Exploring the broader Hilbert space of the standard model

Tom Melia, Hokkaido 2nd Workshop March 5 2026

With Surjeet Rajendran and David E Kaplan

…..With a muon interlude

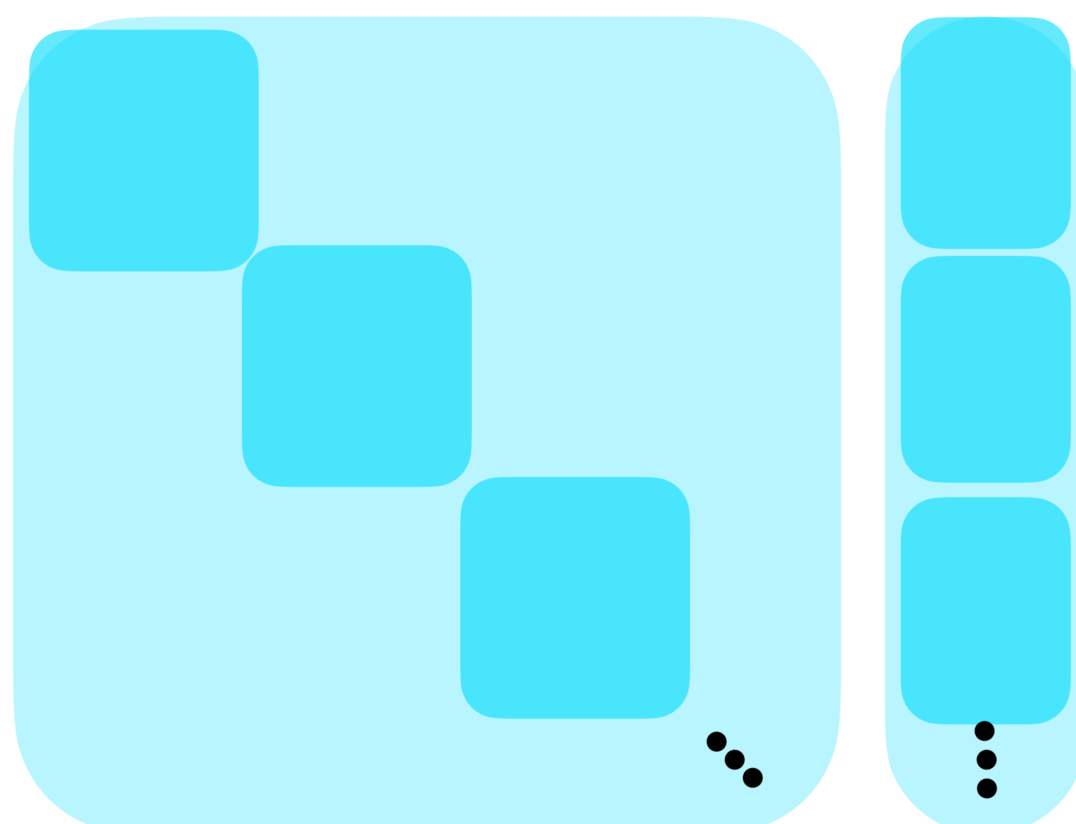
Invitation: a particular point of view on how to think about gauge theories (EM, QCD, Gravity..)

In this setup, the word ‘gauge’ ONLY means the Hilbert space splits up into pieces that evolve in time independently (i.e. incoherently)

$$[H, Q] = 0$$

$$Q|q\rangle = q|q\rangle$$

“Good quantum numbers”



$$H \, |\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle$$

Only consider simple commutation relations (and stick to bosons)

$$[\phi_i(\mathbf{x}), \Pi_j(\mathbf{y})] = i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

And start building up the most general *local* Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\delta_{ij}\Pi_i\Pi_j + \frac{1}{2}(\partial_a\phi_i)G^{ab}_{ij}(\partial_b\phi_j) + \frac{1}{2}\phi_i M_{ij}\phi_j + \ldots$$

In Fourier space, to quadratic order

$$H = \int_{\mathbf{k}} \Pi_i(-\mathbf{k})\Pi_i(\mathbf{k}) + \phi_i(-\mathbf{k})K_{ij}(k)\phi_j(\mathbf{k})$$

$$K_{ij} = M_{ij} + k_a k_b G_{ij}^{ab}$$

Solving eigenvalues

$$H = \int_{\mathbf{k}} \omega_\alpha(k) a_\alpha^\dagger(\mathbf{k}) a_\alpha(\mathbf{k})$$

$$\omega_\alpha(\mathbf{k})^2 = \lambda_\alpha(k)$$

Hilbert space

Field basis

$$\hat{\phi}_i(\mathbf{k})|\phi_i\rangle = \phi_i(\mathbf{k})|\phi_i\rangle$$

Momentum basis

$$\hat{\Pi}_i(\mathbf{k})|\Pi_i\rangle = \Pi_i(\mathbf{k})|\Pi_i\rangle$$

Fock basis

$$a_\alpha(\mathbf{k}) = \frac{1}{2w_\alpha(\mathbf{k})}(w_\alpha(\mathbf{k})\phi(\mathbf{k}) + i\,\Pi_\alpha(\mathbf{k}))$$

Vacuum $a(\mathbf{k})|0\rangle = 0$

States $a^\dagger(\mathbf{k_1})\ldots a^\dagger(\mathbf{k_n})|0\rangle$

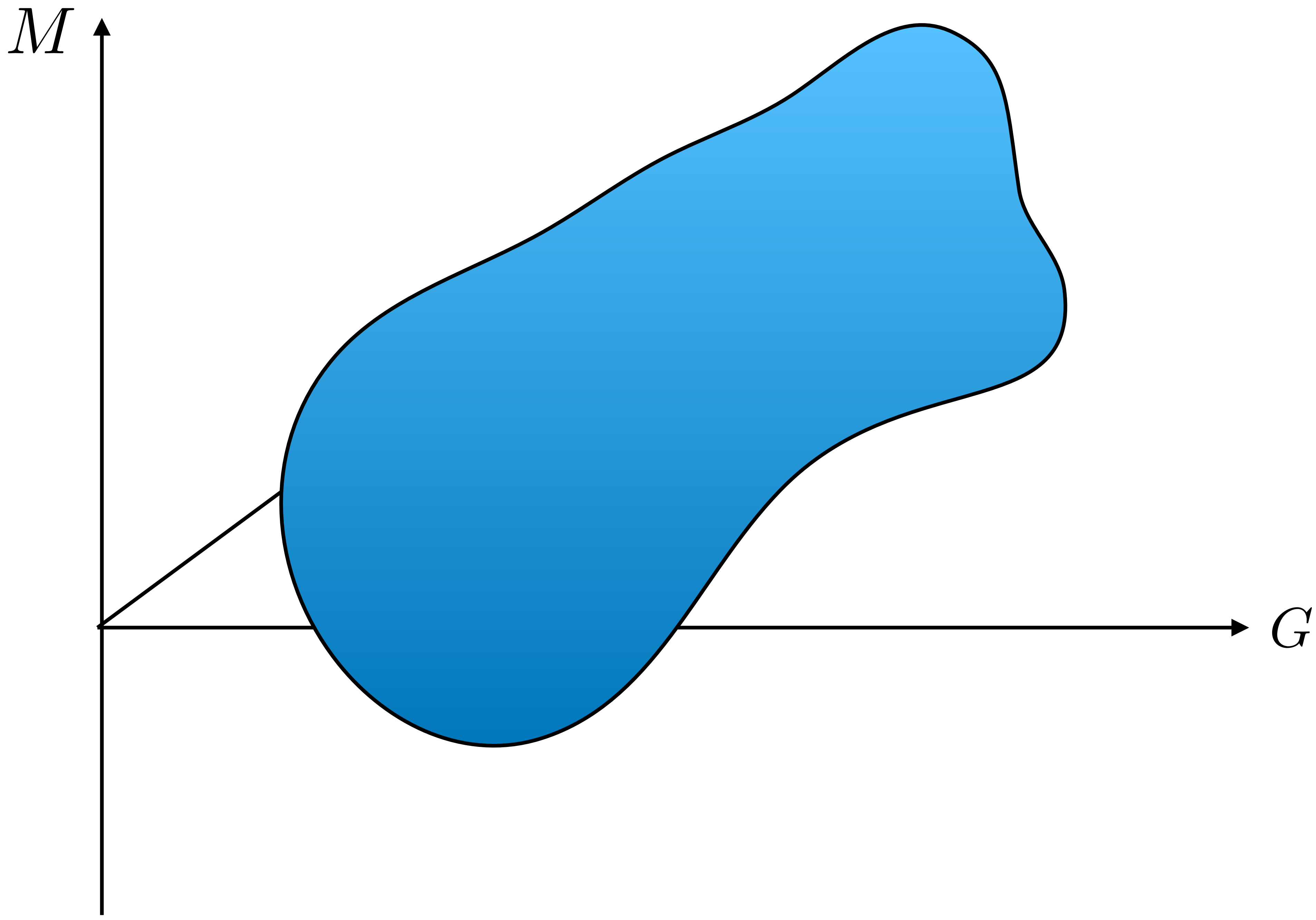
In complete analogy to QM

$$\hat{x}|x\rangle = x|x\rangle$$

$$\hat{p}|p\rangle = p|p\rangle$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

Theory space



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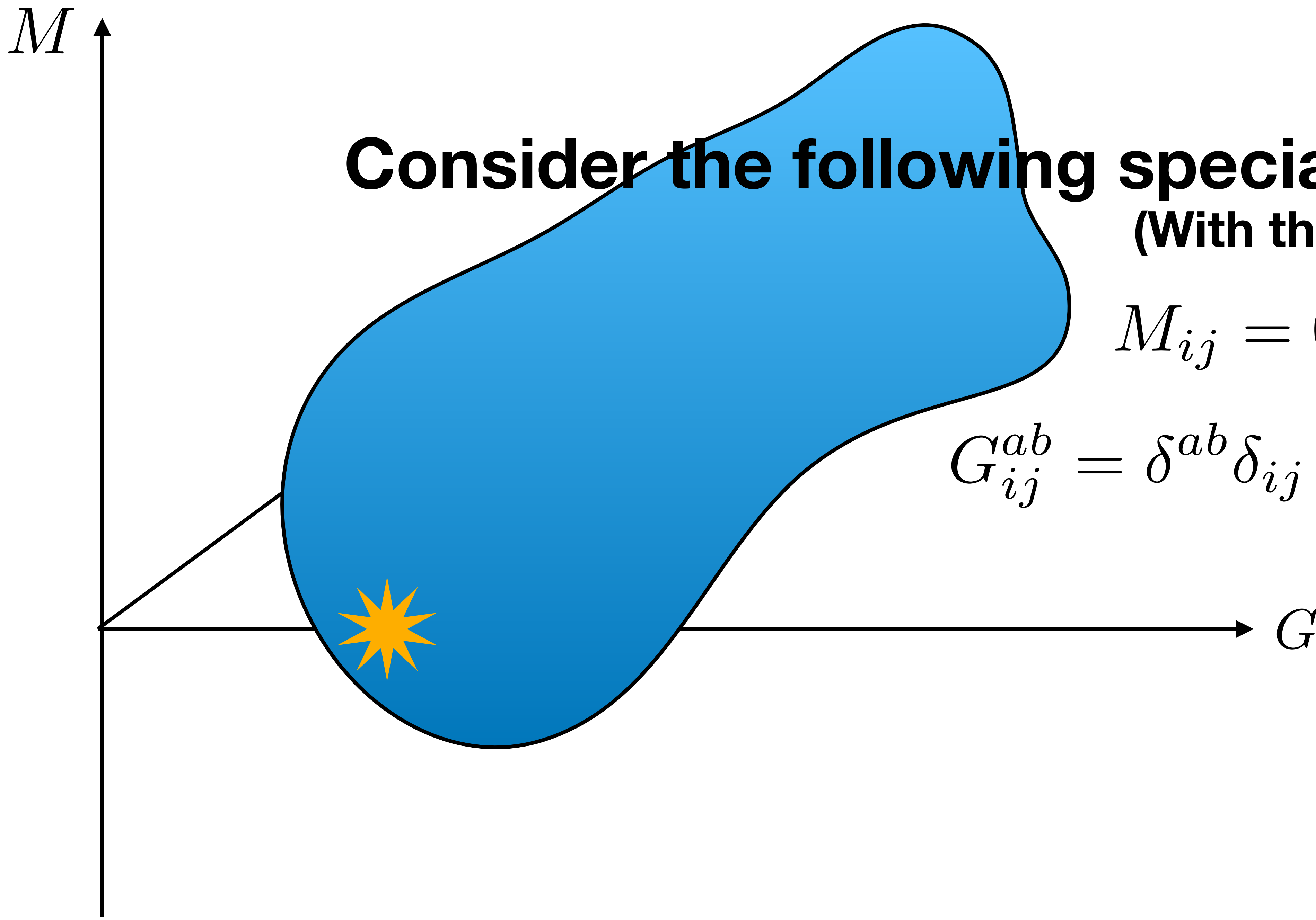


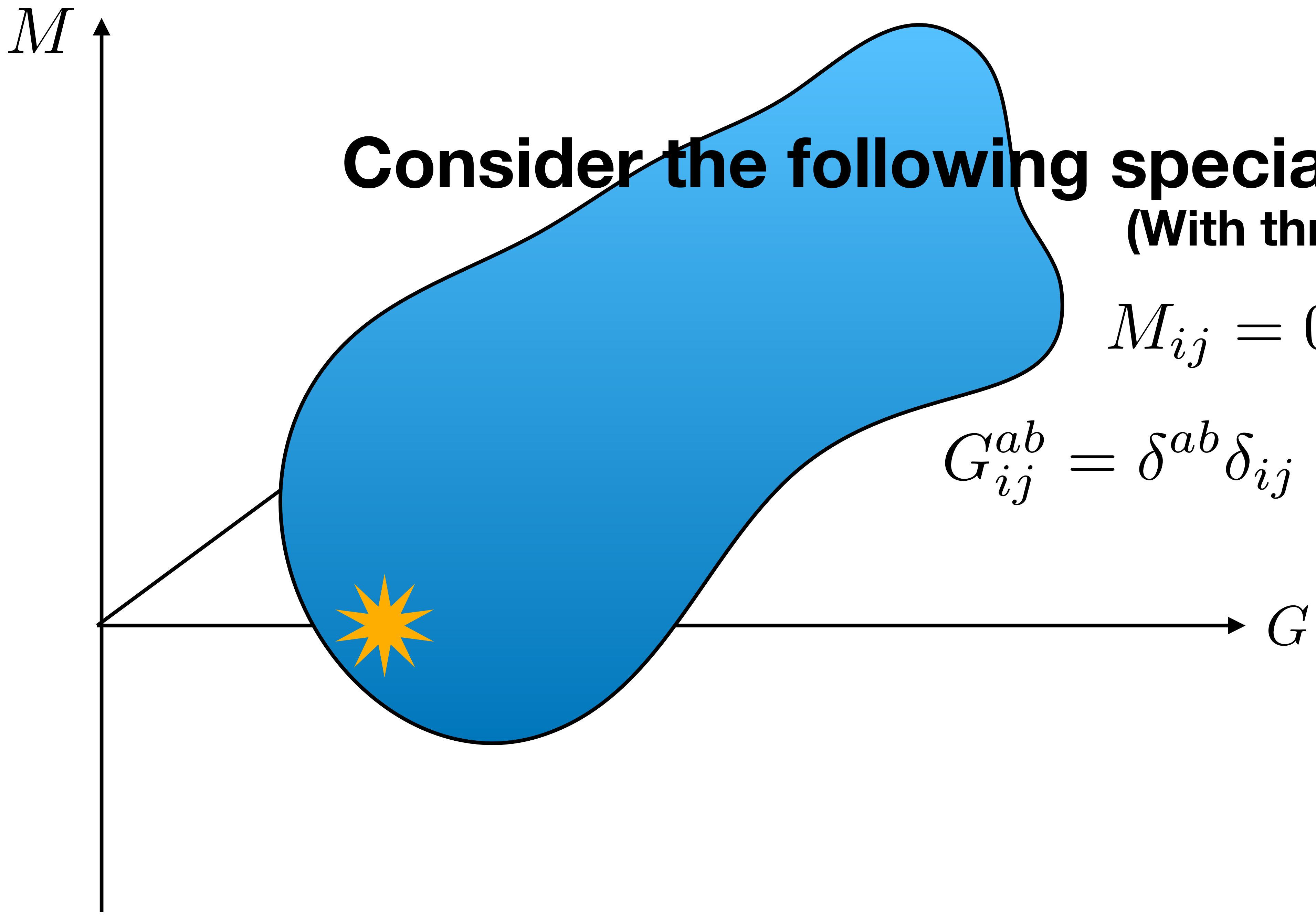
Consider the following special point

(With three fields)

$$M_{ij} = 0$$

$$G^{ab}_{ij} = \delta^{ab}\delta_{ij} - \delta^a_j \delta^b_i$$

Theory space



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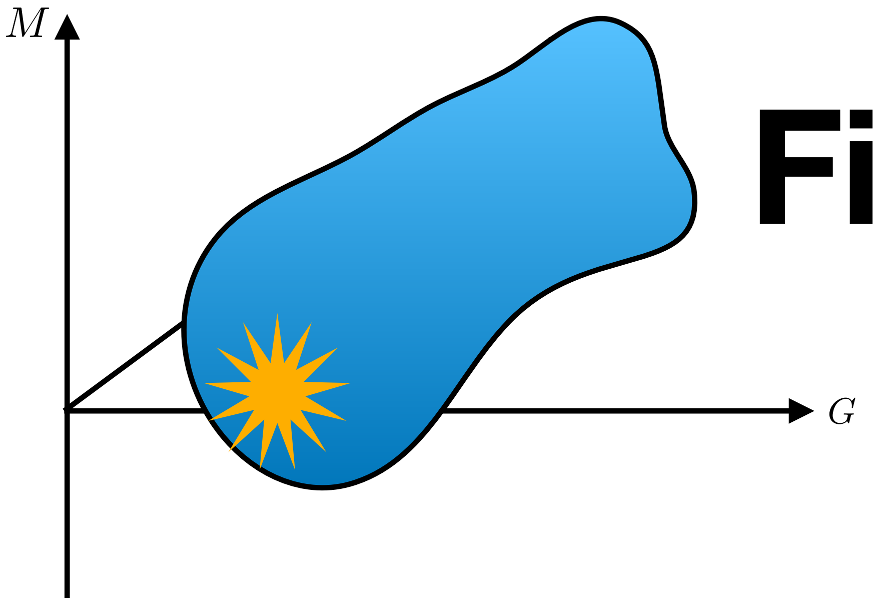
$$M_{ij} = 0$$

$$G^{ab}_{ij} = \delta^{ab}\delta_{ij} - \delta^a_j \delta^b_i$$

$$H = \frac{1}{2}\int d^3x \left(\Pi_i(x)\Pi_i(x) + (\nabla \times \vec{\phi}(x))^2\right) = \frac{1}{2}\int d^3x (\vec{E}^2 + \vec{B}^2)$$



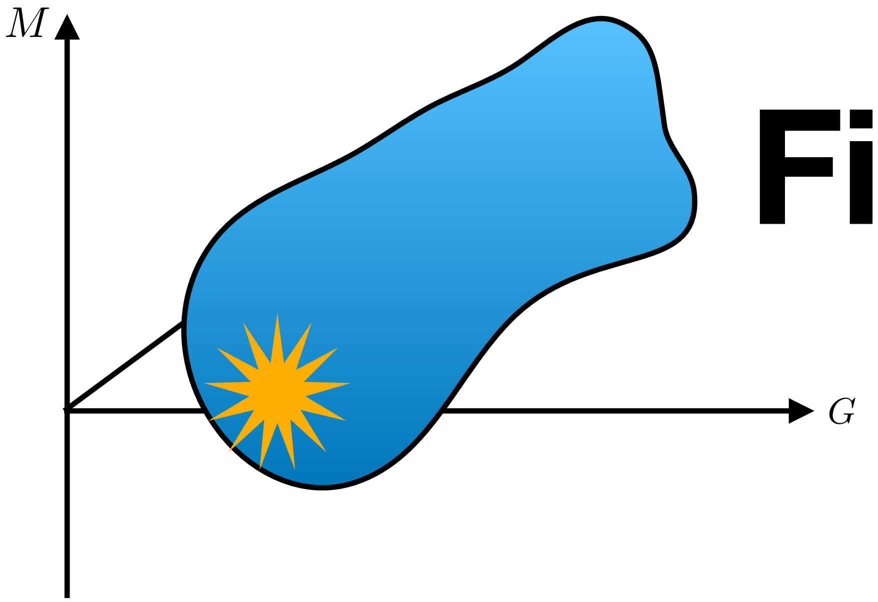
Find Hamiltonian for EM

In Fourier space..

$$H = \int_{\mathbf{k}} \Pi_i(-\mathbf{k})\Pi_i(\mathbf{k}) + \phi_i(-\mathbf{k})K_{ij}(k)\phi_j(\mathbf{k})$$

$$K_{ij} = M_{ij} + k_a k_b G_{ij}^{ab}$$

$$\omega_{T1}(k) = k \qquad \omega_{T2}(k) = k \qquad \omega_L(k) = 0$$



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Longitudinal piece decouples

$$H = \int_k \sum_{\alpha=1,2} \big(E_\alpha(-k)E_\alpha(k) + k^2 A_\alpha(-k)A_\alpha(k)\big) + E_L(-k)E_L(k)$$

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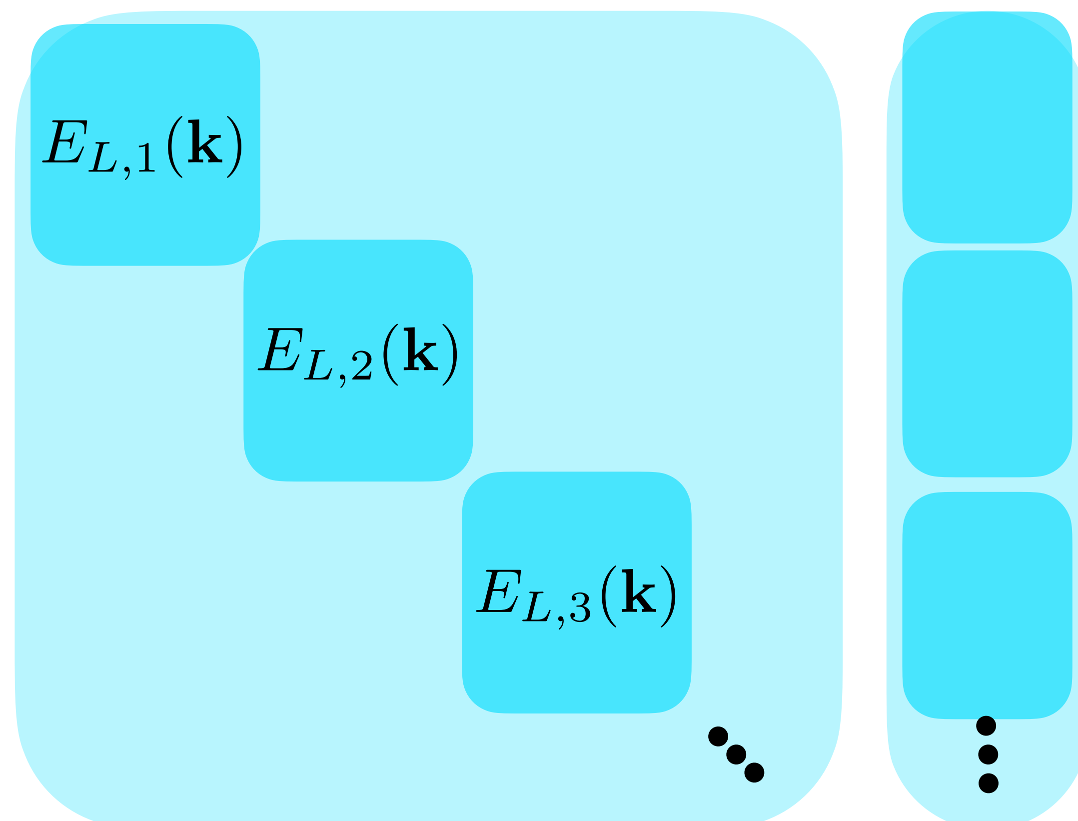
Clearly

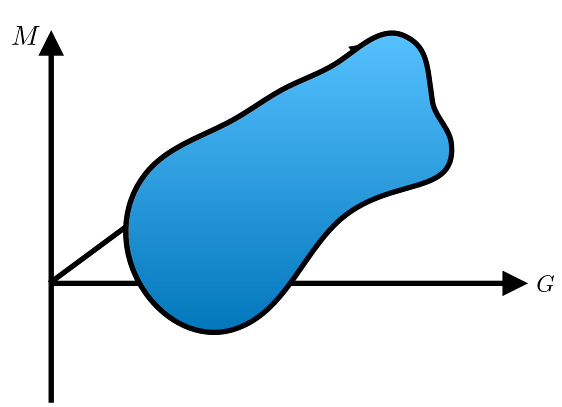
$$[H, E_L(k)] = 0$$

$$\hat{E}_L(k)|E_L(k)\rangle = E_L(k)|E_L(k)\rangle$$

"Good quantum numbers"



Most useful to think in momentum basis, then



$$|E_i(\mathbf{k})\rangle = |E_{T1}(\mathbf{k})\rangle \otimes |E_{T2}(\mathbf{k})\rangle \times |E_L(\mathbf{k})\rangle$$

$$\mathcal{H} = \bigotimes_k \left(\mathcal{H}_{T1,k} \otimes \mathcal{H}_{T2,k} \otimes \mathcal{H}_{L,k}\right)$$

Longitudinal piece decouples

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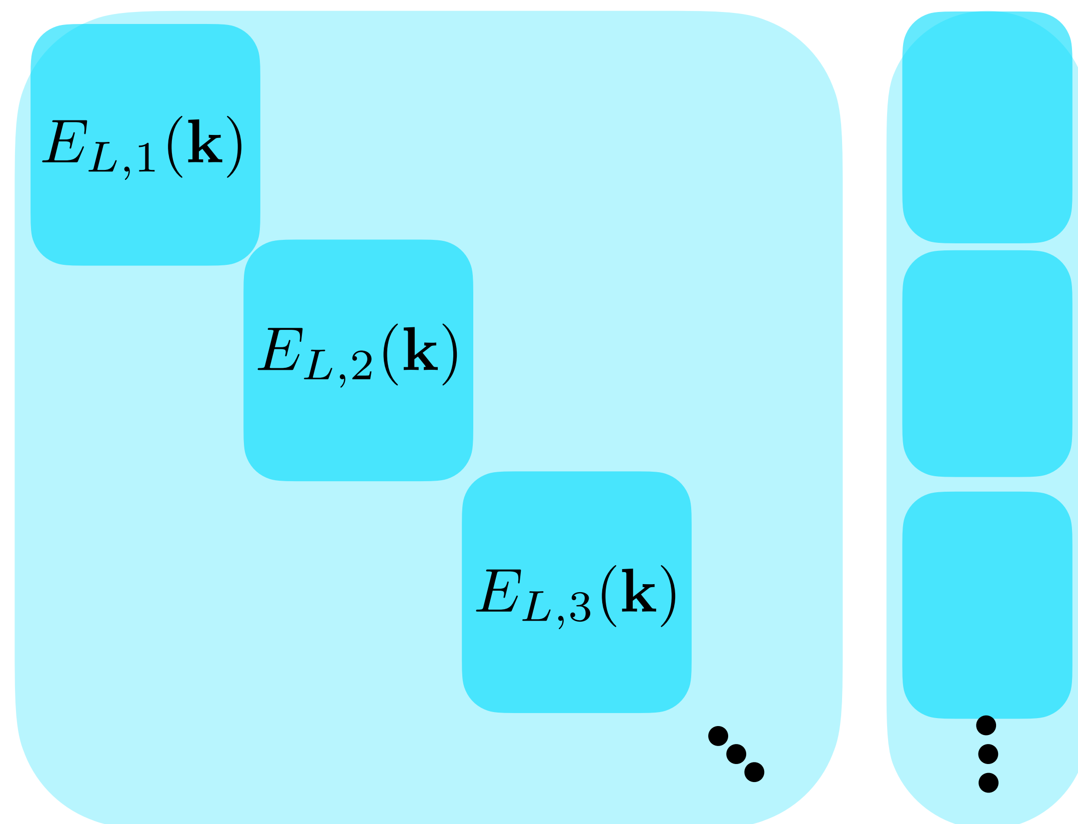

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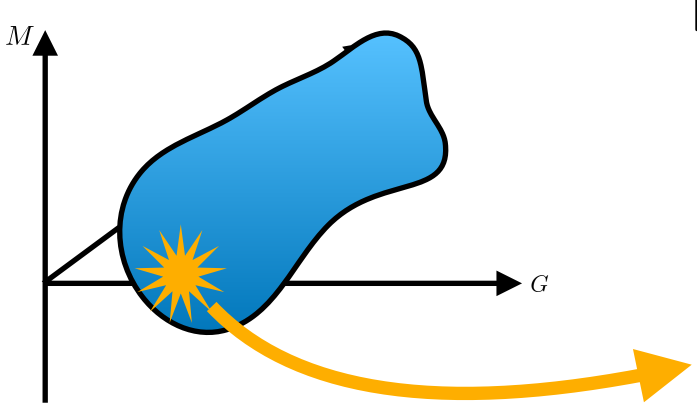

$$|E_i(\mathbf{k})\rangle = |E_{T1}(\mathbf{k})\rangle \otimes |E_{T2}(\mathbf{k})\rangle \times |E_L(\mathbf{k})\rangle$$

$$\mathcal{H} = \bigotimes_k (\mathcal{H}_{T1,k} \otimes \mathcal{H}_{T2,k} \otimes \mathcal{H}_{L,k})$$

$$\mathcal{H} = \bigoplus_{\{E_L(k)\}} \left(\bigotimes_k \mathcal{H}_{T1,k} \otimes \mathcal{H}_{T2,k} \right)$$

Longitudinal piece decouples

$$H = \int_k \sum_{\alpha=1,2} \big(E_\alpha(-k)E_\alpha(k) + k^2 A_\alpha(-k)A_\alpha(k)\big) + E_L(-k)E_L(k)$$

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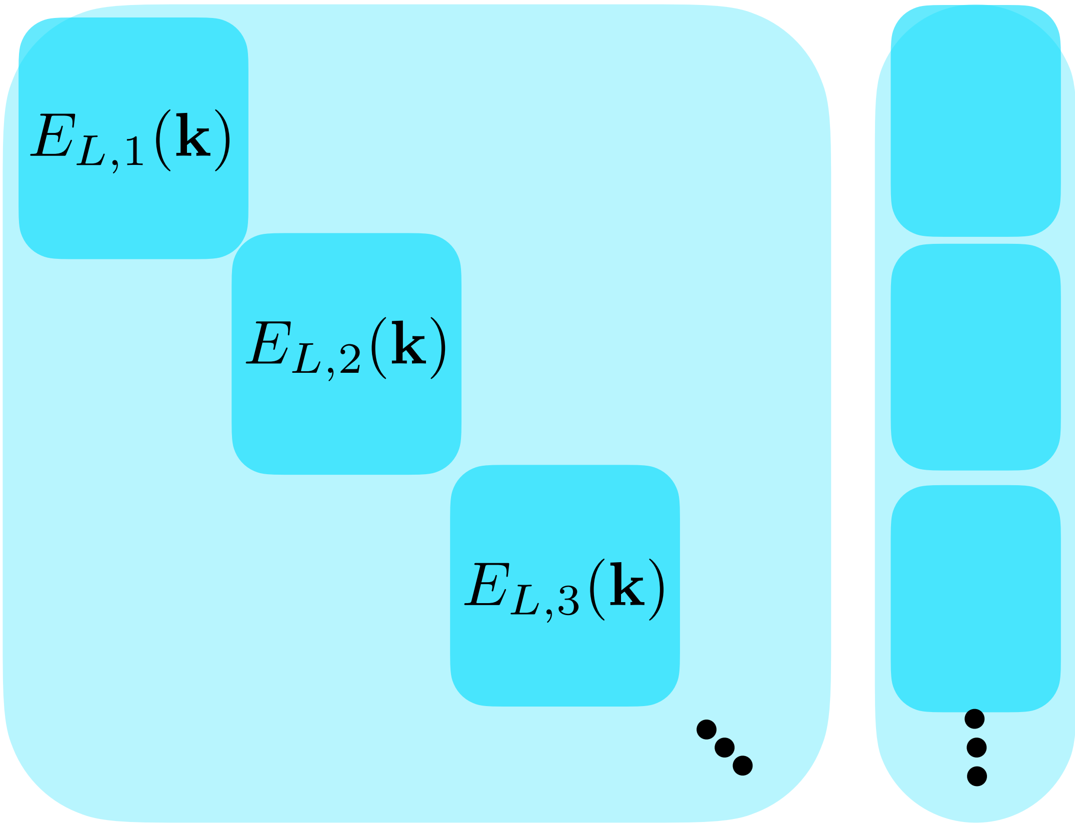

Choosing a sector aka fixing Gauss’ law

Conventionally, always choose $|E_L(\mathbf{k}) = 0\rangle$

Longitudinal piece decouples

$$H = \int_k \sum_{\alpha=1,2} \big(E_\alpha(-k)E_\alpha(k) + k^2 A_\alpha(-k)A_\alpha(k)\big) + E_L(-k)E_L(k)$$

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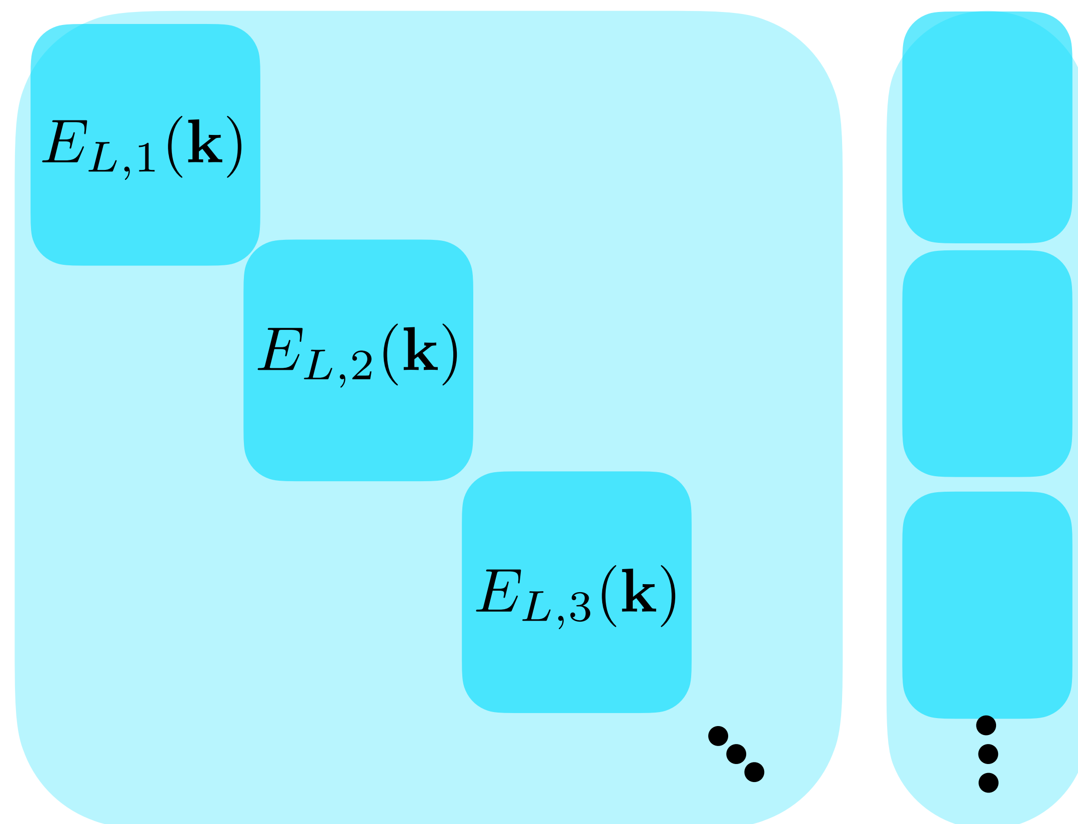
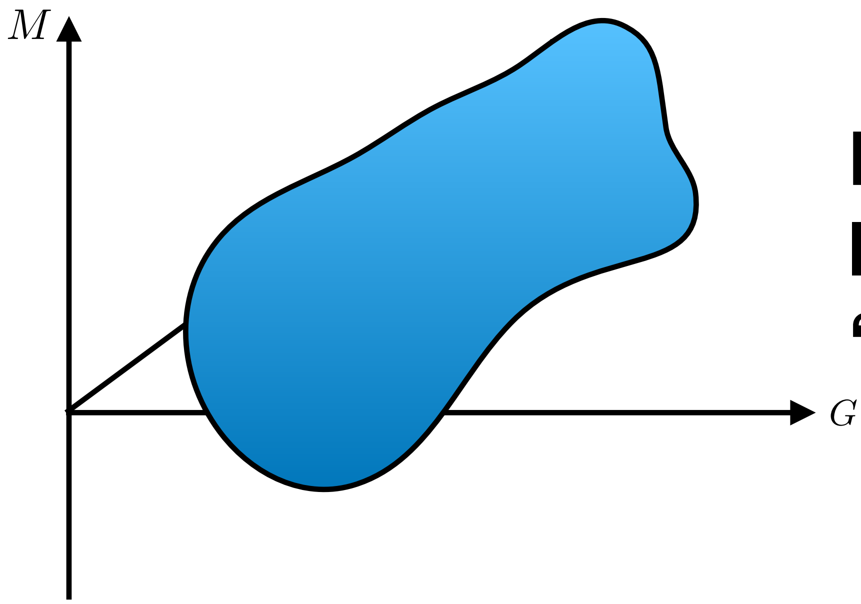


Choosing a sector aka fixing Gauss’ law

More generally

$$H \supset \int_{\mathbf{k}} \frac{k_i E_i(-\mathbf{k}) k_j E_j(\mathbf{k})}{k^2} \leftrightarrow \int d^3\mathbf{x} d^3\mathbf{y} \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

This approach lands on Weyl gauge



No funny business with modified and non-local commutation relations (a la other gauges 'quantise a classical Lagrangian' approach)

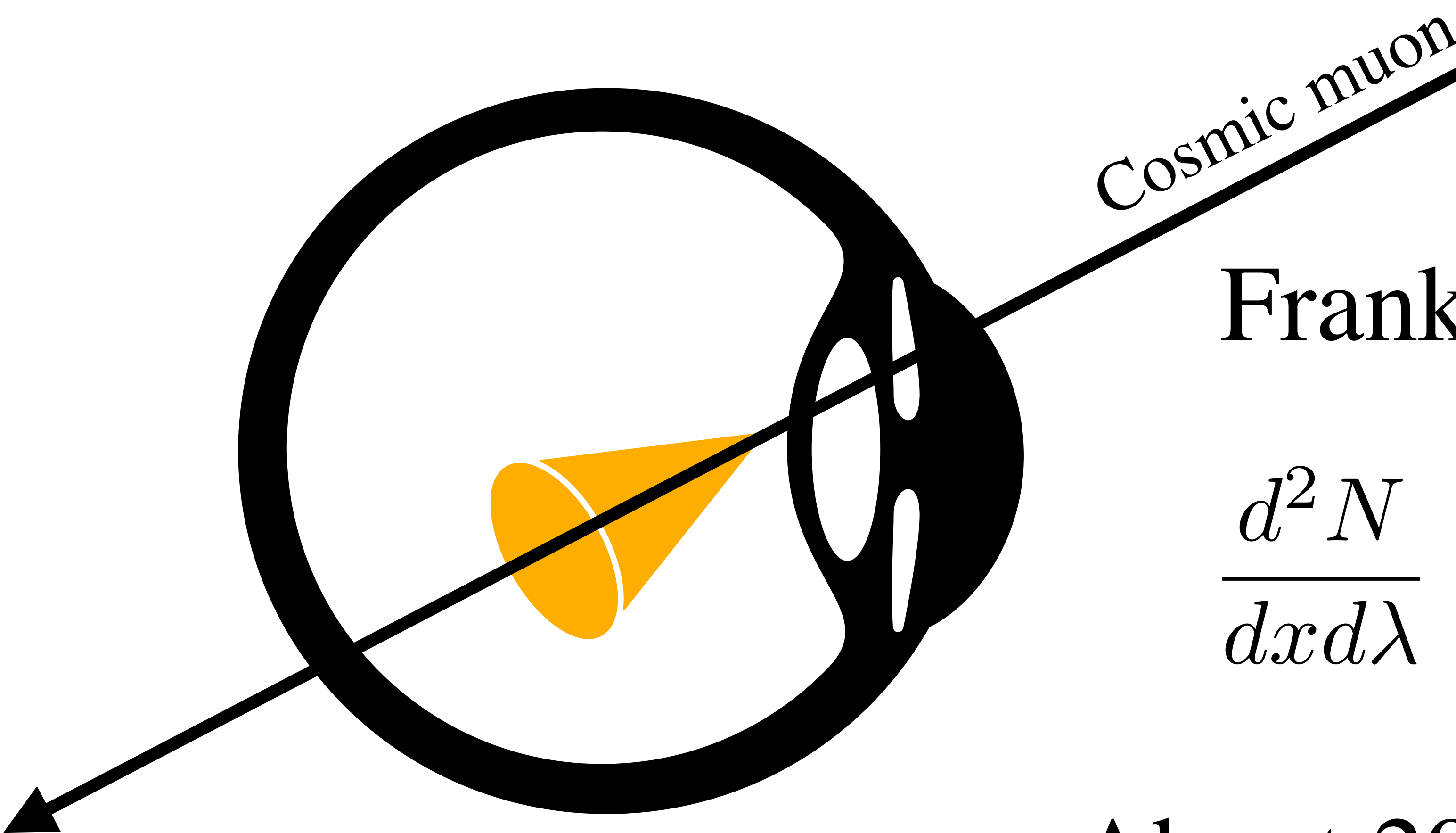
The choice of state has phenomenology of an apparent background charge density

Becomes non-trivial on curved space

Muon interlude


Muon vision
(At Kashiwa science camp)


Can we see a cosmic muon with the naked eye?



Flux: 1/eyeball/min

Frank-Tamm

$$\frac{d^2N}{dx\,d\lambda} = \frac{2\pi\alpha}{\lambda^2}\left(1 - \frac{1}{\beta^2 n^2}\right)$$

About 200 visible Cherenkov photons/cm in eyeball jelly

A few (5-7) photons are detectable

Hecht, S., Schlaer, S. & Pirenne, M. H. Energy, quanta and vision. *J. Opt. Soc. Am.* **38**, 196–208 (1942).

One photon is detectable

Nature Communications volume 7, Article number: 12172 (2016)

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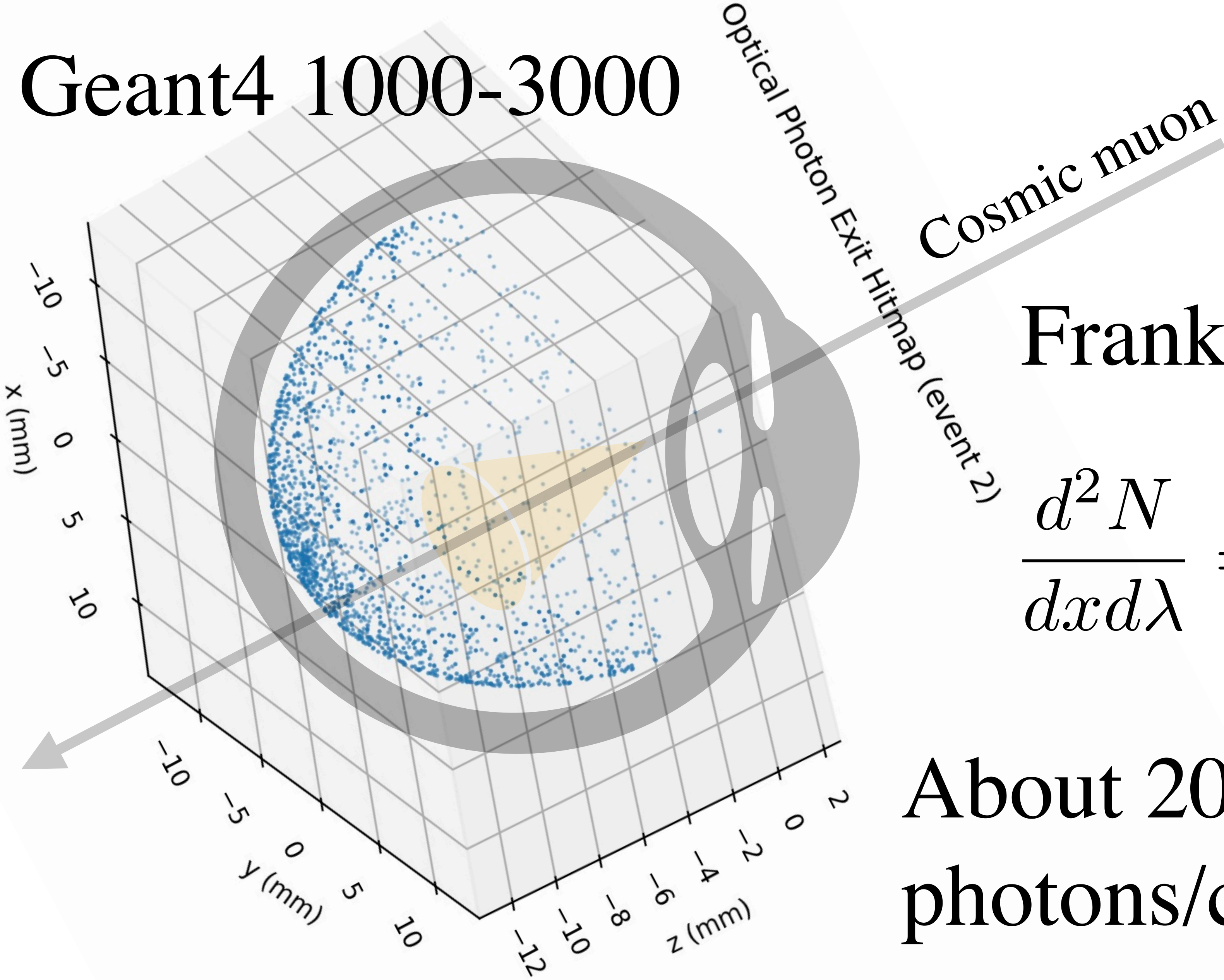

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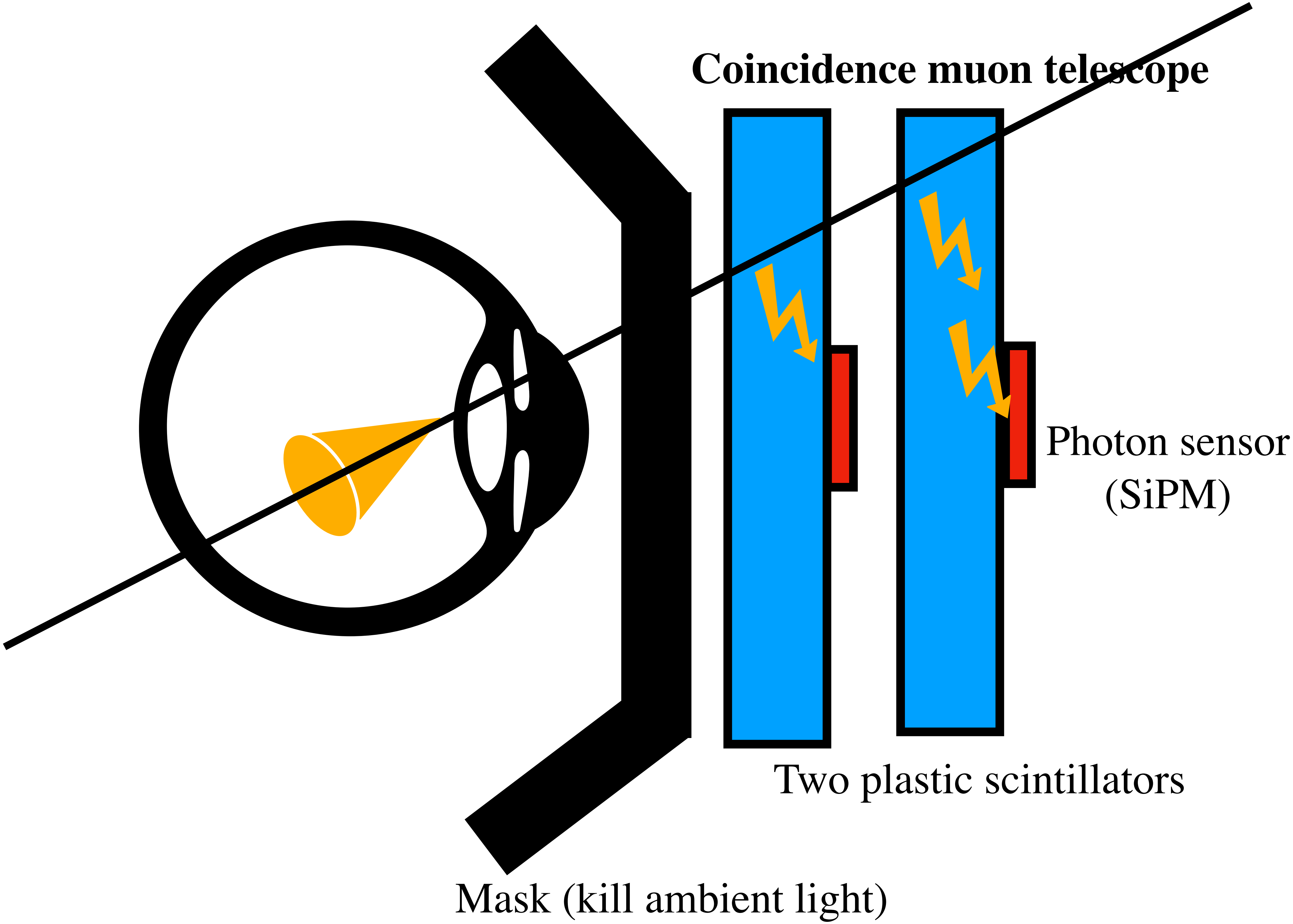
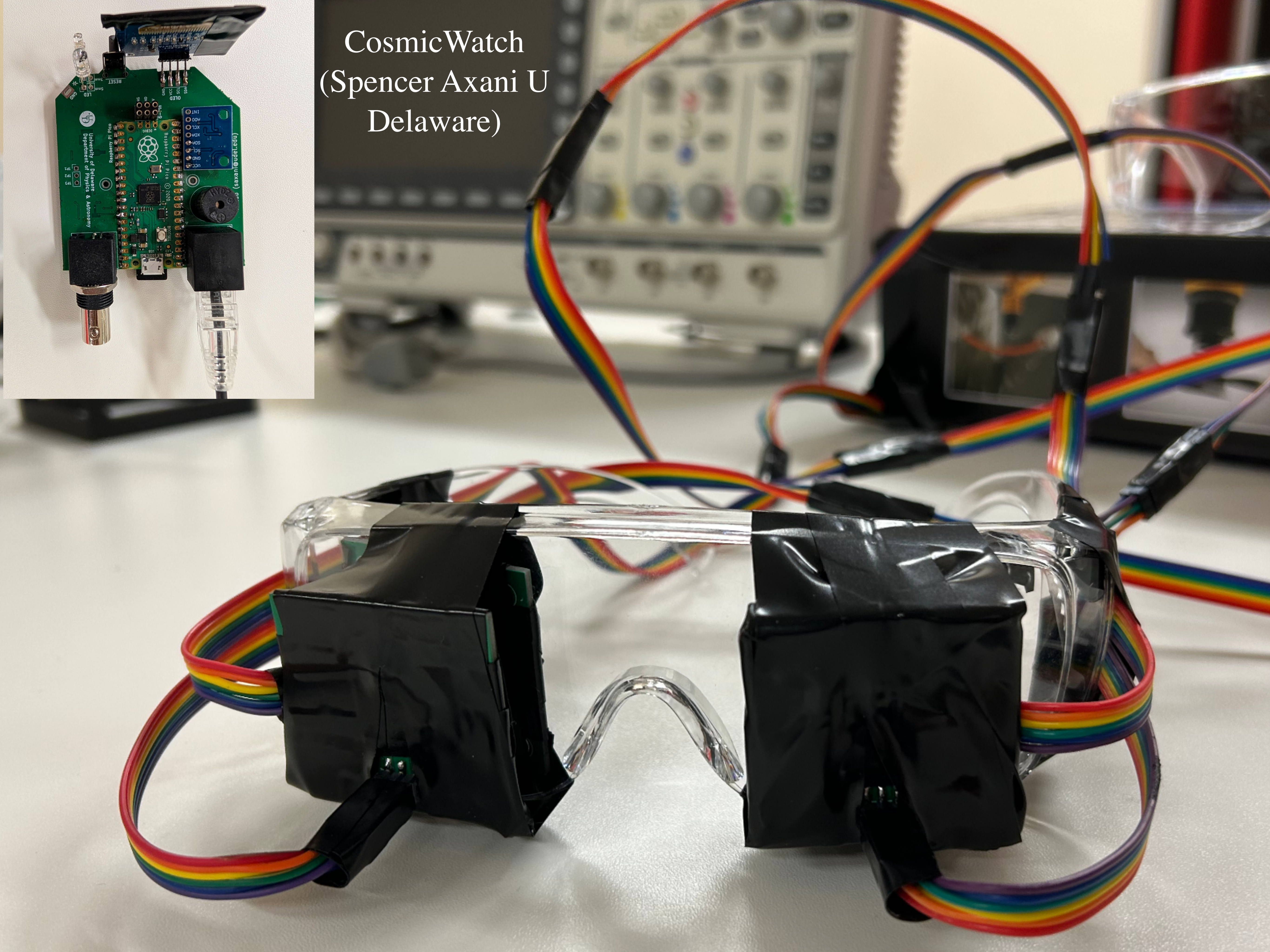
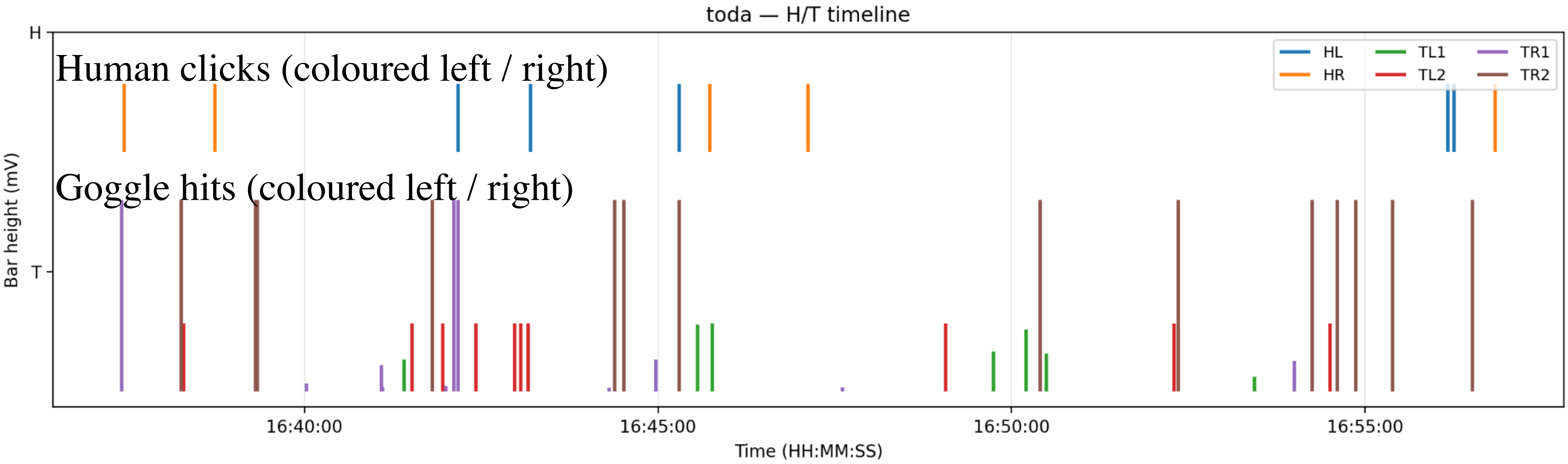
Experimental setup




CosmicWatch
(Spencer Axani U
Delaware)

In a black-out room….



1.7 sigma detection

(+0.6 sigma, 0.6 sigma, 1.6 sigma + two ‘no see-ers’)

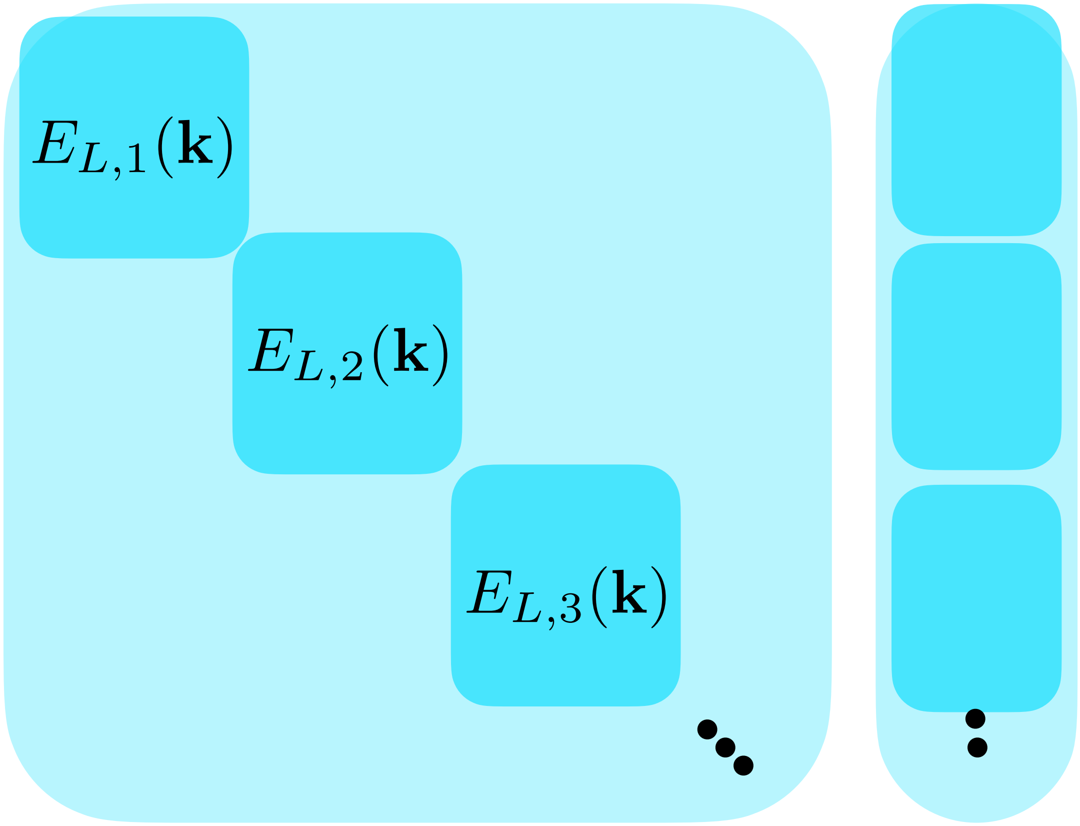
Next time you visit IPMU.. we need more statistics!

End muon interlude

For EM we had a choice of state

$$[H, E_L(k)] = 0$$

$$\hat{E}_L(k)|E_L(k)\rangle = E_L(k)|E_L(k)\rangle$$



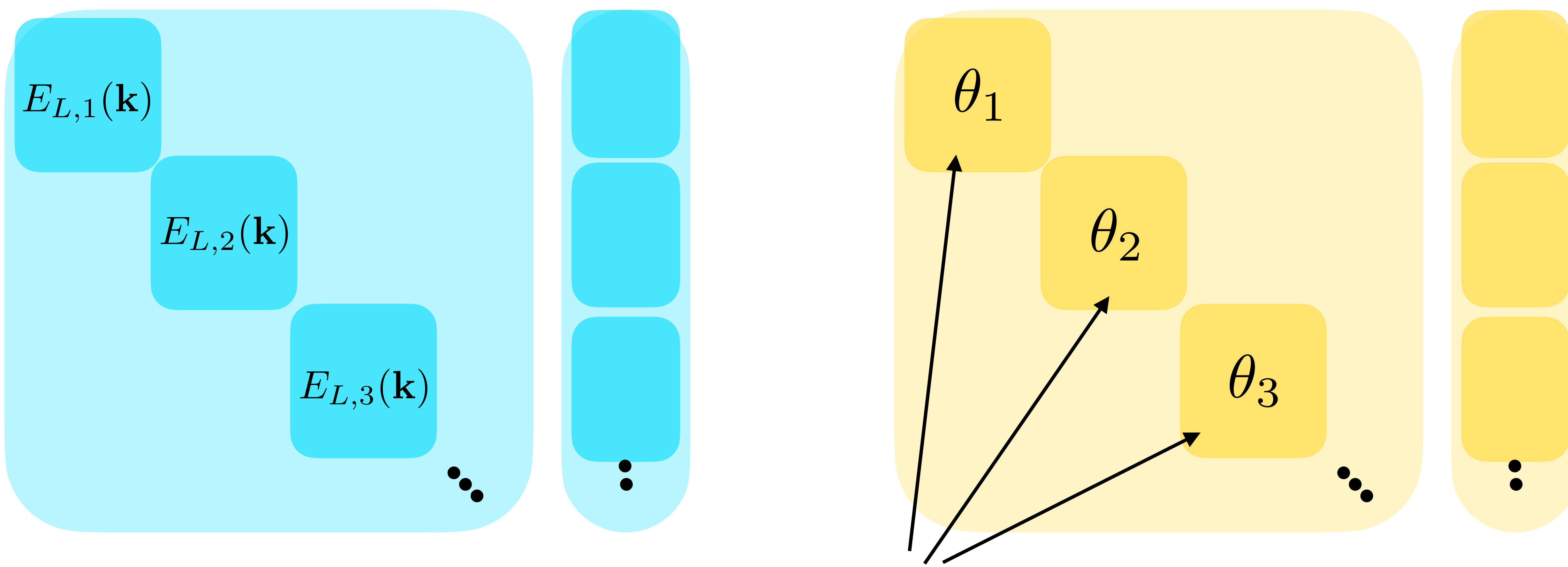
For QCD we have a similar situation

$$[H, E_L(k)] = 0$$

$$\hat{E}_L(k)|E_L(k)\rangle = E_L(k)|E_L(k)\rangle$$

$$[H_{QCD}, G] = 0$$

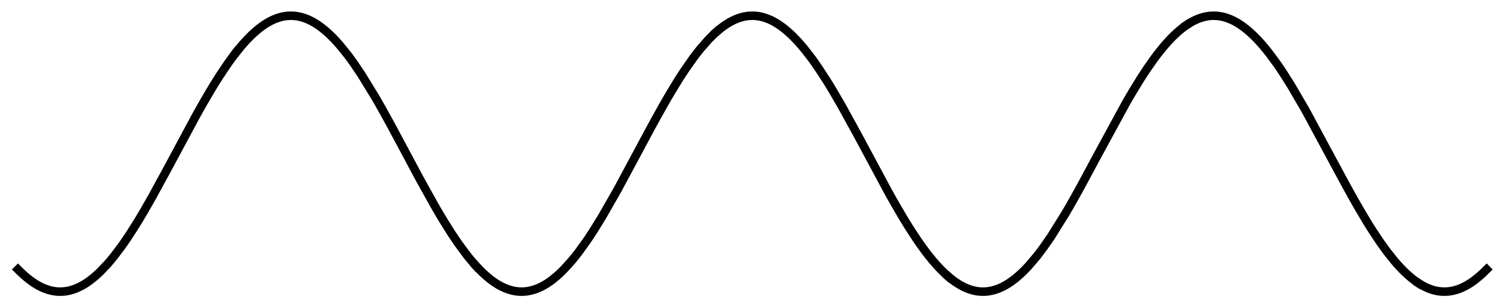
$$G|\theta_{QCD}\rangle = e^{i\theta_{QCD}}|\theta_{QCD}\rangle$$



In each of these blocks of Hilbert space, the nucleon EDM is different

The toy QM analogy

Particle in a periodic potential (or on a ring; identical physics for closed system)



$$H_0 = \frac{1}{2}\hat{P}^2 + \cos\left(\frac{2\pi\hat{X}}{a}\right)$$

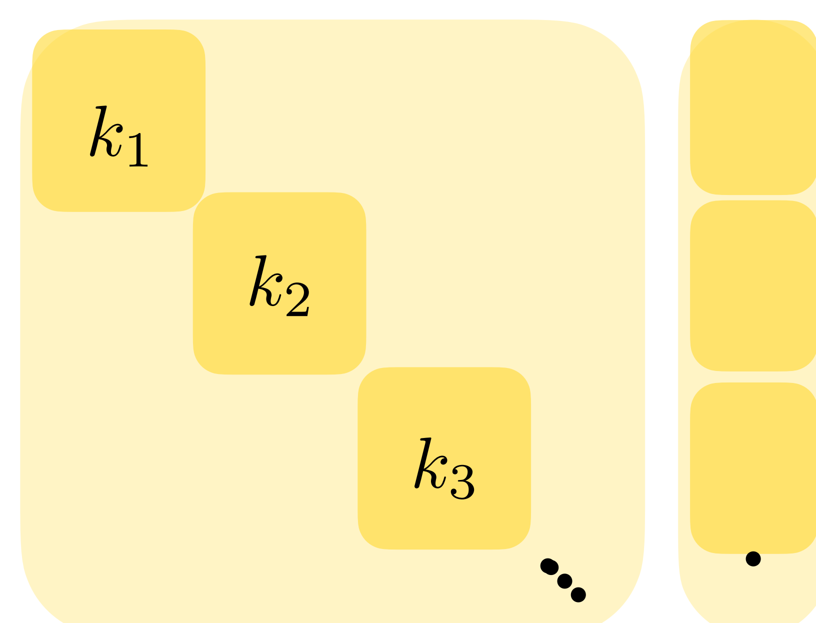
Bloch waves

$$\psi_k(x) = e^{i\frac{kx}{a}}u(x)$$

$$T = e^{i\hat{P}a} \quad (\hat{X} \to \hat{X} + a)$$

$$[\hat{H}, \hat{T}] = 0$$

$$\hat{T}|\psi_k\rangle = e^{ik}|\psi_k\rangle$$



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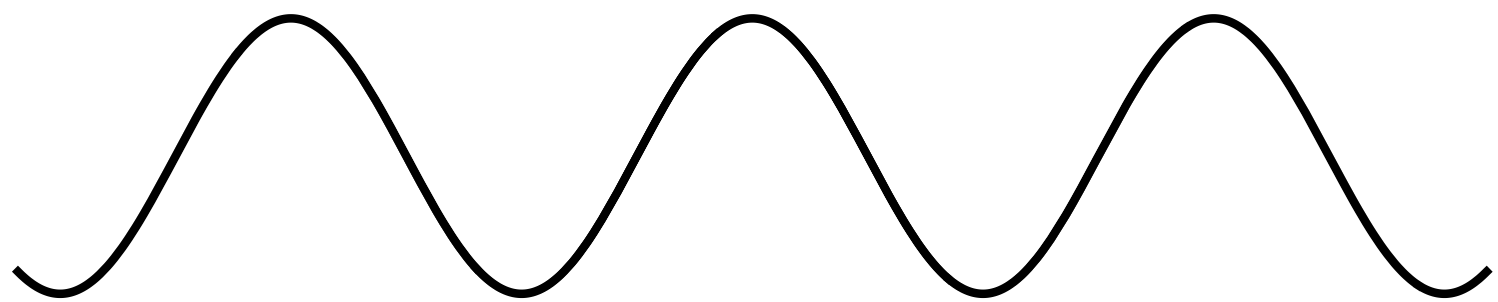


$$H_0 = \frac{1}{2}\hat{P}^2 + \cos\left(\frac{2\pi\hat{X}}{a}\right)$$

$$\psi_k(x) = e^{i\frac{kx}{a}}u(x)$$

Parity:

$$[H_0, P] = 0 \qquad P|\psi_k\rangle = |\psi_{-k}\rangle$$

$$[P, T] \neq 0 \qquad k = 0, \pi$$

Only T *and* P eigenstates

I want to make two points about this system

1. Shifting k to and from H

Defining

$$U_k = e^{ik\hat{X}} \qquad |\psi_k\rangle = e^{ik\hat{X}}|\psi_0\rangle$$

Unitary equivalence

$$H_0 \quad \{|\psi_k\rangle\} \quad \longleftrightarrow \quad U_\delta H_0 U_\delta^\dagger \quad \{U_\delta|\psi_k\rangle\}$$

Where

$$U_\delta H_0 U_\delta^\dagger = H_\delta = \frac{1}{2}(-i\frac{\partial}{\partial x} + \delta)^2 + \cos(\frac{2\pi\hat{X}}{a})$$

$$\{U_\delta|\psi_k\rangle\} = \{|\psi_{k+\delta}\rangle\} = \text{ the same set of states, relabelled}$$

1. Shifting k to and from H

Defining

$$U_k = e^{ik\hat{X}} \qquad |\psi_k\rangle = e^{ik\hat{X}}|\psi_0\rangle$$

Unita

You can always choose a parity even hamiltonian (also for QCD). The physics question is what state you then choose as an initial condition. These choices can break parity. Parity is not ‘a solution’ to a problem of “*why not the parity breaking ones?*”.

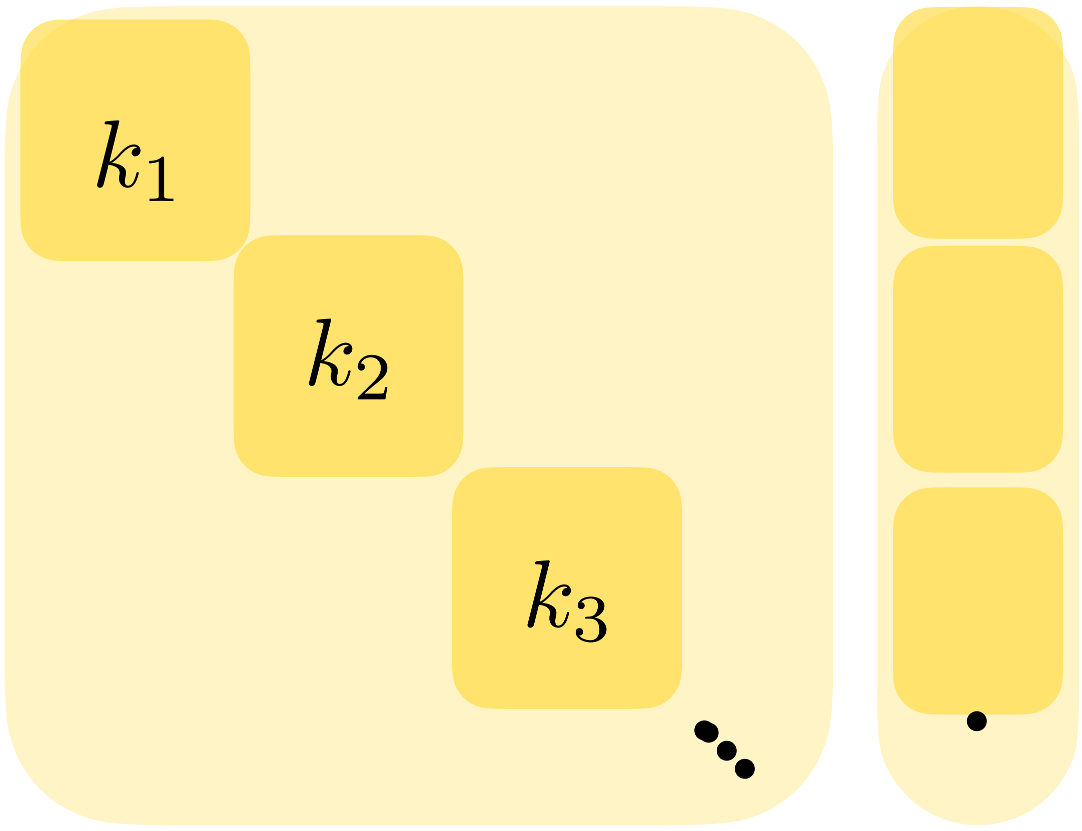
Where

$$U_\delta H_0 U_\delta^\dagger = H_\delta = \frac{1}{2}(-i\frac{\partial}{\partial x} + \delta)^2 + \cos(\frac{2\pi\hat{X}}{a})$$

$$\{U_\delta|\psi_k\rangle\} = \{|\psi_{k+\delta}\rangle\} = \text{ the same set of states, relabelled}$$

2. You do not need to be in T or P eigenstate (even in the 'gauge' case)

$$|\psi >= |\psi_{k_1}\rangle + |\psi_{k_2}\rangle + \ldots$$



Just an incoherent sum. It is not 'gauge invariant'. But it evolves trivially - can be no objection to this as a state.

To conclude

I showed examples in QED and QCD where the concept of the broader Hilbert space is important for phenomenology

There is also a story in gravity. The four momentum constraints similarly split up the broader Hilbert space. The analogue of the QED background charge is a background 'shadow matter' that can look like dark matter

Maybe we can see muons with the naked eye