Quantum Error Correction (QEC) for Dark Matter Detection
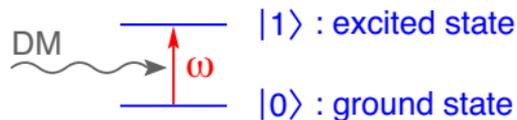
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Particle Physics at Crossroads 2026, Sapporo, Japan (2026.03.02 – 06)

1. Introduction

There are many ideas of using qubits (quantum bits) for DM detection

DM $\xrightarrow{\omega}$ $|1\rangle$: excited state

$|0\rangle$: ground state

- Qubit is a quantum two-level system, essential element for quantum computers

- Qubits can be used to probe EM field induced by wave-like DM
  [Dixit et al. ('21); Chen, Fukuda, Inada, TM, Nitta, Sichanugrist ('22, '24); Engelhardt, Bhoonah, Liu ('23); Chigusa, Hazumi, Herbschleb, Mizuochi, Nakayama ('23); Agrawal et al. ('23); Ito, Kitano, Nakano, Takai ('23); Braggio et al. ('24); Chigusa, Kasamaki, Kusano, TM, Nakayama, Ozawa, Takahashi, Umemoto, Vutha ('25); $\cdots$]

  $\Leftrightarrow$ Qubit may be excited by DM-induced EM field

Expected sensitivity to dark photon DM with superconducting qubits
[Chen, Fukuda, Inada, TM, Nitta, Sichanugrist ('22)]
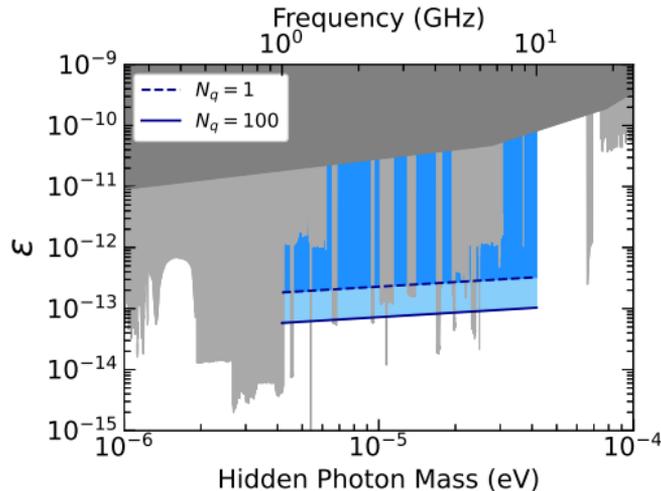
⇒ With qubits, we may probe DM candidates unexplored before

⇒ DM search using superconducting qubit is on-going

How does the sensitivity scale with the number of qubits N?

- $S/\sqrt{B} \propto \sqrt{N}$, if individual qubit is separately treated

- Better sensitivity may be possible using entangled states, but noises can be serious problems

- We propose a protocol with QEC, realizing an improvement of the sensitivity

Outline:

1. Introduction

2. Setup

3. QEC for DM Detection

4. Summary

2. Setup

Effective Hamiltonian of qubit coupled to wave-like DM ($\omega \simeq m_{\mathrm{DM}}$):

$$H \simeq \eta \left(|1\rangle\langle 0| + |0\rangle\langle 1| \right) = \eta X$$

$$X \equiv |1\rangle\langle 0| + |0\rangle\langle 1|, \;\; Y \equiv i|1\rangle\langle 0| - i|0\rangle\langle 1|, \;\; Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$$

η: DM-qubit coupling: $\eta \ll \gamma$ is assumed (γ: noise rate)

$\Rightarrow$ Discovery of DM can be claimed with confirming non-zero η

$$|\Psi(t)\rangle = \cos \eta t \, |0\rangle - i \sin \eta t \, |1\rangle$$

DM oscillation has unknown phase: $\phi_{\mathrm{DM}} \simeq \bar{\phi}_{\mathrm{DM}} \cos(m_{\mathrm{DM}} t + \alpha)$

$\Leftrightarrow H \simeq \eta(X \cos \alpha - Y \sin \alpha)$

The phase α randomizes on the DM coherence time τ_{DM}

$\Rightarrow$ I focus on the case that $\tau_{\mathrm{DM}} \gg$ (coherence time of qubit)

Noise: I consider bit-flip noise (random Pauli-X channel) $|0\rangle \leftrightarrow |1\rangle$

$\Rightarrow$ Signal and parallel noise are hard to distinguish

$\Rightarrow$ Other major noises may be reduced to (effective) bit-flip by a QEC-like operation

Lindblad equation (for N-qubit case)

$$\dot{\rho} = -i \left[\eta \sum_{i=1}^{N} X_i, \rho \right] + \frac{1}{2}\gamma \sum_{i=1}^{N} (X_i \rho X_i - \rho) \equiv \mathcal{L}[\rho]$$

X_i: Pauli-X operator acting onto i-th qubit

Evolution based on the above Lindblad equation:

$$\rho(t) = \bigotimes_i \rho_i(t) \; \Leftarrow \; \text{No qubit-qubit coupling, } \rho(0) = |00\cdots\rangle\langle 00\cdots|$$

Evolution of the density matrix of i-th qubit:

$$\dot{\rho}_i = -i\eta\,[X_i, \rho_i] + \frac{1}{2}\gamma\,(X_i \rho_i X_i - \rho_i)$$

$$\rho_i = \frac{1}{2}\big[\mathbb{1} + \rho_X(t)X_i + \rho_Y(t)Y_i + \rho_Z(t)Z_i\big]$$

$$\Rightarrow \rho_X(t) = 0,\quad \rho_Y(t) = -e^{-\gamma t}\sin 2\eta t,\quad \rho_Z(t) = e^{-\gamma t}\cos 2\eta t$$

Separate counting of excited qubits (M measurement cycles):

$$N_{\text{excite}} \equiv M \sum_{i=1}^{N} \frac{\langle \mathbb{1} - Z_i \rangle}{2} = \frac{NM}{2}(1 - \rho_Z) \simeq \frac{NM}{2}(2\eta^2 t^2 + \gamma t) + \cdots$$

$$\Rightarrow S \sim NM\eta^2 t^2,\quad B \sim NM\gamma t$$

$$\frac{S}{\sqrt{B}} \sim \sqrt{N} \times \frac{\sqrt{M}\eta^2 t^2}{\sqrt{\gamma t}} \quad \Rightarrow \quad \Delta^{(\text{sep.})}\eta^2 \sim \frac{\gamma^2}{\sqrt{NM}} \quad (\text{with } t \sim \gamma^{-1})$$

3. QEC for DM Detection

Without the noise, state is symmetric under exchanges of qubits

$$e^{-iHt} |0\rangle^{\otimes N} \simeq \bigotimes_i e^{-i\eta X_i t} |0\rangle^{\otimes N} \simeq |0\rangle^{\otimes N} - i\eta t \sum_i X_i |0\rangle^{\otimes N} + O(\eta^2)$$

If no noise, the state stays in "Code space" neglecting terms of $O(\eta^2)$:

Code space (logical qubit): $\{|0\rangle^{\otimes N}, |W\rangle\} \equiv \{|0_L\rangle, |1_L\rangle\}$

$$|W\rangle \equiv \frac{1}{\sqrt{N}} \sum_i X_i |0\rangle^{\otimes N} = \frac{1}{\sqrt{N}} \big(|100\cdots\rangle + |010\cdots\rangle + \cdots\big)$$

$$\Rightarrow H \simeq \sqrt{N}\,\eta\, X_L \quad \text{with} \quad X_L \equiv (|1_L\rangle\langle 0_L| + |0_L\rangle\langle 1_L|)$$
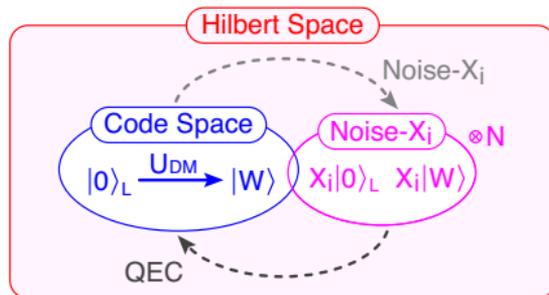
X_i-noise acts only on i-th qubit

$$|0\rangle^{\otimes N} \xrightarrow{X_i} |i\rangle \equiv X_i |0\rangle^{\otimes N} = |0\cdots 010\cdots 0\rangle$$

Noises kick out the state from the code space

$$\langle i|W\rangle = \frac{1}{\sqrt{N}} \qquad \Rightarrow$$

QEC may bring the state back to the code space

Protocol of our proposal:

1. Specify the "subspace" to which the state is likely to belong

   $\Rightarrow$ Information about the error (noise) location, if error occurs

2. Correct error without destroying logical information

   $\Rightarrow a X_i \left| 0_L \right\rangle + b X_i \left| W \right\rangle \xrightarrow{\text{QEC}} a \left| 0_L \right\rangle + b \left| W \right\rangle$

Various unitary operations are necessary

$\Rightarrow$ I assume that any unitary operation onto the system is possible (without discussing implementation cost)

$\Rightarrow$ DM detection using qubits in high-fidelity quantum computers

Outline of the protocol for DM detection with QEC

1. Divide τ into many unit intervals with duration $\Delta t \ll \tau \sim \gamma^{-1}$

2. Every unit time intervals, perform error correction

3. Readout at $t \sim \gamma^{-1}$
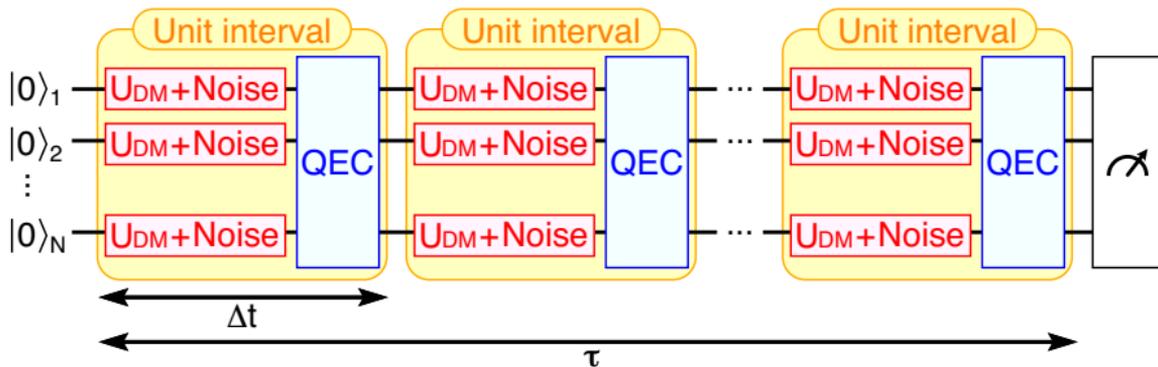
4. Repeat 1 – 3 (M times)

QEC step 1: POVM measurement

1-1 Expand the Hilbert space, introducing extra "ancilla" qubits

$$|\Psi\rangle_{\text{sensor}} \rightarrow |\Psi\rangle_{\text{sensor}} \otimes |\Psi'\rangle_{\text{ancilla}}$$

1-2 Specify the list of the "measurement operators" M_a

$$\sum_a M_a^\dagger M_a = \mathbb{1}$$

1-3 Perform the following unitary operation "U_M" (which exists)

$$U_M \left[|\Psi\rangle \otimes |0\rangle_{\text{ancilla}} \right] = \sum_a M_a |\Psi\rangle \otimes |a\rangle_{\text{ancilla}}$$

$$= M_1 |\Psi\rangle \otimes |1\rangle_{\text{ancilla}} + M_2 |\Psi\rangle \otimes |2\rangle_{\text{ancilla}} + \cdots$$

1-4 Measure the ancilla state to probe the sensor state

Our choice of M_a to probe the sensor state $|\Psi\rangle$

- Projection on the code space

  $\Rightarrow\ M_0 = |0_L\rangle\langle 0_L| + |W\rangle\langle W|$

- Approximate projection on the space spanned by $\{X_i\,|0_L\rangle\,,X_i\,|W\rangle\}$

  $\Rightarrow\ M_i \simeq X_i\big(\,|0_L\rangle\langle 0_L| + |W\rangle\langle W|\,\big)X_i + O(N^{-1/2})$

- $\cdots$

We can find a set of operators satisfying:

$$\sum_a M_a^\dagger M_a = \mathbb{1}$$

QEC step 2: Recovery operation

- $M_0 = |0_L\rangle\langle 0_L| + |W\rangle\langle W|$: Projection on the code space

  $\Rightarrow$ No recovery operation needed: $R_0 = \mathbb{1}$

- $M_i \simeq X_i \big(|0_L\rangle\langle 0_L| + |W\rangle\langle W| \big) X_i + O(N^{-1/2})$:

  Approximate projection on the space spanned by $\{X_i |0_L\rangle , X_i |W\rangle\}$

  $\Rightarrow R_i \simeq \big(|0_L\rangle\langle 0_L| X_i + |W\rangle\langle W| X_i + \text{h.c.} \big) + \cdots$

  $$R_i : \begin{cases} |0_L\rangle \leftrightarrow X_i |0_L\rangle + O(N^{-1/2}) \\ |W\rangle \leftrightarrow X_i |W\rangle + O(N^{-1/2}) \end{cases}$$

- $\cdots$

$\Rightarrow$ After the recovery operation, the state comes back to the code space

Density matrix after all the operations (assuming $\rho(t) \in$ Code space):

$$\rho(t + \Delta t) \simeq \sum_a R_a M_a \Big\{ \rho(t) + \mathcal{L}[\rho(t)]\Delta t \Big\} M_a^\dagger R_a^\dagger \simeq \rho(t) + \dot{\rho}\Delta t$$

The noise rate does not scale with N (assuming ideal QEC):

$$\dot{\rho}|_{N \gg 1} \simeq -i\sqrt{N}\,\eta\,[\,X_L, \rho\,] + \frac{1}{2}\gamma\,(X_L \rho X_L - \rho) + \frac{1}{4}\gamma\,(Z_L \rho Z_L - \rho)$$

$$\Rightarrow\ S \sim N\eta^2 t^2, \quad B \sim \gamma t \quad \text{(single shot)}$$

Sensitivity after M measurement cycles (taking optimal $t \sim \gamma^{-1}$):

$$\frac{S}{\sqrt{B}} \sim N \times \frac{\sqrt{M}\eta^2 t^2}{\sqrt{\gamma t}} \ \Rightarrow\ \Delta\eta^2 \sim \frac{\gamma^2}{N\sqrt{M}} \ \Leftrightarrow\ \Delta^{(\text{sep.})}\eta^2 \sim \frac{\gamma^2}{\sqrt{NM}}$$

The sensitivity estimation with $t \sim \gamma^{-1}$ breaks down when $N \gtrsim (\gamma/\eta)^2$

$\Leftrightarrow$ If $\sqrt{N}\eta \gtrsim \gamma$, many cycles of Rabi oscillation occurs for $t \sim \gamma^{-1}$

$\Leftrightarrow$ The discovery of the DM is possible when $N \gtrsim (\gamma/\eta)^2 M^{-1}$

Best protocol for $N \gtrsim N_* \equiv (\gamma/\eta)^2$:

1. Divide N sensors into N/N_* groups of N_* sensors

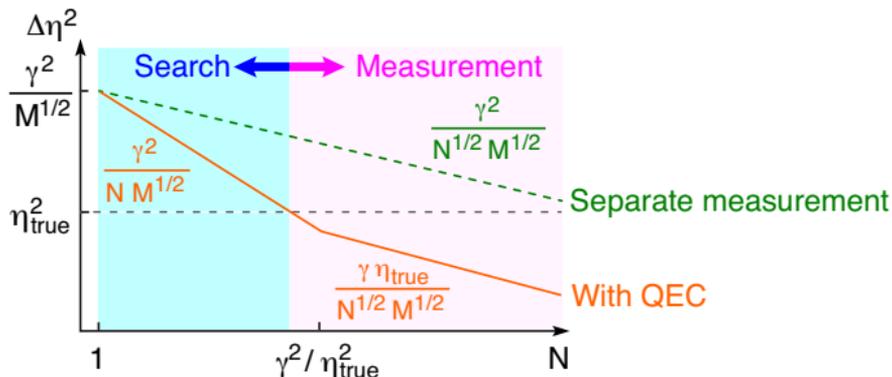2. Apply protocol so far to each group

Expected sensitivity

$$\frac{S}{\sqrt{B}} \sim N_* \times \frac{\sqrt{M}\eta^2 t^2}{\sqrt{\gamma t}} \times \sqrt{\frac{N}{N_*}} \sim \sqrt{NM} \times \frac{\gamma\,\eta\,t^2}{\sqrt{\gamma t}}$$

$$\Leftrightarrow \left. \frac{S}{\sqrt{B}} \right|^{\text{(sep.)}} \sim \sqrt{NM} \times \frac{\eta^2\,t^2}{\sqrt{\gamma t}}$$

For $N \to \infty$, we do not beat the standard quantum limit (SQL)

$\Leftrightarrow$ QEC helps to reach the discovery threshold earlier

Expected sensitivity (with M measurement cycle)

Our protocol is useful particularly for the DM detection

$\Leftrightarrow$ Because of the random reset of the phase α, observable should be proportional to η^2

4. Summary

In DM detection with qubits, we may use quantum properties of qubits

$\Rightarrow$ QEC may help to improve the sensitivity

Need lots of new developments to realize our proposal

- Error rates in gate operation and readout should be small

- Need to design and implement necessary unitary operations

- Difficulties may be overcome with R&D for realizing high-fidelity (fault-tolerant) quantum computers

More progresses in quantum technologies should happen

$\Rightarrow$ It is interesting to think about their applications

Backup: $H = \eta X$

$|0\rangle \leftrightarrow |1\rangle$ transition is induced by EM field (with frequency m)

$\Rightarrow H = \omega \, |1\rangle\langle 1| + 2\eta \left(|0\rangle\langle 1| + |1\rangle\langle 0| \right) \cos(mt + \alpha)$

1st term: Energy difference between $|0\rangle$ and $|1\rangle$

2nd term: $|0\rangle \leftrightarrow |1\rangle$ by external EM field (with coupling strength η)

$\Rightarrow i\dfrac{d}{dt} |\Psi_S(t)\rangle = H \, |\Psi_S(t)\rangle$

With Pauli operators:

$\Rightarrow H = -\dfrac{1}{2}\omega(Z - 1) + 2\eta \, X \cos(mt + \alpha)$

$X \equiv |1\rangle\langle 0| + |0\rangle\langle 1| \,, \;\; Y \equiv i \, |1\rangle\langle 0| - i \, |0\rangle\langle 1| \,, \;\; Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$

In the interaction picture with $H_0 = \omega |1\rangle\langle 1|$:

$$H_I = e^{-iH_0 t} \left[2\eta\, X \cos(mt + \alpha) \right] e^{iH_0 t}$$

$$\simeq \eta\big[X \cos\{(\omega - m)t + \alpha\} - Y \sin\{(\omega - m)t + \alpha\} \big]$$

Terms proportional to $\sin(\omega + m)t$ and $\cos(\omega + m)t$ are neglected

We can always redefine X and Y s.t.

$$H = \eta X$$

$\Leftrightarrow$ For the case of DM-induced EM field, α is randomized

$\Rightarrow$ α is approximately a constant within the DM coherence time

Backup: Evolution of Single Qubit

Effective Hamiltonian of qubit coupled to DM (with $m_{\mathrm{DM}} \simeq \omega$, $\alpha = 0$):

$$H \simeq \eta\, X$$

$$X \equiv |1\rangle\langle 0| + |0\rangle\langle 1|\,, \;\; Y \equiv i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|\,, \;\; Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$$
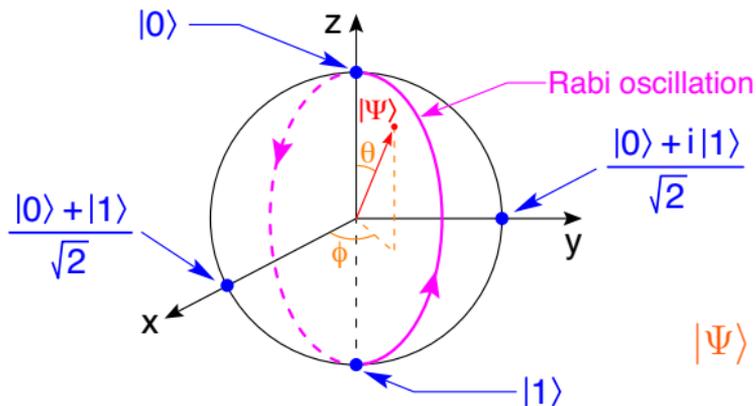
Schrödinger equation

$$i\frac{d}{dt}\,|\Psi\rangle = H\,|\Psi\rangle$$

$$|\Psi(t)\rangle = f_g(t)\,|0\rangle + f_e(t)\,|1\rangle$$

$$\Rightarrow\; i\frac{d}{dt}\begin{pmatrix} f_g(t) \\ f_e(t) \end{pmatrix} = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}\begin{pmatrix} f_g(t) \\ f_e(t) \end{pmatrix}$$

Unitary evolution with DM-induced interaction

$$\begin{pmatrix} f_g(t) \\ f_e(t) \end{pmatrix} = U_{\text{DM}} \begin{pmatrix} f_g(0) \\ f_e(0) \end{pmatrix} \simeq \begin{pmatrix} \cos \eta t & -i \sin \eta t \\ -i \sin \eta t & \cos \eta t \end{pmatrix} \begin{pmatrix} f_g(0) \\ f_e(0) \end{pmatrix}$$

$$|\Psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

Such an excitation process can be used for DM detection

$$P_{g \to e}(t) \simeq (\eta t)^2$$

Backup: Quantum Circuit for GHZ

$U_{\rm DM}$ induces pure phase rotations of its eigenstates

$$U_{\rm DM} \simeq \begin{pmatrix} \cos\delta & -i\sin\delta \\ -i\sin\delta & \cos\delta \end{pmatrix} \quad \text{with} \quad \delta \equiv \eta t \ll 1$$

$$\Rightarrow \ U_{\rm DM} \left|\pm\right\rangle = e^{\mp i\delta} \left|\pm\right\rangle \quad \text{with} \quad \left|\pm\right\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|0\right\rangle \pm \left|1\right\rangle\right)$$

$$\Rightarrow \ U_{\rm DM}^{\otimes N} \left|\pm\right\rangle^{\otimes N} = e^{\mp iN\delta} \left|\pm\right\rangle^{\otimes N}$$

We want to make the accumulated phase physical

$\Rightarrow$ We consider the following quantum (unitary) operation

$$U_{\rm GHZ} : \begin{pmatrix} \left|0\right\rangle^{\otimes N} \\ \left|1\right\rangle^{\otimes N} \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} \left|+\right\rangle^{\otimes N} + \left|-\right\rangle^{\otimes N} \\ \left|+\right\rangle^{\otimes N} - \left|-\right\rangle^{\otimes N} \end{pmatrix}$$

Basic unitary operations (quantum gates)

- Z gate

  $$Z = |g\rangle\langle g| - |e\rangle\langle e| \ \Rightarrow |+\rangle \xrightarrow{Z} |-\rangle \ \text{with} \ |\pm\rangle \equiv \frac{1}{\sqrt{2}}\big(|g\rangle \pm |e\rangle\big)$$
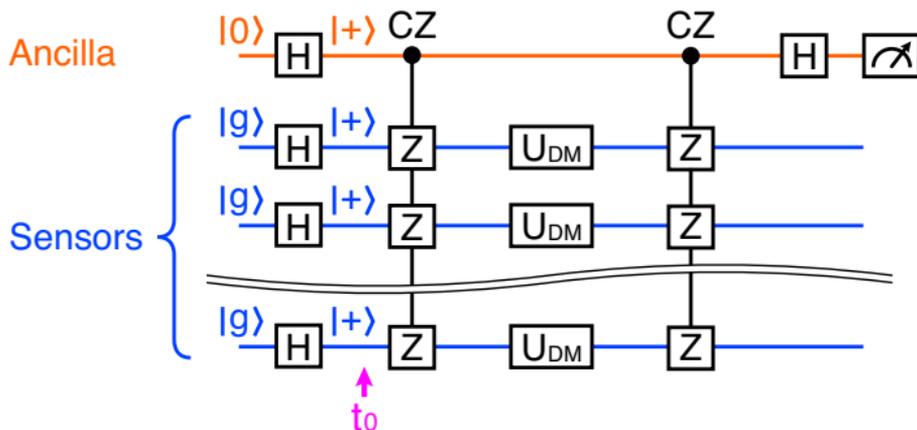
- Hadamard gate

  $$H = |+\rangle\langle g| + |-\rangle\langle e| \ \Rightarrow |g\rangle \xrightarrow{H} |+\rangle, \ \ |e\rangle \xrightarrow{H} |-\rangle$$

- Controlled Z gate

  $$CZ = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes Z$$

  $$\Rightarrow \ \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) \otimes |+\rangle \xrightarrow{CZ} \frac{1}{\sqrt{2}}|0\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |-\rangle$$
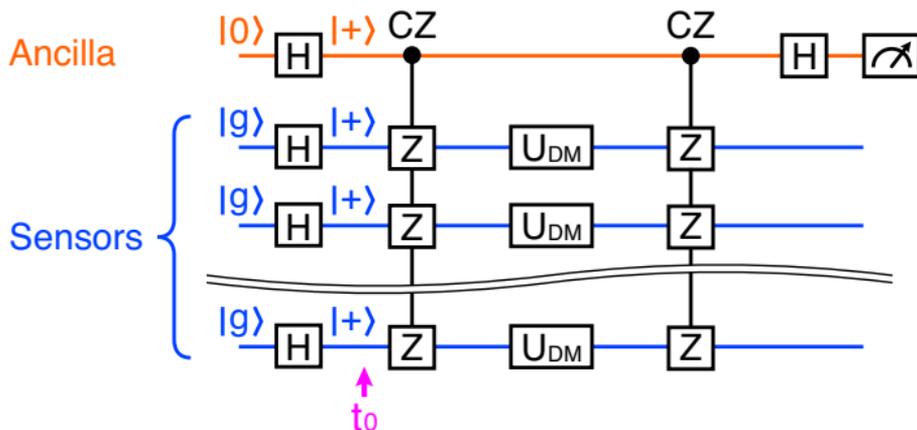
One measurement cycle for the signal enhancement

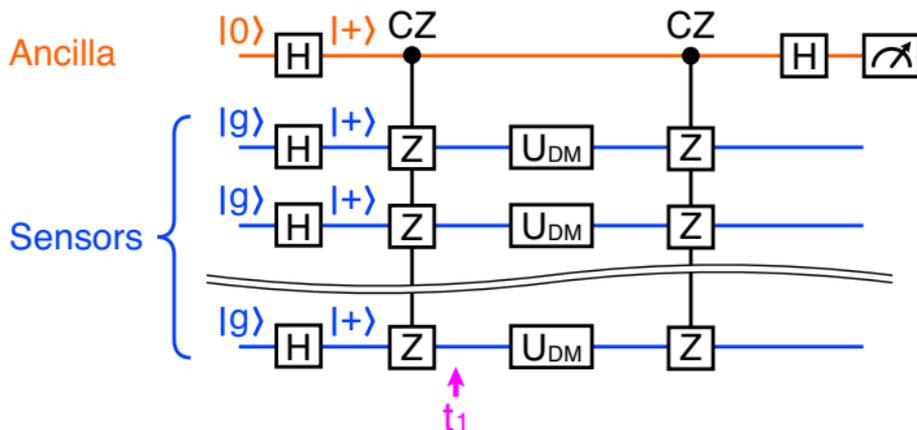The above is an example of the quantum circuit

$\Rightarrow$ Let us first see how it works when $\alpha = 0$

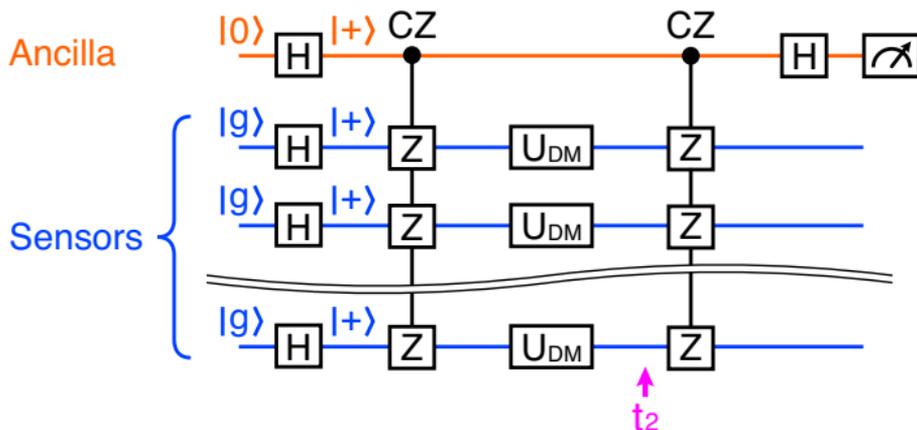# One measurement cycle for the signal enhancement

$$|\Psi(t_0)\rangle = |+\rangle \otimes |+\rangle^{\otimes N_q} = \frac{1}{\sqrt{2}}|0\rangle \otimes |+\rangle^{\otimes N_q} + \frac{1}{\sqrt{2}}|1\rangle \otimes |+\rangle^{\otimes N_q}$$

One measurement cycle for the signal enhancement
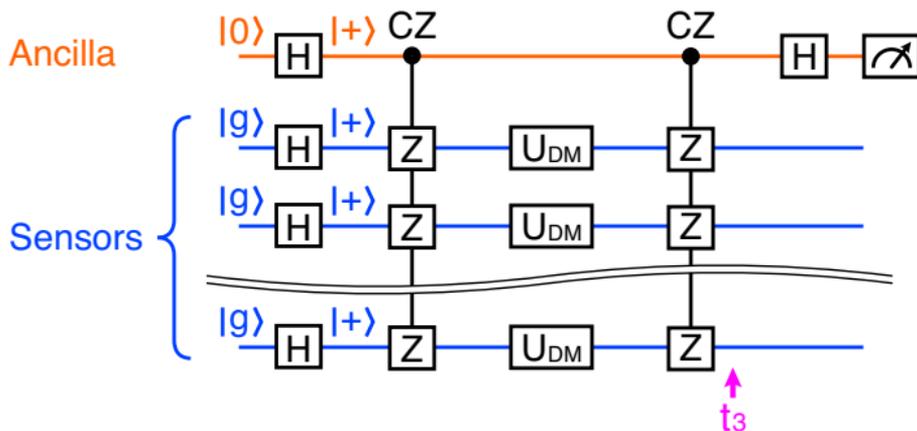

$$|\Psi(t_1)\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |+\rangle^{\otimes N_q} + \frac{1}{\sqrt{2}}|1\rangle \otimes |-\rangle^{\otimes N_q}$$

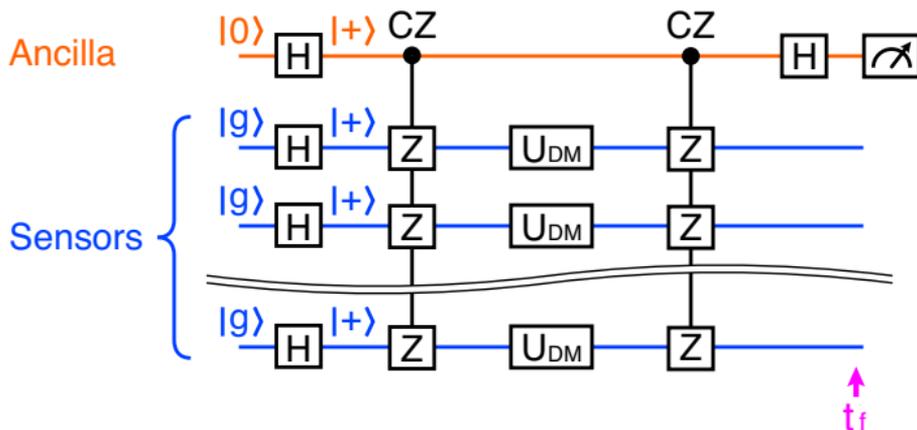# One measurement cycle for the signal enhancement

$$|\Psi(t_2)\rangle = \frac{1}{\sqrt{2}}e^{iN_q\delta}|0\rangle \otimes |+\rangle^{\otimes N_q} + \frac{1}{\sqrt{2}}e^{-iN_q\delta}|1\rangle \otimes |-\rangle^{\otimes N_q}$$

One measurement cycle for the signal enhancement

$$|\Psi(t_3)\rangle = \frac{1}{\sqrt{2}} e^{iN_q\delta}|0\rangle \otimes |+\rangle^{\otimes N_q} + \frac{1}{\sqrt{2}} e^{-iN_q\delta}|1\rangle \otimes |+\rangle^{\otimes N_q}$$

$$= \left(\cos N_q\delta \, |+\rangle + i \sin N_q\delta \, |-\rangle \right) \otimes |+\rangle^{\otimes N_q}$$

One measurement cycle for the signal enhancement

$$|\Psi(t_{\mathrm{f}})\rangle = \big(\cos N_{\mathrm{q}}\delta \, |0\rangle + i \sin N_{\mathrm{q}}\delta \, |1\rangle \big) \otimes |+\rangle^{\otimes N_{\mathrm{q}}}$$

$\Rightarrow$ Ancilla qubit can be excited: $P_{0\to 1} \simeq \sin^2 N_{\mathrm{q}}\delta \simeq N_{\mathrm{q}}^2 \delta^2$

The phase α is unknown in the actual search, but...
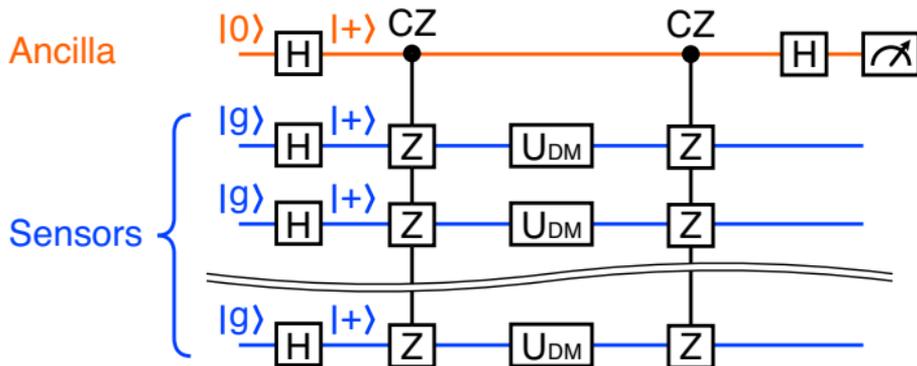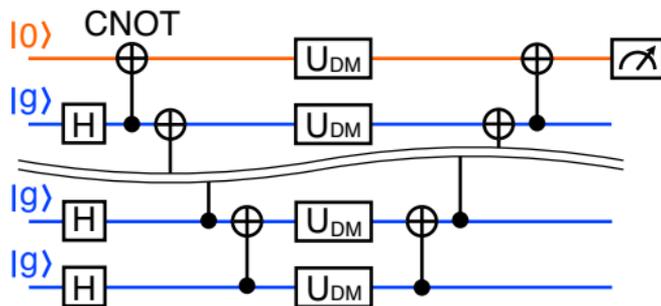

$$P_{0\to 1} \simeq N_q^2 \delta^2 \cos^2 \alpha \to \frac{1}{2} N_q^2 \delta^2$$

$\Rightarrow$ Signal rate can be of $O(N_q^2)$

$\Rightarrow$ The number of gate operation can be $O(N_q)$

Circuit only with nearest neighbor interactions

$\Rightarrow$ (# of gates) $\sim O(N_q)$

CNOT (Controlled-NOT) $= |g\rangle\langle g| \otimes 1 + |e\rangle\langle e| \otimes X$

$\Rightarrow P_{0\to1} \simeq \dfrac{1}{2}N_q^2\delta^2$

$\Rightarrow$ (# of signals) $\sim O(N_q^2)$

$\Rightarrow$ (# of errors & noises) $\sim O(N_q) \ll$ (# of signals), for $N_q \gg 1$

Backup: POVM

POVM measurement (with ancilla)

- We introduce ancilla, to which a projection measurement is possible

  $$|\text{Sensor}\rangle \rightarrow |\text{Sensor}\rangle \otimes |\text{Ancilla}\rangle$$

- We use the following unitary gate U_M:

  $$U_M |\psi\rangle \otimes |0\rangle = \sum_a M_a |\psi\rangle \otimes |a\rangle \quad \text{with} \quad \langle a|a'\rangle = \delta_{a,a'}$$

  $\{M_a\}$: operators on sensor qubits, satisfying $\sum_a M_a^\dagger M_a = \mathbb{1}$

If the ancilla is observed as $|a\rangle$

- $|\psi\rangle \rightarrow \dfrac{M_a |\psi\rangle}{\sqrt{\langle\psi|M_a^\dagger M_a|\psi\rangle}}$ with probability $\langle\psi|M_a^\dagger M_a|\psi\rangle$

- M_a and M_b (with $a \neq b$) are not necessarily orthogonal

Unitarity of U_M:

$$U_M \left|i\right\rangle \otimes \left|0\right\rangle = \sum_a M_a \left|i\right\rangle \otimes \left|a\right\rangle = \sum_{j,a} \left|j\right\rangle \otimes \left|a\right\rangle U_{(j,a);(i,0)}$$

The matrix U with $U_{(j,a);(i,0)} = \left\langle j|M_a|i\right\rangle$ can be unitary

$$\left[U^\dagger U\right]_{(i,0);(j,0)} = \sum_{k,a} \left\langle i|M_a^\dagger|k\right\rangle \left\langle k|M_a|j\right\rangle = \sum_a \left\langle i|M_a^\dagger M_a|j\right\rangle = \delta_{ij}$$

$$U = \left[\begin{array}{|c|c|c|c|c|} \hline U_{(j,a;\,i,0)} & U_{(j,a;\,i,1)} & U_{(j,a;\,i,2)} & \cdots & U_{(j,a;\,i,N)} \\ \hline \end{array} \right]$$

$\Rightarrow$ We can properly choose $U_{(j,b);(i,a\neq 0)}$

List of measurement operators for our POVM measurement

- $M_0 = |0\rangle\langle 0| + |W\rangle\langle W|$

  $$|W\rangle \equiv \frac{1}{\sqrt{N}} \sum_i X_i |0\rangle$$

- $M_i = \sqrt{\dfrac{N-1}{N}} \left(\big|i_{(1)}\big\rangle_\perp \big\langle i_{(1)}\big|_\perp + \big|i_{(2)}\big\rangle_\perp \big\langle i_{(2)}\big|_\perp \right)$

  $$\big|i_{(1)}\big\rangle_\perp \equiv \sqrt{\frac{N}{N-1}} X_i |0\rangle - \frac{1}{\sqrt{N-1}} |W\rangle$$

  $$\big|i_{(2)}\big\rangle_\perp \equiv \sqrt{\frac{N}{(N-1)(N-2)}} X_i \sum_{j\neq i} X_j |0\rangle - \sqrt{\frac{2}{N-2}} |S\rangle$$

  $$|S\rangle \equiv \frac{1}{\sqrt{2N(N-1)}} \sum_{i\neq j} X_i X_j |0\rangle$$

List of measurement operators (cont.)

- $M_S = |S\rangle\langle S|$

- $M_R = \sqrt{\mathbb{1} - M_0^\dagger M_0 - \sum_i M_i^\dagger M_i - M_S^\dagger M_S}$

  $\Leftrightarrow \mathbb{1} - M_0^\dagger M_0 - \sum_i M_i^\dagger M_i - M_S^\dagger M_S \geq 0$

We utilize POVM measurement with $\{M_0, M_i, M_S, M_R\}$

Notice: $\displaystyle\sum_{a=0,i,S,R} M_a^\dagger M_a = \mathbb{1}$

$\Rightarrow$ Recovery operation applied, depending on the measurement result

Recovery operation

- $M_0 = |0\rangle\langle 0| + |W\rangle\langle W|$: Projection on the code space

  $\Rightarrow$ No recovery operation needed: $R_0 = \mathbb{1}$

- $M_i \simeq X_i\big(|0\rangle\langle 0| + |W\rangle\langle W|\big)X_i + O(N^{-1/2})$:

  Approximately, projection on the space spanned by $\{X_i|0\rangle, X_i|W\rangle\}$

  $\Rightarrow R_i = \big(|0\rangle\langle i_{(1)}|_{\perp} + |W\rangle\langle i_{(2)}|_{\perp} + \text{h.c.}\big) + \mathbb{1}_{\perp}$

- $M_S = |S\rangle\langle S|$

  $\Rightarrow R_S = \big(|W\rangle\langle S| + \text{h.c.}\big) + \mathbb{1}_{\perp}$

- M_R: Unimportant for our analysis

  $\Leftrightarrow M_R\big\{\rho(t) + \mathcal{L}[\rho(t)]\Delta t\big\}M_R = 0$, if $\rho(t) \in$ Code space

Backup: Reduction of Noises to the Bit-Flip

Consider the model with the following Lindbladian superoperator:

$$\mathcal{L}[\rho] = -i\left[\eta\sum_{i=1}^{N}(X_i\cos\alpha - Y_i\sin\alpha), \rho\right]$$

$$+ \sum_{i=1}^{N}\left(\frac{1}{2}\gamma_X\mathcal{D}_{X_i}[\rho] + \frac{1}{2}\gamma_Z\mathcal{D}_{Z_i}[\rho] + \gamma_{\mathrm{D}}\mathcal{D}_{|0\rangle\langle1|_i}[\rho] + \cdots\right)$$

$$\mathcal{D}_L[\rho] \equiv L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$$

The present model is more complicated than the original one

$$\mathcal{L}^{(\mathrm{orig.})}[\rho] = -i\left[\eta\sum_i X_i, \rho\right] + \frac{1}{2}\gamma_X\sum_i\mathcal{D}_{X_i}[\rho]$$

$\Rightarrow$ We may use a QEC to reduce the present model to the original one

We consider a logical qubit consisting of three physical qubits:

$$|0\rangle_{3L} \equiv \frac{1}{\sqrt{2}}(|+\rangle_{3L} + |-\rangle_{3L}), \quad |1\rangle_{3L} \equiv \frac{1}{\sqrt{2}}(|+\rangle_{3L} - |-\rangle_{3L})$$

$$|\pm\rangle_{3L} \equiv |\pm\rangle^{\otimes 3} \equiv \left[\frac{1}{\sqrt{2}}\big(|0\rangle \pm |1\rangle\big) \right]^{\otimes 3} \equiv |\pm\rangle |\pm\rangle |\pm\rangle$$

Code space of the logical qubit: $\mathcal{H}_{3L} \equiv \{|0\rangle_{3L}, |1\rangle_{3L}\}$

- Bit-flip noise (X_i) keeps the state inside $\mathcal{H}_{3L}$

  $$X_i |0\rangle_{3L} = |1\rangle_{3L} \quad \Rightarrow \quad X_i |\pm\rangle_{3L} = \pm |\pm\rangle_{3L}$$

- Other noises, as well as Y_i term in H, kick the state out from $\mathcal{H}_{3L}$

  E.g.: $Z_1 |+\rangle |+\rangle |+\rangle = |-\rangle |+\rangle |+\rangle$

- If $|\psi\rangle \in \mathcal{H}_{3L}$: $X_1 X_2 |\psi\rangle = X_2 X_3 |\psi\rangle = X_3 X_1 |\psi\rangle = |\psi\rangle$

POVM measurement with the following measurement operators

- $P_L = \dfrac{1}{4}(1 + X_1X_2 + X_2X_3 + X_3X_1)$

  Projection operator to the code space

- $P_1 = \dfrac{1}{4}(1 - X_1X_2 + X_2X_3 - X_3X_1)$

  Error in the first physical qubit: $\{\left|-\right\rangle\left|+\right\rangle\left|+\right\rangle , \ \left|+\right\rangle\left|-\right\rangle\left|-\right\rangle\}$

- We also introduce P_2 and P_3

  $P_L + P_1 + P_2 + P_3 = 1$

If the measurement result is $P_{1,2,3}$, we apply $Z_{1,2,3}$

  $Z_1\left|\mp\right\rangle\left|\pm\right\rangle\left|\pm\right\rangle = \left|\pm\right\rangle\left|\pm\right\rangle\left|\pm\right\rangle$

Protocol for three-qubit system, with density matrix $\tilde{\rho}$:

$$\tilde{\rho}(t + \Delta t_3) \simeq \sum_{I=L,1,2,3} R_I P_I \Big\{ \tilde{\rho}(t) + \mathcal{L}[\tilde{\rho}(t)] \Delta t_3 \Big\} P_I R_I$$
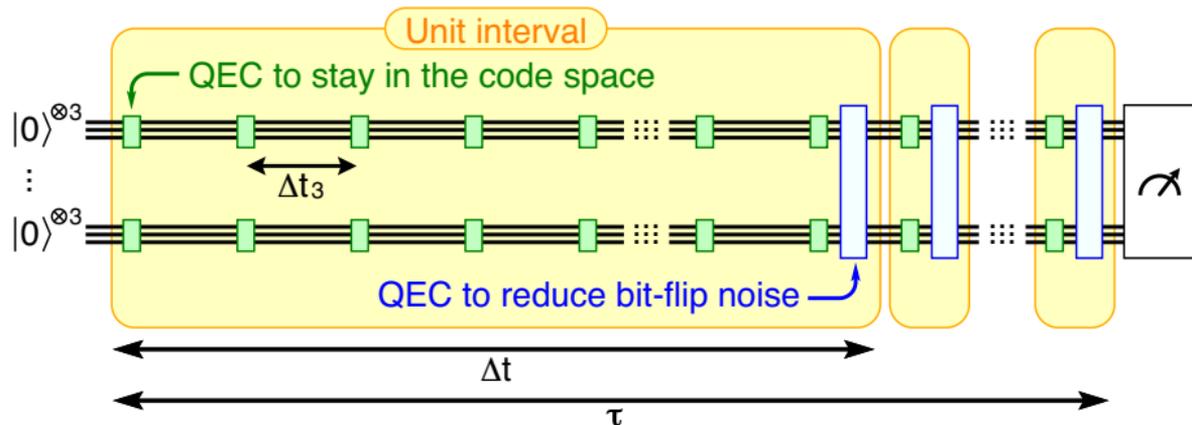
$$R_L = \mathbb{1}, \quad R_{1,2,3} = Z_{1,2,3}$$

After some calculation, we can find:

$$\dot{\tilde{\rho}} \simeq -i \left[3\eta \cos\alpha X_{3L}, \tilde{\rho} \right] + \frac{3}{2} \left(\gamma_X + \gamma_Z + \gamma_{\mathrm{D}} \right) \mathcal{D}_{X_{3L}}[\tilde{\rho}]$$

$$X_{3L} = |1\rangle_{3L} \langle 0|_{3L} + |0\rangle_{3L} \langle 1|_{3L}$$

$\Rightarrow$ The system is equivalent to the original one

Schematically...

Effective number of the sensor qubits decreases by a factor of a few

$\Leftrightarrow$ Scaling behavior of $\delta\eta^2 \propto N^{-1}T^{1/2}$ does not change