Algebraic Structure in the Exact Renormalization Group

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Outline

1. Itroduction
2. Setup and notation
3. Assumption: scaling of normal-ordered correlation functions
4. Main results: existence, explicit form, generator
5. Proof sketch: how $R_{t,t'}$ is extracted
6. Summary and future plans

1. Itroduction

- Sonoda (2015) defined the most general renormalization group transformation and showed that it reproduces Polchinski (1984), Ball et al. (1995), and Morris (1998).
- We explicitly construct **the finite exact renormalization group transformation $R_{t,t'}$ as a functional-differential operator** acting on the Boltzmann factor.
- We consider a d-dimensional Euclidean field theory for a real scalar field ϕ.

2.1 About the notation

- Inner product for functionals f and g:

$$(f, g) \equiv \int d^d x f(x) g(x)$$

- For an operator A with kernel $A(x, y)$,

$$(f, Ag) \equiv \int d^d x \int d^d y f(x) A(x, y) g(y), \quad (Af, g) = (f, A^T g).$$

- For an operator A and the functional derivative,

$$(A\phi, \frac{\delta}{\delta\phi}) \equiv \int d^d x (A\phi)(x) \frac{\delta}{\delta\phi(x)}, \quad \left(\frac{\delta}{\delta\phi}, A\frac{\delta}{\delta\phi}\right) \equiv \int d^d x \int d^d y \frac{\delta}{\delta\phi(x)} A(x, y) \frac{\delta}{\delta\phi(y)}.$$

2.2 Setup

Cutoff propagators C_Λ and C'_Λ have the **same momentum dependence** and differ only by the cutoff functions K and k:

$$C_\Lambda(p) = \frac{K\left(\frac{|p|}{\Lambda}\right)}{|p|^{d-2[\phi]}}, \qquad C'_\Lambda(p) = \frac{k\left(\frac{|p|}{\Lambda}\right)}{|p|^{d-2[\phi]}}.$$

(Their kernels are $C_\Lambda(x,y) = \int \frac{d^dp}{(2\pi)^d} C_\Lambda(p) e^{-ip\cdot(x-y)}$ and similarly for C'_Λ; inverses are defined by $C_\Lambda(p) C_\Lambda^{-1}(p) = 1$ etc.)

Cutoff functions:

$$K(s) = 1 \ (s \in [0,1]), \quad K(s) \to 0 \ (s \to \infty), \quad K \text{ non-decreasing, positive-definite,}$$
$$k(s) = 0 \ (s \in [0,1]), \quad k(s) > 0 \ (s > 1), \quad k \text{ positive-definite.}$$

2.3 the exact RG transformation $R_{t,t'}$

Renormalization group: a family of transformations $R_{t,t'}$ mapping an action $S_{t'}$ with cutoff Λ to an action S_t at another scale, i.e. a map on theory space $\mathcal{S}$. Here t is the transformation parameter:

$$R_{t,t'} : \quad -S_{t'}[\phi] \mapsto -S_t[\phi]$$

- Finite ERG can be written as a similarity transformation of the scale transformation (implicit in Sonoda (2013)).
- Its infinitesimal generator is the flow equation (e.g. Polchinski equation).

If there exists a fixed point $S^* := \{S \in \mathcal{S} \mid R_t e^{-S} = e^{-S}\}$, then the $\Lambda \to \infty$ limit can be computed and one can construct the field theory rigorously. This is our motivation.

3. Assumption and what is derived

- The true Assumption is **the scaling relation for normal-ordered correlation functions**.
- From that Assumption, one derives the following scaling identity for the normal-ordered generating functional (with respect to C'_Λ):

$$e^{\widetilde{W}_t[J]} = e^{\int_{t'}^{t} d\tau \left(D_\tau J, \frac{\delta}{\delta J}\right)} e^{\widetilde{W}_{t'}[J]}, \quad (t \geq t' \geq 0) \tag{3.1}$$

- Scale operators:

$$(D_t J)(x) \equiv \left(\Delta_t - d - x^\mu \frac{\partial}{\partial x^\mu}\right) J(x), (D_t^T f)(x) \equiv \left(\Delta_t + x^\mu \frac{\partial}{\partial x^\mu}\right) f(x) \quad (3.2$$

- Everything that follows (existence of $R_{t,t'}$, its explicit operator form, and the generator G_t) is a consequence.

4.1 finite ERG acts on the Boltzmann factor

From (3.1), the finite exact renormalization group transformation acting on the Boltzmann factor itself is obtained:

$$e^{-S_t[\phi]} = R_{t,t'} e^{-S_{t'}[\phi]}, \quad R_{t,t'} : \; e^{-S_{t'}[\phi]} \mapsto e^{-S_t[\phi]} \tag{4.1}$$

Once this scaling structure is available, the $\Lambda \to \infty$ limit becomes straightforward to evaluate within this framework.

4.2 What we found: explicit operator form of $R_{t,t'}$

Define the coarse-graining operator $\mathcal{R}_{C'_\Lambda}$, the scale transformation $\mathcal{D}_{t,t'}$, by

$$\mathcal{R}^{-1}_{C'_\Lambda} \equiv e^{-\frac{1}{2}\left(\frac{\delta}{\delta\phi},\, C'_\Lambda \frac{\delta}{\delta\phi}\right)}, \quad \mathcal{R}_{C'_\Lambda} \equiv e^{\frac{1}{2}\left(\frac{\delta}{\delta\phi},\, C'_\Lambda \frac{\delta}{\delta\phi}\right)}, \quad \mathcal{D}_{t,t'} \equiv e^{\int_{t'}^{t} d\tau \left(\frac{\delta}{\delta\phi},\, \mathcal{K} D^T_\tau \mathcal{K}^{-1}\phi\right)} \tag{4.2}$$

Cutoff operator $\mathcal{K}$ and its inverse $\mathcal{K}^{-1}$ (as integral kernels):

$$\mathcal{K}(x,y) \equiv \int \frac{d^dp}{(2\pi)^d} K\left(\frac{|p|}{\Lambda}\right) e^{-ip\cdot(x-y)}, \mathcal{K}^{-1}(x,y) \equiv \int \frac{d^dp}{(2\pi)^d} \frac{e^{-ip\cdot(x-y)}}{K\left(\frac{|p|}{\Lambda}\right)} \tag{4.3}$$

Then

$$R_{t,t'} = \mathcal{R}_{C'_\Lambda}\, \mathcal{D}_{t,t'}\, \mathcal{R}^{-1}_{C'_\Lambda}. \tag{4.4}$$

This representation is more general than in previous studies.

4.3 Interpretation of $\mathcal{R}_{C'_\Lambda}$

- $\mathcal{R}_{C'_\Lambda}$ is the coarse-graining operator given as a heat operator (differential Weierstrass transform).
- Its inverse $\mathcal{R}_{C'_\Lambda}^{-1}$ gives the normal ordering with respect to C'_Λ.
- This provides an algebraic way to organize "coarse-graining + rescaling" as a similarity transformation[Kupiainen, 1986].

4.4 generator and the flow equation

Define the generator of an infinitesimal ERG transformation:

$$G_t \equiv \frac{\partial}{\partial \epsilon} R_{t+\epsilon,t}\bigg|_{\epsilon=0}, \qquad R_{t,t'} = \exp\left\{\int_{t'}^{t} d\tau\, G_\tau\right\}, \tag{4.5}$$

and one obtains

$$G_t = \left(\mathcal{K}^{-1} D_t \mathcal{K}\,\frac{\delta}{\delta\phi},\ C'_\Lambda \frac{\delta}{\delta\phi} + \phi\right) \ +\ \dot{c}_t, \tag{4.6}$$

where $\dot{c}_t$ is a t-dependent constant term coming from the Jacobian of the scale transformation (it multiplies the Boltzmann factor by an overall factor and can be absorbed into the normalization of the partition function).

5.1 Proof sketch: the main route

1. Define the normal-ordered generating functional $e^{\widetilde{W}[J]}$.
2. Restrict to $J = KC_\Lambda^{-1}\phi$ to convert the J-space scaling into a configuration-space scale transformation $\mathcal{D}_{t,t'}$.
3. Compare two representations of $e^{\widetilde{W}[KC_\Lambda^{-1}\phi]}$ to extract the operator acting on the Boltzmann factor, i.e. define $R_{t,t'}$.
4. Differentiate $R_{t+\epsilon,t}$ at $\epsilon = 0$ to obtain G_t.

5.2 Step 1: define $e^{\widetilde{W}[J]}$

Define the normal-ordered generating functional by

$$e^{\widetilde{W}[J]} \equiv \frac{1}{W} \int \mathcal{D}\phi \left(\mathcal{R}_{C'_\Lambda}^{-1} e^{(J, K^{-1}\phi)} \right) e^{-S_t[\phi]}, \quad W \equiv \int \mathcal{D}\phi e^{-S_t[\phi]} \tag{5.1}$$

- $\mathcal{R}_{C'_\Lambda}^{-1}$ is interpreted as a functional derivative with respect to the integration variable ϕ.
- It acts only on $e^{(J, K^{-1}\phi)}$ inside the parentheses.

5.3 Step 2: restrict to $J = \mathcal{K} C_\Lambda^{-1} \phi$

Substituting $J = \mathcal{K} C_\Lambda^{-1} \phi$ into (3.1), one can write

$$e^{\widetilde{W}_t[\mathcal{K} C_\Lambda^{-1}\phi]} = \mathcal{D}_{t,t'}\, e^{\widetilde{W}_{t'}[\mathcal{K} C_\Lambda^{-1}\phi]}, \quad \mathcal{D}_{t,t'} \equiv \exp\left\{ \int_{t'}^{t} d\tau \left(\frac{\delta}{\delta\phi}, \mathcal{K} D_\tau^T \mathcal{K}^{-1} \phi \right) \right\}. \tag{5.2}$$

This is the configuration-space scale transformation induced by the J-space scaling.

5.4 Step 3: return to the Boltzmann factor and define $R_{t,t'}$

From (5.1) with $J = KC_\Lambda^{-1}\phi$ (integration variable ϕ'):

$$e^{\widetilde{W}[KC_\Lambda^{-1}\phi]} = \frac{1}{W}\int \mathcal{D}\phi' \left(\mathcal{R}_{C'_\Lambda}^{-1} e^{(\phi, C_\Lambda^{-1}\phi')}\right) e^{-S_t[\phi']} \tag{5.3}$$

Comparing the t and t' expressions through (5.2), one is led to define

$$R_{t,t'} \equiv \mathcal{R}_{C'_\Lambda} \mathcal{D}_{t,t'} \mathcal{R}_{C'_\Lambda}^{-1} \tag{5.4}$$

so that $e^{-S_t[\phi]} = R_{t,t'} e^{-S_{t'}[\phi]}$ follows.

The renormalization group transformation is often regarded as a semigroup, but in this definition an inverse exists, hence it becomes a group.

5.5 Step 4: derive G_t

Compute $R_{t+\epsilon,t}$ and expand:

$$R_{t+\epsilon,t} = 1 + \epsilon \left[\left(\mathcal{K}^{-1} D_t \mathcal{K} \frac{\delta}{\delta\phi},\, C'_\Lambda \frac{\delta}{\delta\phi} + \phi \right) + \dot{c}_t \right] + O(\epsilon^2). \tag{5.5}$$

Thus

$$G_t = \frac{\partial}{\partial\epsilon} R_{t+\epsilon,t} \Big|_{\epsilon=0}. \tag{5.6}$$

6. Summary

- Previous research:
  - scaling identity for $e^{\widetilde{W}[J]}$ derive ERG transformation $e^{-S_t[\phi]} = R_{t,t'} e^{-S_{t'}[\phi]}$, Sonoda (2015).
- What we construct:
  - explicit operator representation as a similarity transformation,

$$R_{t,t'} = \mathcal{R}_{C'_\Lambda} \mathcal{D}_{t,t'} \mathcal{R}^{-1}_{C'_\Lambda}$$

  - infinitesimal generator reproduces the known flow equation,

$$G_t = \left(K^{-1} D_t K \frac{\delta}{\delta \phi}, C'_\Lambda \frac{\delta}{\delta \phi} + \phi \right)$$

Future plans

- At a fixed point, define the normal-ordered generating functional by $e^{\widetilde{W}_*[J]} \equiv \left\langle : e^{(J, K^{-1}\phi)} :_{C'_\Lambda} \right\rangle_{S_*}$. Then the fixed-point Boltzmann factor can be reconstructed by the inverse Weierstrass transform:

$$e^{-S_*[\phi]} = e^{-\frac{1}{2}\left(\frac{\delta}{\delta\phi}, C'_\Lambda \frac{\delta}{\delta\phi}\right)} e^{\widetilde{W}_*[K C_\Lambda^{-1} \phi]}$$

- Sonoda (2019) constructed a gauge-invariant Gradient Flow ERG (GFERG). By redefining the field and differential operators, GFERG can be written in a similar form.