

Revisiting Instantons Physics in Broken Phase

Takafumi Aoki

ICRR, the University of Tokyo

Ongoing work with M. Ibe and S. Shirai.

(Gauge-Theory) Instanton

Topological soliton in non-abelian (e.g. $SU(2)$) gauge theories on 4D Euclidean space.

- Classified by **winding number** $w = 0, \pm 1, \dots$
- Local minima of the action with

$$S_{\text{Euclidean}} = \frac{8\pi^2}{g^2} |w|.$$

- Winding Number:

$$\begin{aligned} w &= (\text{constant}) \times \int d^4x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} \\ &= 0, \pm 1, \pm 2, \dots \quad (\text{for } S_{\text{Euclidean}} < \infty). \end{aligned}$$

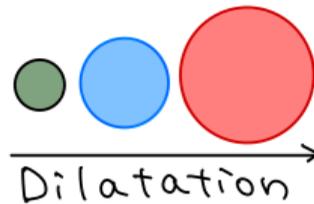
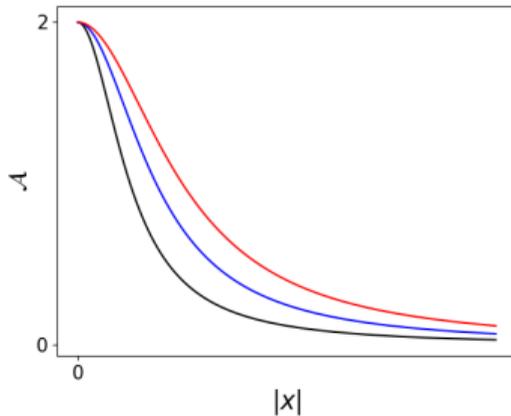
Instanton

Dilatation Zero Mode

The action is classically independent of the **size** ρ .

i.e. dilatation is a **zero mode**, which do not cost the action.

$$\text{Instanton in Singular gauge: } A_{\mu}^a = \bar{\eta}_{\mu\nu}^a \frac{x_{\nu}}{x^2} \mathcal{A}(|x|), \quad \mathcal{A}(|x|) = \frac{2\rho^2}{x^2 + \rho^2}.$$



Instanton Effects in Path Integral

Semiclassical Approximation

Note: \hbar -expansion $\simeq g^2$ -expansion \rightarrow breakdown in $g \gg 1$.

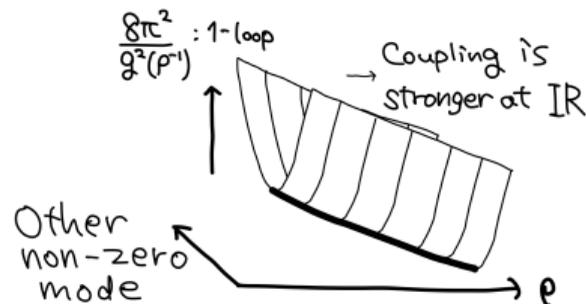
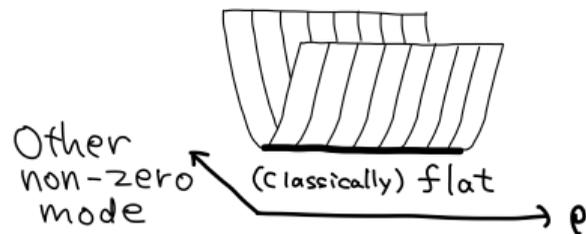
\hbar -expansion of $Z = \int \mathcal{D}A \exp(-S[A]/\hbar)$:

0. $S \simeq$ Classical minima.
1. $S \simeq$ Classical minima
+ [Field fluctuations around the minima] 2 .

Integrating out fluctuations [e.g. G. 't Hooft (1976)],

bare $g \rightarrow$ renormalized $g(\rho^{-1})$.

i.e. ρ is **not** a flat modulus in quantum level.



Instanton in Broken Phase

What If Gauge Symmetry is Spontaneously Broken?

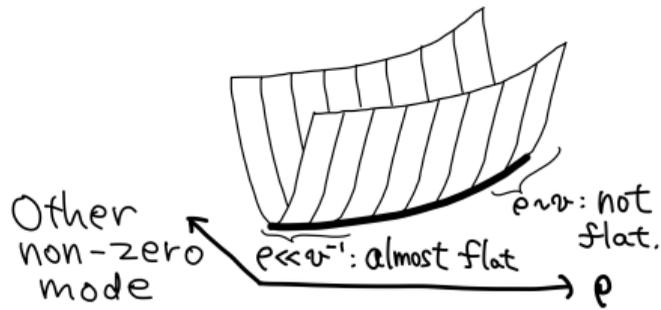
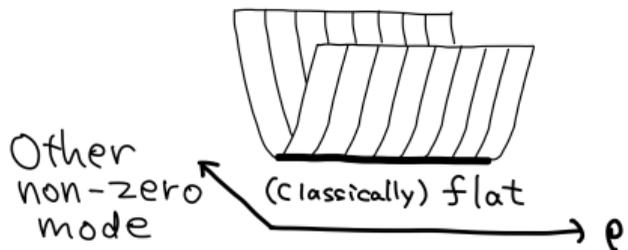
[I. Affleck (1980)]

$$\mathcal{L}_{\text{YMH}} = \frac{1}{2g^2} \text{Tr} F_{\mu\nu} F_{\mu\nu} + |D_\mu H|^2 + \frac{\lambda}{4} (H^\dagger H - v^2)^2. \quad (\text{SU}(2) \text{ gauge theory with SU}(2) \text{ doublet } H.)$$

- Instanton action increases as the size increases.

Why?: $|H| \sim v$ at $|x| \gtrsim v^{-1} \rightarrow |D_\mu H|^2$ picks up the instanton tail at $|x| \gtrsim v^{-1}$.

- There is **no strict minimum** with non-trivial winding number..
- However, small instantons ($\rho \ll v^{-1}$) “do not see” symmetry breaking, effectively.



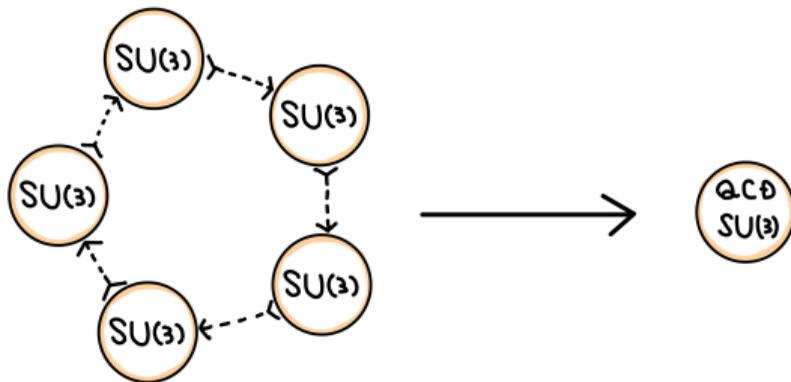
Application of Instanton in Broken Phase

Axion Mass Enhancement?: [P. Agrawal & K. Howe (2017)]

Product group toy model: Gauge $[SU(3)]^{n_s}$ symmetry with n_s axions.

$$\mathcal{L} = \sum_{i=1}^{n_s} \text{tr} \left[-\frac{1}{2} F_i F_i + \left(\theta_i + \frac{a_i}{f_i} \right) \frac{g_i^2}{16\pi^2} F_i \tilde{F}_i \right] + (\text{scalars for symmetry breaking})$$

Bi-fundamental scalars break gauge symmetry: $[SU(3)]^{n_s} \rightarrow SU(3)_{\text{QCD}}$



Application of Instanton in Broken Phase

Small instanton effects from each $SU(3)_i$: $\Delta V(a_i) \propto -v^4 \exp\left(-\frac{8\pi^2}{g_i^2(v)}\right) \cos\left(\frac{a_i}{f_i}\right)$.
(v : breaking scale)

Axion mass enhancement in a toy model. [P. Agrawal & K. Howe (2017), C. Csaki et al (2020)]

Note:

There is **a difficulty** in enhancing the axion mass by small instantons, while “naturally” solving the strong CP problem [TA, M.Ibe, S.Shirai and K.Watanabe (2024)].

Constrained Instanton

- Instantons with $\rho \lesssim v^{-1}$ have non-negligible effects.
- To compute their effects, those (non-minimal) configurations should be extracted.

Minimization with constraint (of size) helps us picking up $\rho \neq 0$ configurations.

Solution at $|x| \ll v^{-1}$ in singular gauge:

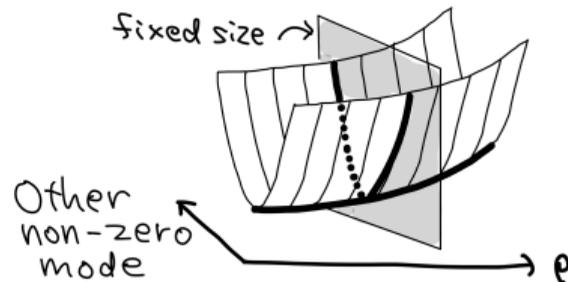
[I. Affleck (1980)]

$$A_{\mu}^a = \bar{\eta}_{\mu\nu}^a \frac{x_{\nu}}{x^2} \left[\frac{2\rho^2}{x^2 + \rho^2} + O(\rho^2 v^2) \right],$$

$$H = \begin{pmatrix} 0 \\ v \end{pmatrix} \left[\sqrt{\frac{x^2}{x^2 + \rho^2}} + O(\rho^2 v^2) \right].$$

$$S_{\text{YMH}} = \frac{8\pi^2}{g^2} + 2\pi^2 \rho^2 v^2 + O(\rho^4 v^4).$$

Overlapping between instanton tail and $|H| \sim v$, at $|x| \gtrsim v^{-1}$.



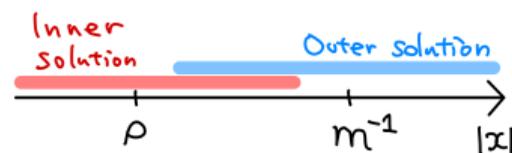
Constrained Instanton: Explicit Construction

Profile Functions \mathcal{A} and \mathcal{H} :

$$A_{\mu}^a = \bar{\eta}_{\mu\nu}^a \frac{x_{\nu}}{x^2} \mathcal{A}(|x|), \quad H = \begin{pmatrix} 0 \\ \nu \end{pmatrix} \mathcal{H}(|x|)$$

Recipe for analytically expanded solution of \mathcal{A} (similar for \mathcal{H}): (Note: $m = gv/\sqrt{2}$)

1. Rewrite EOM at **inner** ($|x| \ll m^{-1}$) and **outer** ($|x| \gg \rho$) regions in terms of dimensionless variables, $|x|/\rho$ and $m|x|$, respectively.
2. Solve at **inner/outer** regions, order by order of perturbation in $\rho m \ll 1$.
3. **Matching inner/outer** solutions at $\rho \ll |x| \ll m^{-1}$.



Difficulty in Constructing Consistent Solution?

On matching **inner/outer** solutions,

- M. Nielsen and N. K. Nielsen (1999) indicated that **matching fails** at $(\rho m)^2$ -order corrections, for **almost all the conventionally-used choices** of constraint.

If this mismatch is unavoidable,

- It is a **conceptual concern** for the constrained instanton procedure.
- Semi-classical computation of higher precision becomes difficult.

Conclusion in our work:

- The **inner/outer** solutions **match consistently almost independently to choices** of the constraint, through careful treatment of **expansion parameters**.
- We derived explicit $(\rho m)^2$ corrections and a framework covering higher orders.

Summary

- **Constrained instantons** is an instanton-like configuration and is the minimum of action on the constrained surface of a fixed size.
- Nielsen and Nielsen (1999) indicated that ρv -expanded consistent solutions do not exist for almost every constraint, due to **inner/outer mismatch**.
- We clarified that the **matching works well almost independently to the choice of the constraint**, if we treat the expansion parameters properly.
- We derived explicit $(\rho m)^2$ corrections and a framework covering higher orders.

BACKUP

Constrained Instanton: Procedure

Minimization with Constraint in Classical Level: **Lagrange Multiplier Method**

$$S_{\text{total}}[A, H; \sigma] = S_{\text{YMH}}[A, H] + \sigma(S_{\text{constraint}} - f(\rho)),$$

$$S_{\text{constraint}} = \int d^4x \mathcal{O}_{\text{constraint}}.$$

Procedure:

- **Lagrange Multiplier Method:**

For a fixed value of $f(\rho)$, find a stationally point of $S_{\text{total}}[A, H; \sigma]$.

Constraint term:

- An example: $\mathcal{O}_{\text{constraint}} = \left(\frac{1}{2} \text{Tr} F \tilde{F}\right)^2$, $\int d^4x \mathcal{O}_{\text{constraint}} = \frac{384\pi^2}{7} \rho^{-4}$

Constrained Instanton: Procedure

Procedure of Constraint in Quantum Level

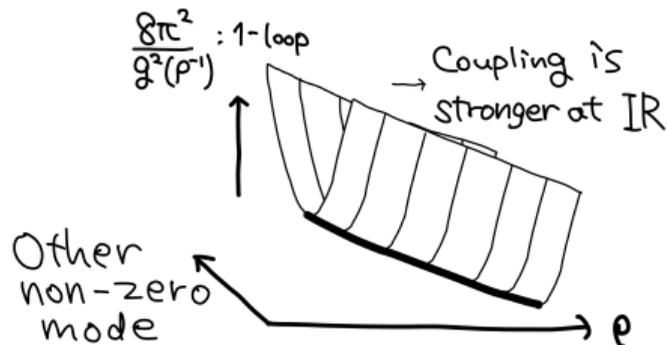
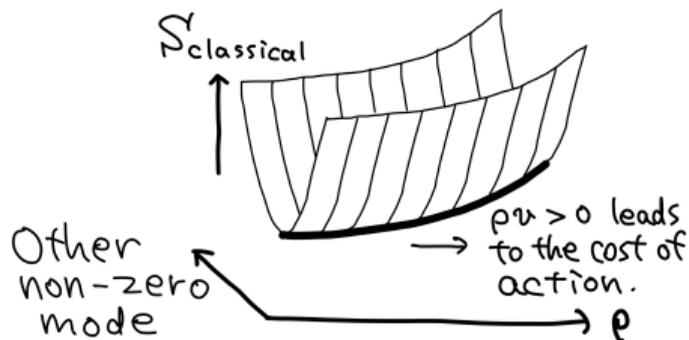
Constrained instanton procedure = “**insertion of 1**” to path integral.

- (Entire configuration space) = \sum_{ρ} (slice of configurations with the fixed size ρ)

Two size-dependent effects compete (in asymptotically free case),

Classical: Action increases as ρv increases.

Quantum effect: Effective coupling $g(\rho^{-1})$ depends on the instanton size ρ .



Constrained Instanton: Procedure

Procedure of Constraint in Quantum Level [Gervais, Neveu and Virasoro (1977)]

Constrained instanton procedure can be understood as “**insertion of 1**” to Z .

$$\begin{aligned}
 Z &= \int \mathcal{D}\Phi \, df \, \delta(f - S_{\text{constraint}}) \exp(-S_{\text{YMH}}) \\
 &= \int \mathcal{D}\Phi \, df \, \delta(f - S_{\text{constraint}}) \exp\left(\overbrace{-S_{\text{YMH}} - \sigma(S_{\text{constraint}} - f(\rho))}^{-S_{\text{total}}}\right) \quad (\text{Zero is just added.}) \\
 &= \dots = \int \overbrace{\mathcal{D}\varphi}^{\phi =: \varphi + (\text{stationary point})} df \frac{d\mu}{2\pi} \exp\left(\overbrace{-S_{\text{total}}^{\text{classical}}(f)}^{\text{stationary-point value}} - i\mu \int \frac{\delta S_{\text{constraint}}}{\delta\Phi} \varphi - \frac{1}{2} \int \varphi \frac{\delta^2 S_{\text{total}}}{\delta\Phi^2} \varphi + \mathcal{O}(\varphi^3)\right) \\
 &= \dots = \int \mathcal{D}\tilde{\varphi} \, df \exp\left(-S_{\text{total}}^{\text{classical}}(f) - \frac{1}{2} \int \tilde{\varphi} \frac{\delta^2 S_{\text{total}}}{\delta\Phi^2} \tilde{\varphi} + \mathcal{O}(\tilde{\varphi}^3)\right) \int \frac{d\mu}{2\pi} \exp\left(-\frac{\mu^2}{2} \left(-\frac{\partial f}{\partial\sigma}\right)\right) \\
 &\simeq \int df \left(-\frac{\partial f}{\partial\sigma}\right)^{-1/2} \exp(-S_{\text{total}}^{\text{classical}}(f)) \int \mathcal{D}\tilde{\varphi} \exp\left(-\frac{1}{2} \int \tilde{\varphi} \frac{\delta^2 S_{\text{total}}}{\delta\Phi^2} \tilde{\varphi}\right)
 \end{aligned}$$

Constrained Instanton: Explicit Construction

Order-by-order solutions:

1. Leading-order (LO) in ρm :

$$\mathcal{A}(|x|) = \begin{cases} \frac{2\rho^2}{x^2 + \rho^2} & |x| \ll m^{-1} \\ (\text{const}) \times K_2(m|x|) & |x| \gg \rho \end{cases} \quad m = gv / \sqrt{2}.$$

$$\left(\text{“Massive free field” solution: } K_2(m|x|) \sim \begin{cases} \frac{2}{m^2 x^2} & |x| \ll m^{-1} \\ \sqrt{\frac{\pi}{2mx}} e^{-m|x|} & |x| \gg m^{-1} \end{cases} \right)$$

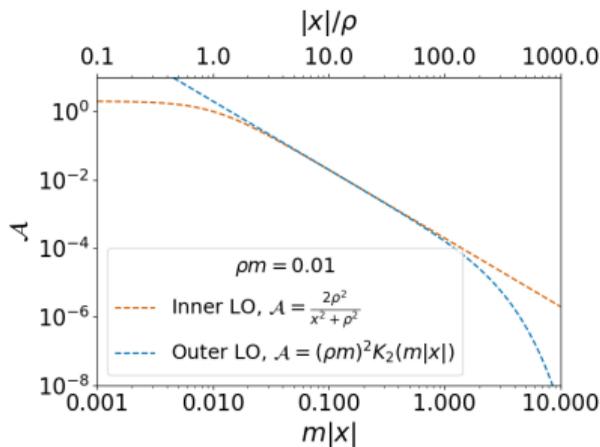
2. Higher-order (in ρm) **inner/outer** solutions can be obtained iteratively.

Constrained Instanton: Explicit Construction

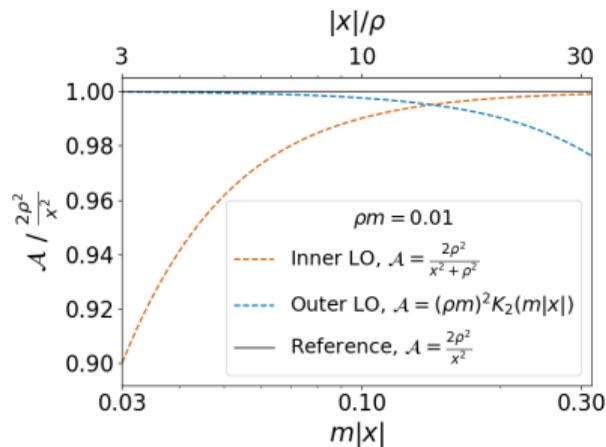
3. Matching of LO (in ρm) inner/outer solutions

$$\mathcal{A}(|x|) = \begin{cases} \frac{2\rho^2}{x^2 + \rho^2} & |x| \ll m^{-1} \\ c_0 \times K_2(m|x|) & |x| \gg \rho \end{cases} \quad m = gv / \sqrt{2}.$$

Matching: $c_0 = (\rho m)^2$



$\rho \ll |x| \ll m^{-1}$



Do inner/outer solutions match consistently at higher-order, too?

Our Work: Matching

More closely, the right thing to do is the **double expansion** in $|x|/\rho$ in addition to ρm .

Key consequence: refinement of **outer** ($|x| \gg \rho$) solution.

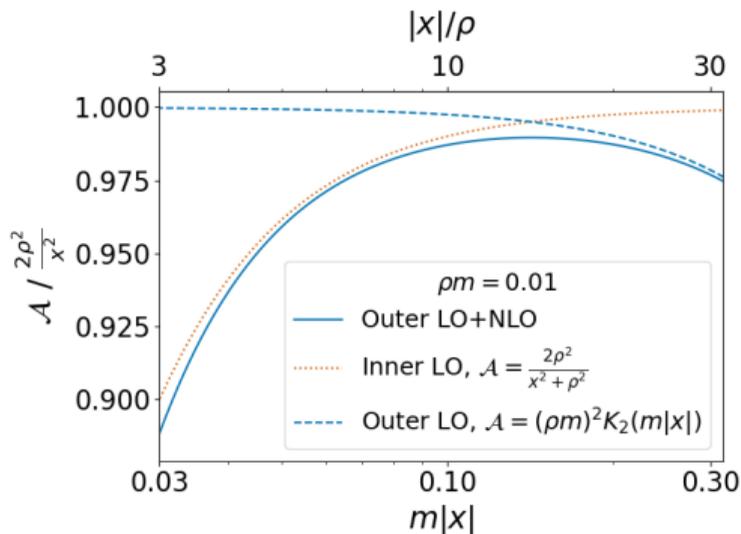
- Naive intuition: “ $\mathcal{A} \propto K_2(m|x|)$ (massive **free** field), at far away from $x = 0$.”
- This holds at $|x| \gg m^{-1}$, while **fails at the matching region** $\rho \ll |x| \ll m^{-1}$.

More precisely,

- $\mathcal{A}(|x|) \propto K_2(m|x|)$ **uniformly to all orders** (of ρm) for **far-outer region**, $|x| \gg v^{-1}$.
- Whereas for $\rho \ll |x| \ll v^{-1}$, the behavior is **order dependent**.

Our Work: Matching

Example: Outer ($x \gg \rho$) solution at LO (- - -) and LO+NLO (—)



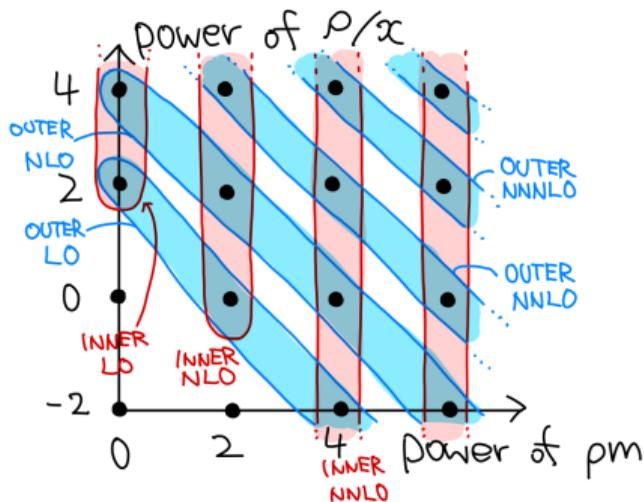
1. NLO **outer** solution is merely an $O(\rho^2 m^2) = O(10^{-4})$ correction to LO **outer** solution at $|x| \gtrsim m^{-1}$.
2. However, NLO correction is **essential at matching region** $\rho \ll |x| \ll m^{-1}$.
3. NLO **outer** solution is needed to match to **inner** solutions **already at inner LO** (.....).

The difference in order of ρm between **inner** and **outer** solutions is crucial in matching.

→ This is handled by the double expansion in $|x|/\rho$ in addition to ρm .

Our Work: Matching

Matching can be done through **double expansion** in ρm and ρ/x .



Example: **LO (of ρm) outer solution**

Outer LO

$$\mathcal{A}(|x|) \sim (\rho m)^2 K_2(mx)$$

$$= (\rho m)^2 \left[\frac{2}{m^2 x^2} - \frac{1}{2} + O(m^2 x^2) \right] \quad (x \ll m^{-1})$$

$$= \underbrace{\frac{2\rho^2}{x^2}}_{\text{Inner LO}} - \underbrace{\frac{1}{2}(\rho m)^2}_{\text{Inner NLO}} + \underbrace{O(\rho^4 m^4)}_{\text{Inner NNLO}}$$

Highlight: Behavior of the outer solution must be treated carefully.

- $\mathcal{A}(|x|) \propto K_2(mx)$ **uniformly to all orders** (of ρm) for **far-outer region**, $x \gg v^{-1}$.
- Whereas for $\rho \ll x \ll v^{-1}$, the behavior is **order dependent**.

Our Work: Matching

Explicitly, double expansions with respect to ρm and ρ/x match as

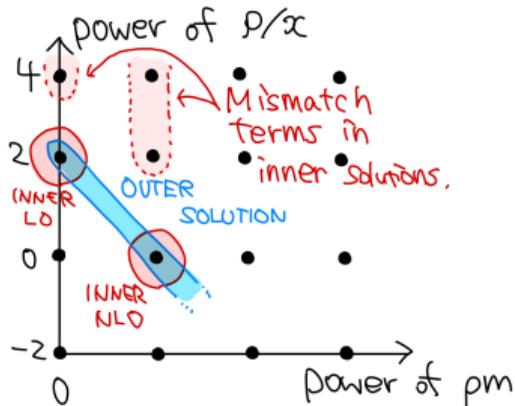
$$\begin{aligned} & \mathcal{A}_{\text{inner}}^{(\text{LO})}(\rho^2/x^2) + \mathcal{A}_{\text{inner}}^{(\text{NLO})}(\rho^2/x^2) \\ &= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m_A)^2 \left[-\left(c_2 - \frac{1}{12}\right) \hat{r}^2 - 6c_2 + \left(12c_2 \log \frac{\rho^2}{x^2} - 36c_2 - \frac{1}{12}\right) \frac{\rho^2}{x^2} + \mathcal{O}\left(\frac{\rho^4}{x^4}\right) \right]. \end{aligned}$$

$$\begin{aligned} & \mathcal{A}_{\text{outer}}^{(\text{LO})}(m^2 x^2) + \mathcal{A}_{\text{outer}}^{(\text{NLO})}(m^2 x^2) \\ &= \frac{2\rho^2}{x^2} - \frac{2\rho^4}{x^4} + (\rho m)^2 \left[-\frac{1}{2} + \frac{\rho^2}{x^2} \ln \frac{\rho^2}{x^2} + c_{\text{out}} \frac{\rho^2}{x^2} \right] + \mathcal{O}(\rho^4 m^4). \end{aligned}$$

- Matching is possible by adjusting c_2 and c_{out} . For example, $c_1 = 1/12$.
- Other constants are consistently determined by, for example, $S < \infty$ at $r = 0$

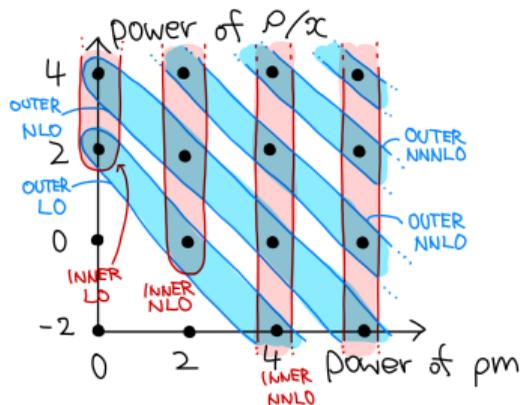
Our Work: Matching

Previous Work



- Diverse behavior of outer solution at the matching region is missing.
- $O_{\text{constraint}}$ **is severely restricted** to avoid the appearing mismatch.

Our Work



- Double expansion at matching region.
- Matching works well **independently to the choice of $O_{\text{constraint}}$** .