Delta t\mathcal{L})(\hat{\rho}_t)$$

- $\hat{\gamma}_t$: a quantum state which follow the Lindblad eq.
- Petz map for $\mathcal{T}_{\Delta t}$:

$$\mathcal{P}_{\mathcal{T}_{\Delta t},\hat{\gamma}_t}(\cdot) = \hat{\gamma}_t^{1/2}(1 + \Delta t\mathcal{L})^\dagger\left(\hat{\gamma}_{t+\Delta t}^{-1/2}(\cdot)\hat{\gamma}_{t+\Delta t}^{-1/2}\right)\hat{\gamma}_t^{1/2}$$

# Reverse process of the Lindblad dynamics(Cont'd)

- Expanding it around $t$ and truncate the terms which order is higher than $\mathcal{O}(\Delta t^2)$:

$$-\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{\tilde{H}}_t, \hat{\rho}\right] + \sum_k \left(\hat{\tilde{L}}_{k,t}\hat{\rho}\hat{\tilde{L}}_{k,t}^\dagger - \frac{1}{2}\left\{\hat{\tilde{L}}_{k,t}^\dagger\hat{\tilde{L}}_{k,t}, \hat{\rho}\right\}\right)$$

where we have set $\hat{G}_t = \hat{\gamma}_t^{1/2}$, $\dot{\hat{G}}_t = d\hat{G}_t/dt$ and

$$\hat{\tilde{H}}_t = -\frac{1}{2}\left(\hat{G}_t\hat{H}_t\hat{G}_t + i\hbar\dot{\hat{G}}_t\hat{G}_t^{-1} + \frac{i\hbar}{2}\sum_k \hat{G}_t\hat{L}_{k,t}^\dagger\hat{L}_{k,t}\hat{G}_t^{-1}\right) + h.c.$$

$$\hat{\tilde{L}}_{k,t} = \hat{G}_t\hat{L}_{k,t}^\dagger\hat{G}_t^{-1}$$

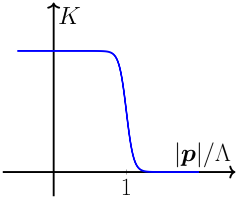
- We have got a reverse process for Lindblad dynamics.

# ERG eq for density matrix: setup 1

- Finite temperature scalar field theory

$$S_\Lambda = \frac{1}{2}\int_{-\beta/2}^{\beta/2} dt \int_{\boldsymbol{p}} K_{\Lambda,\boldsymbol{p}}^{-1} \left( \frac{\partial \phi(t,\boldsymbol{p})}{\partial t} \frac{\partial \phi(t,-\boldsymbol{p})}{\partial t} + \omega_{\boldsymbol{p}}^2 \phi(t,\boldsymbol{p})\phi(t,-\boldsymbol{p}) \right) + S_{int,\Lambda}$$

- $\Lambda$: effective energy scale
- $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$
- $K_{\Lambda,\boldsymbol{p}} = K(|\boldsymbol{p}|/\Lambda)$: cutoff function.
- partition function

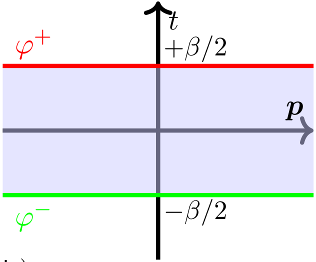



$$Z_\Lambda = \int \mathcal{D}\phi(-\beta/2 \le t \le \beta/2)\delta[\phi(+\beta/2) - \phi(-\beta/2)]e^{-S_\Lambda}$$

# ERG eq for density matrix: setup 2

- density matrix

$$\rho_\Lambda[\varphi^+,\varphi^-] = \frac{1}{Z_\Lambda}\int \mathcal{D}\phi(-\beta/2 < t < \beta/2) e^{-S_\Lambda|_{\phi(\pm\beta/2)=\varphi^\pm}}$$




- requirement:
  1. $-\Lambda\frac{dZ_\Lambda}{d\Lambda} = A_\Lambda Z_\Lambda$ ($A_\Lambda$ : a quantity which is independent to fields)
  2. $-\Lambda\frac{\partial e^{-S_{int,\Lambda}}}{\partial\Lambda}\Big|_{\varphi^+=\varphi^-=\varphi} = -\Lambda\frac{\partial\, e^{-S_{int,\Lambda}}\big|_{\varphi^+=\varphi^-=\varphi}}{\partial\Lambda}$
  3. $-\Lambda\frac{\partial\, e^{-S_{int,\Lambda}}\big|_{\varphi^+=\varphi^-=\varphi}}{\partial\Lambda} =$
     $-\frac{1}{2}\int_{\boldsymbol{p}}\frac{\dot{K}_{\Lambda,\boldsymbol{p}}}{2\omega_{\boldsymbol{p}}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)\frac{\delta^2\, e^{-S_{int,\Lambda}}\big|_{\varphi^+=\varphi^-=\varphi}}{\delta\varphi_{\boldsymbol{p}}\delta\varphi_{-\boldsymbol{p}}} - \frac{1}{2}\int_{-\beta/2}^{\beta/2} dtds\int_{\boldsymbol{p}}\dot{K}_{\Lambda,\boldsymbol{p}}\Delta(t,s)\frac{\delta^2\, e^{-S_{int,\Lambda}}\big|_{\varphi^+=\varphi^-=\varphi}}{\delta\phi_q(t,\boldsymbol{p})\delta\phi_q(s,-\boldsymbol{p})}$
     (cf. Polchinski eq)

# ERG eq for density matrix: result

$$
\begin{aligned}
&-\Lambda\frac{\partial}{\partial\Lambda}\rho_\Lambda[\varphi^+,\varphi^-] \\
&=\int_{\boldsymbol{p}}\Bigg[-\frac{1}{2}\frac{\dot K_{\Lambda,\boldsymbol{p}}}{2\omega_{\boldsymbol{p}}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)\left(\frac{\delta}{\delta\varphi^+}+\frac{\delta}{\delta\varphi^-}\right)^2-\frac{1}{2}\frac{\dot K_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}\left(\frac{\delta}{\delta\varphi^+}+\frac{\delta}{\delta\varphi^-}\right)(\varphi^++\varphi^-) \\
&\qquad-\frac{\chi}{4}\frac{\dot K_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}(\varphi^+-\varphi^-)\left(\frac{\delta}{\delta\varphi^+}-\frac{\delta}{\delta\varphi^-}\right)+\frac{1-\chi}{4}\frac{\dot K_{\Lambda,\boldsymbol{p}}}{K^2_{\Lambda,\boldsymbol{p}}}\omega_{\boldsymbol{p}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)(\varphi^+-\varphi^-)^2\Bigg]\rho[\varphi^+,\varphi^-]
\end{aligned}
$$

$$
\begin{aligned}
-\Lambda\frac{\partial e^{-S_{int,\Lambda}}}{\partial\Lambda}&=-\frac{1}{2}\int_{-\beta/2}^{\beta/2}dtds\int_{\boldsymbol{p}}\dot K_{\Lambda,\boldsymbol{p}}\Delta(t,s)\frac{\delta^2 e^{-S_{int,\Lambda}}}{\delta\phi_q(t,\boldsymbol{p})\delta\phi_q(s,-\boldsymbol{p})} \\
&\quad+\frac{1}{2}\int_{\boldsymbol{p}}\left[-\frac{\dot K_{\Lambda,\boldsymbol{p}}}{2\omega_{\boldsymbol{p}}}\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)\left(\frac{\delta}{\delta\varphi^+}+\frac{\delta}{\delta\varphi^-}\right)^2-\frac{\chi}{2}\frac{\dot K_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}(\varphi^+-\varphi^-)\left(\frac{\delta}{\delta\varphi^-}-\frac{\delta}{\delta\varphi^-}\right)\right]e^{-S_{int,\Lambda}}
\end{aligned}
$$

- $\chi\rightarrow$a parameter which depends on RG scheme.

# ERG as a quantum operation

- General evolution of a quantum state $\Rightarrow$ trace-preserving completely positive (TPCP) map
- The renormalization group procedure constructs semi-group: $R_{\Lambda_3,\Lambda_2} \circ R_{\Lambda_2,\Lambda_1} = R_{\Lambda_3,\Lambda_1}$
- TPCP map forming a semi-group $\Leftrightarrow$ Lindblad equation

$$\frac{\partial \rho}{\partial t} = -i[X, \rho] + \sum_m \left( L_m \rho L_m^\dagger - \frac{1}{2}\left\{ L_m^\dagger L_m, \rho \right\} \right)$$

- Q. Are there cases where the ERG equation for the density matrix becomes the Lindblad equation?

# ERG as a quantum operation(Cont'd)

- A. Yes, there are.
- The range of $\chi$ is determined by $\beta\omega_{\boldsymbol{p}}$.
- $\chi = 0$ is the special case that is allowed for any $\beta\omega_{\boldsymbol{p}}$.
  - Unitary part of RG ("Hamiltonian")

$$X_\Lambda = -\frac{1}{4}\int_{\boldsymbol{p}} \frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}\left(\hat{\pi}_{-\boldsymbol{p}}\hat{\phi}_{\boldsymbol{p}} + \hat{\phi}_{\boldsymbol{p}}\hat{\pi}_{-\boldsymbol{p}}\right) = -\frac{i}{4}\int_{\boldsymbol{p}}\left(\hat{a}^\dagger_{\Lambda,\boldsymbol{p}}\hat{a}^\dagger_{\Lambda,-\boldsymbol{p}} - \hat{a}_{\Lambda,\boldsymbol{p}}\hat{a}_{\Lambda,-\boldsymbol{p}}\right)$$

  $\rightarrow$disentanger! (cf. [Kuwahara et al., 2024])

  - Non-unitary part ("jump operator")

$$L_1(\boldsymbol{p}) = \sqrt{\frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{2K_{\Lambda,\boldsymbol{p}}}\left(\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)+1\right)}\,\hat{a}^{(0)}_{\Lambda,\boldsymbol{p}}, \quad L_2(\boldsymbol{p}) = \sqrt{\frac{-\dot{K}_{\Lambda,\boldsymbol{p}}}{2K_{\Lambda,\boldsymbol{p}}}\left(\coth\left(\frac{\beta\omega_{\boldsymbol{p}}}{2}\right)-1\right)}\,\hat{a}^{(0)\dagger}_{\Lambda,\boldsymbol{p}}$$

$$\hat{a}^{(0)}_{\Lambda,\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\boldsymbol{p}}}{K_{\Lambda,\boldsymbol{p}}}}\hat{\phi}_{\boldsymbol{p}} + i\sqrt{\frac{K_{\Lambda,\boldsymbol{p}}}{\omega_{\boldsymbol{p}}}}\hat{\pi}_{\boldsymbol{p}}\right)$$

# (Score based)diffusion model: learning

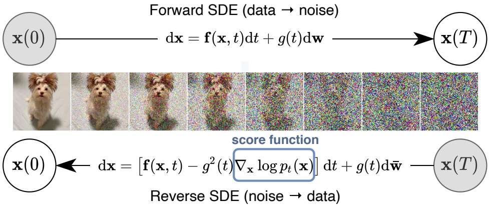



Figure: Conceptual diagram. adapted from [Song et al., 2020]

- Learning process
  1. $P_{emp,0}(x)$: Empirical distribution constructed from data that reflects the statistical features of the data.
  2. Diffusing $P_{emp,0}(x)$ yields Gaussian noise at time $T$: $P_{emp,t}(x)$
  3. Learn a neural network $s^i_{\theta,t}$ that approximates $s^i_{\theta,t} \sim \frac{1}{P_{emp,t}} \sum_j \frac{\partial(D^{ij} P_{emp,t})}{\partial x^j}$.

# (Score based)diffusion model: generating process

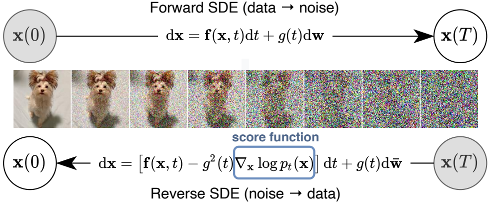



Figure: Conceptual diagram. adapted from [Song et al., 2020]

- Generating process
  1. Prepare Gaussian noise $q_T(x)$, which we take as an "initial" distribution for reverse diffusion.
  2. "Evolve" $t : T \to 0$ by using the reverse diffusion equation whose drift correction is approximated by a neural network:

$$-\frac{\partial q_t(x)}{\partial t} = \sum_i \frac{\partial}{\partial x_t^i} \left[ \bar{f}^i_{\theta,t}(x_t) q(x_t, t) \right] + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_t^i \partial x_t^j} \left[ D^{ij}(x_t, t) p(x_t, t) \right]$$

$$\text{whre} \quad \bar{f}^i_{\theta,t}(x_t) = f^i_t(x_t) - s^i_{\theta,t}(x_t)$$

  3. We get a new sample from $q_0(x)$.