Description of curved spaces by finite-size matrices

Tatsuya Seko (Shizuoka U.)

Collaborators:

Keiichiro Hattori (Shizuoka U.), Asato Tsuchiya (Shizuoka U.)

KEK Theory Workshop 2025

This work was supported by JST SPRING, Japan Grant Number JPMJSP2167.

Background

- Matrix models are expected to give a nonperturbative formulation of superstring theory.
- The type IIB matrix model: [Ishibashi, Kawai, Kitazawa, Tsuchiya (1997)]

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[A_a\,, A_b][A^a, A^b] - \frac{1}{2}\bar{\psi}\Gamma^a[A_a\,, \psi]\right)$$

$A_a\ (a = 1, \ldots, 10)\,,\ \psi : N \times N$ Hermitian matrices

  - Spacetime emerges from the degrees of freedom of matrices.
  - This model is expected to include gravity.

$\Downarrow$

Curved spacetime should be described in the type IIB matrix model.

Background

- The covariant derivative interpretation of matrix models:

  [Hanada, Kawai, Kimura (2006)]

$$\boxed{A_{(a)} = i\nabla_{(a)}}$$

  - This enables to describe curved spaces in matrix models.
  - The Einstein eq. is obtained from EOM of the type IIB matrix model.

- The size of the matrices $A_{(a)} = i\nabla_{(a)}$ is infinite.

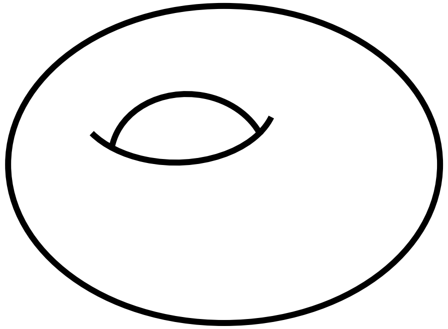
$$\Downarrow$$

  - Regularization is needed to compute quantum effects.
  - It is necessary to make the size of the matrices finite to apply the covariant derivative interpretation to numerical simulations.

Research overview

- We regularize the matrices $A_{(a)} = i\nabla_{(a)}$ to finite-size matrices.

a manifold M



infinite-size matrices

$$A_{(a)} = i\nabla_{(a)}$$

finite-size matrices

$$\begin{pmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & \ddots & \vdots \\ p_{N1} & \cdots & p_{NN} \end{pmatrix}$$

$\Longleftrightarrow$ covariant derivative interpretation

$\Longleftrightarrow$ our study

- M: a closed connected $2n$-dimensional Kahler manifold
- We use the Berezin-Toeplitz (BT) quantization for the regularization.

cf. 2-dimensional case : [Hattori, Mizuno, Tsuchiya (2024)]

Outline

- Background
- Research overview
- Covariant derivative interpretation
- BT quantization

Preparation (Review)

- **Regularization of covariant derivatives**

Our study

- Summary and future directions

Covariant derivative interpretation

- The covariant derivative interpretation: $A_{(a)} = i\nabla_{(a)}$
- $\nabla_{(a)}\ (a = 1, \dots, 2n)$

[Hanada, Kawai, Kimura (2006)]

$$\boxed{\nabla_{(a)}\varphi(x,g) \coloneqq {R_{(a)}}^{b}(g^{-1})\nabla_b\varphi(x,g)}$$

① $\nabla_{(a)}$ acts on a regular rep. field $\varphi(x,g)$ $\big(x \in M,\ g \in \mathrm{Spin}(2n)\big)$.

$\longrightarrow$ For $h \in \mathrm{Spin}(2n)$, $\hat{h}\varphi(x,g) = \varphi(x, h^{-1}g)$

② $\nabla_{(a)}$ is ∇_b multiplied by ${R_{(a)}}^{b}(g^{-1})$. $\left(\nabla_b \coloneqq e_b^{\mu}\left(\partial_\mu + \frac{1}{2}\Omega_\mu^{cd}\mathcal{O}_{cd}\right)\right)$

$\longrightarrow$ the vector rep. matrix of $\mathrm{Spin}(2n)$

⇨ The index (a) of $\nabla_{(a)}$ does not transform under $\mathrm{Spin}(2n)$.

⇨ $\nabla_{(a)}$ can be regarded as matrices.

BT quantization

- The BT quantization is a method for regularizing a field to a finite-size matrix.

ϕ : a field on a Kahler manifold M

(a section of a homomorphism bundle)

→ a bundle whose fiber is a vector space of linear maps: $V \to V'$

projecting onto the spaces of zero modes of the Dirac op. D & D'

→ By the Atiyah-Singer index theorem, these spaces are **finite**-dimensional.

$T^{(V',V)}(\phi)$: a finite-size ($=$ dimKer$D' \times$ dimKer$D$) matrix

- p: the topological charge of the gauge field in D & D'

$p \to \infty \quad \Rightarrow \quad$ the matrix size of $T^{(V',V)}(\phi) \to \infty$

BT quantization

- $T^{(V',V)}(\phi)$'s behavior in the large p limit: [Adachi, Ishiki, Kanno (2023)]

- $\lim_{p\to\infty} \left| T^{(V'',V')}(\phi')T^{(V',V)}(\phi) - T^{(V'',V)}(\phi'\phi) \right| = 0$
- $\lim_{p\to\infty} \left| i\hbar_p^{-1}\left[T(f), T^{(V',V)}(\phi)\right] - T^{(V',V)}(\{f,\phi\}) \right| = 0$

Here,

- $\hbar_p \propto \frac{1}{p}$
- $f \in C^\infty(M)$
- $\left[T(f), T^{(V',V)}(\phi)\right] \coloneqq T^{(V',V')}(f)T^{(V',V)}(\phi) - T^{(V',V)}(\phi)T^{(V,V)}(f)$
- $\{f,\phi\} \coloneqq W^{ab}(\partial_a f)(\nabla_b \phi)$, $\quad W^{ab}$: Poisson tensor

Outline

- Background
- Research overview
- Covariant derivative interpretation
- BT quantization

Preparation (Review)

- **Regularization of covariant derivatives**

Our study

- Summary and future directions

Regularization of covariant derivatives

- We will regularize the matrices $i\nabla_{(a)}$ to finite-size matrices.
- $\mathcal{P}_{(a)}$: the matrix regularization of $i\nabla_{(a)}$

    ① $\mathcal{P}_{(a)}$ acts on $T(\varphi)$, matrix regularization of a regular rep. field φ.

    ② $\mathcal{P}_{(a)}T(\varphi) = T\left(i\nabla_{(a)}\varphi\right)$ in the large p limit.

    ③ $\mathcal{P}_{(a)}$ is Hermitian for finite p.

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)$$

- $X^A(x)\ (A = 1,\dots,D)$:
    - embedding coordinate functions of $2n$-dim Kahler manifold M into D-dim Euclidean space R^D
    - $(\partial_b X^A)(\partial_c X^A) = \delta_{bc}$

Regularization of covariant derivatives

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\big(\partial_{(a)}X^A\big)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\big[T\big(\partial_{(a)}X^A\big),T(X^A)\big]T(\varphi)$$

We will show that $\mathcal{P}_{(a)}$ provides the matrix regularization of $i\nabla_{(a)}$.

<u>3 steps</u>

① Obtain $T(\varphi)$, matrix regularization of a regular rep. field φ

② Obtain $T(X^A)$, matrix regularization of the embedding coordinate functions X^A

③ Show that $\mathcal{P}_{(a)}T(\varphi) = T\big(i\nabla_{(a)}\varphi\big)$ in the large p limit.

Regularization of covariant derivatives

- $T(\varphi)$: matrix regularization of a regular rep. field $\varphi(x,g)$

By the Peter-Weyl theorem,

$$\varphi(x,g) = \sum_{r:\,irr.} \underbrace{\tilde{\varphi}^{\langle r\rangle}_{i(j)}(x)}_{\text{field in rep.}r} \underbrace{\sqrt{d_r}\, R^{\langle r^*\rangle}_{i(j)}(g)}_{\text{basis}}$$

- d_r: the dimension of rep.r
- $R^{\langle r^*\rangle}_{i(j)}(g)$: the rep. matrix of r^*

- Under Spin$(2n)$, the index i of $\tilde{\varphi}^{\langle r\rangle}_{i(j)}$ transforms as rep.r, while (j) does not.

$$T(\varphi) = \begin{array}{c} \\ d_r\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ d_{r'}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \end{array} \begin{pmatrix} \vdots \\ T^{(r,1)}\left(\tilde{\varphi}^{\langle r\rangle}_{(1)}\right) \\ \vdots \\ T^{(r,1)}\left(\tilde{\varphi}^{\langle r\rangle}_{(d_r)}\right) \\ T^{(r',1)}\left(\tilde{\varphi}^{\langle r'\rangle}_{(1)}\right) \\ \vdots \\ T^{(r',1)}\left(\tilde{\varphi}^{\langle r'\rangle}_{(d_{r'})}\right) \\ \vdots \end{pmatrix}$$

Regularization of covariant derivatives

- $T(\varphi)$: matrix regularization of a regular rep. field $\varphi(x,g)$

By the Peter-Weyl theorem,

$$\varphi(x,g) = \sum_{r:\,irr.} \underbrace{\tilde{\varphi}^{\langle r\rangle}_{i(j)}(x)}_{\text{field in rep.}r} \underbrace{\sqrt{d_r}\, R^{\langle r^*\rangle}_{i(j)}(g)}_{\text{basis}}$$

- Under Spin($2n$), the index i of $\tilde{\varphi}^{\langle r\rangle}_{i(j)}$ transforms as rep.r, while (j) does not.

$$T(\varphi) = \begin{array}{c} \\ d_r\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ d_{r'}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \end{array} \begin{pmatrix} \vdots \\ T^{(r,1)}\left(\tilde{\varphi}^{\langle r\rangle}_{(1)}\right) \\ \vdots \\ T^{(r,1)}\left(\tilde{\varphi}^{\langle r\rangle}_{(d_r)}\right) \\ T^{(r',1)}\left(\tilde{\varphi}^{\langle r'\rangle}_{(1)}\right) \\ \vdots \\ T^{(r',1)}\left(\tilde{\varphi}^{\langle r'\rangle}_{(d_{r'})}\right) \\ \vdots \end{pmatrix}$$

cutoff : Ξ

- Consider rep.r whose Casimir is less than Ξ.
- Take the limit in which $\Xi \to \infty$ & $p \to \infty$ while keeping $\Xi \ll p$.

Regularization of covariant derivatives

- $T(X^A)$: matrix regularization of the embedding coordinate functions X^A

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\left(\partial_{(a)}X^A\right)\underline{[T(X^A),T(\varphi)]} + \frac{1}{2}\hbar_p^{-1}\left[T\left(\partial_{(a)}X^A\right),T(X^A)\right]T(\varphi)$$

$$\coloneqq T(X^A)T(\varphi) - T(\varphi)T^{(1,1)}(X^A)$$

$$T(X^A) = \begin{pmatrix} \ddots & & & & & & \mathbf{0} \\ & \underbrace{T^{(r,r)}(X^A) \quad \ddots \quad T^{(r,r)}(X^A)}_{d_r} & & & & & \\ & & & \underbrace{T^{(r',r')}(X^A) \quad \ddots \quad T^{(r',r')}(X^A)}_{d_{r'}} & & & \\ \mathbf{0} & & & & & & \ddots \end{pmatrix}$$

Regularization of covariant derivatives

- Proof that $\mathcal{P}_{(a)}T(\varphi) = T\big(i\nabla_{(a)}\varphi\big)$ in the large p limit.

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\big(\partial_{(a)}X^A\big)[T(X^A),T(\varphi)] + \underbrace{\frac{1}{2}\hbar_p^{-1}\big[T\big(\partial_{(a)}X^A\big),T(X^A)\big]T(\varphi)}_{O(1/p)}$$

$$= T\big(\partial_{(a)}X^A\big)\,T(i\{X^A,\varphi\}) + O(1/p)$$

$$= T\big(\partial_{(a)}X^A\big)\,T\left(iW^{cd}(\partial_c X^A)(\nabla_d\varphi)\right) + O(1/p)$$

$$= T\left(i\underbrace{\big(\partial_{(a)}X^A\big)W^{cd}(\partial_c X^A)}_{= R_{(a)}{}^d(g^{-1})}(\nabla_d\varphi)\right) + O(1/p)$$

$$= T\big(i\nabla_{(a)}\varphi\big) + O(1/p)$$

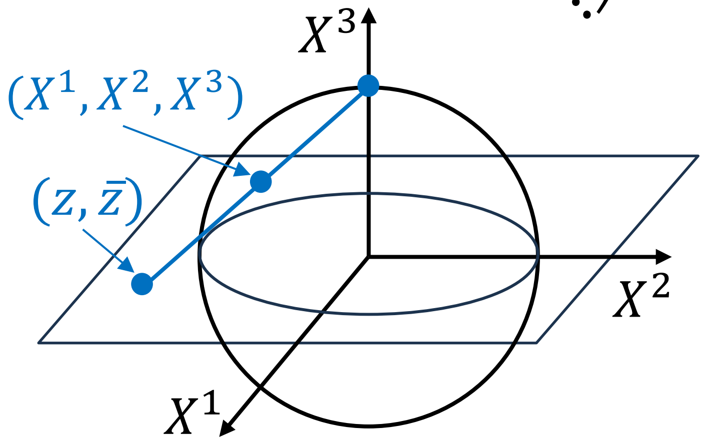
$\Rightarrow$ $\mathcal{P}_{(a)}$ gives matrix regularization of $i\nabla_{(a)}$.

Example: S^2

- $X^A(z,\bar{z})$ $(A=1,2,3)$: embedding coordinate functions of S^2 into R^3

$$T(X^A)=\begin{pmatrix} \ddots & & & \\ & T^{\left(s-\frac{1}{2},\,s-\frac{1}{2}\right)}(X^A) & & \mathbf{0} \\ & & T^{(s,s)}(X^A) & \\ & \mathbf{0} & & T^{\left(s+\frac{1}{2},\,s+\frac{1}{2}\right)}(X^A) \\ & & & & \ddots \end{pmatrix}$$

- s: integer or half integer
- $X^{\pm} \coloneqq X^1 \pm iX^2$
- $T^{(s,s)}(X^{\pm}) = \frac{2}{p+2s+1} J_{\pm}$, $J_{\pm} \coloneqq J_1 \pm i J_2$
- $T^{(s,s)}(X^3) = \frac{2}{p+2s+1} J_3$
- J_i $(i=1,2,3)$: $(p+2s)$-dim rep. of the $\mathfrak{su}(2)$ generator



Example: S^2

$$\mathcal{P}_{(\pm)}T(\varphi) \coloneqq \pm i\hbar_p^{-1}T\big(\partial_{(\pm)}X^A\big)\underline{[T(X^A),T(\varphi)]} \mp \frac{i}{2}\hbar_p^{-1}\big[T\big(\partial_{(\pm)}X^A\big),T(X^A)\big]T(\varphi)$$

$$\coloneqq T(X^A)T(\varphi) - T(\varphi)T^{(0,0)}(X^A)$$

- $\mathcal{P}_{(3)} \coloneqq \left[\mathcal{P}_{(+)}\,,\,\mathcal{P}_{(-)}\right]$
- In the large p limit,

$$\left[2\mathcal{P}_{(+)}\,,\,2\mathcal{P}_{(-)}\right] = 2\big(2\mathcal{P}_{(3)}\big)\,, \quad \left[2\mathcal{P}_{(3)}\,,\,2\mathcal{P}_{(\pm)}\right] = \pm 2\mathcal{P}_{(\pm)}$$

$\Rightarrow$ $2\mathcal{P}_{(\pm)}$ and $2\mathcal{P}_{(3)}$ form $\mathfrak{su}(2)$ algebra.

cf. $\left[2i\nabla_{(+)}\,,\,2i\nabla_{(-)}\right] = 2(-2\mathcal{O}_{+-})\,, \quad \left[-2\mathcal{O}_{+-}\,,\,2i\nabla_{(\pm)}\right] = \pm 2i\nabla_{(\pm)}$

Summary and future directions

- **Summary**
  - We have regularized $i\nabla_{(a)}$ on a Kahler manifold M to finite-size matrices $\mathcal{P}_{(a)}$ by the BT quantization.

$$\mathcal{P}_{(a)}T(\varphi) \coloneqq -\hbar_p^{-1}T\big(\partial_{(a)}X^A\big)[T(X^A),T(\varphi)] + \frac{1}{2}\hbar_p^{-1}\big[T\big(\partial_{(a)}X^A\big),T(X^A)\big]T(\varphi)$$

$\Rightarrow$ This enables to describe curved spaces by finite-size matrices.

- **Future directions**
  - Calculation of 1-loop effective action (the mass of higher-spin fields)
  - Applying the covariant derivative interpretation to the results of numerical simulations to extract the geometry from matrix models