2 March 2026
KEK Cosmo

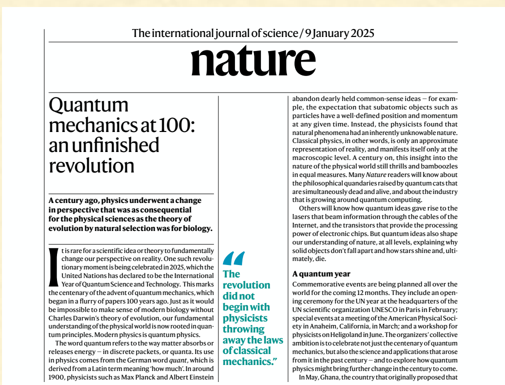
Bell's Inequality and Some Ramifications

— 1. Basics: EPR & Bell —

Izumi Tsutsui

IPNS , KEK
CST, Nihon Univ.

2025-2026: 100 Years from the birth of Quantum Mechanics


The international journal of science / 9 January 2025

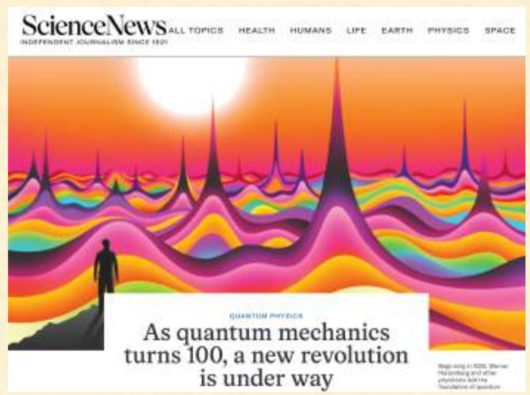
nature

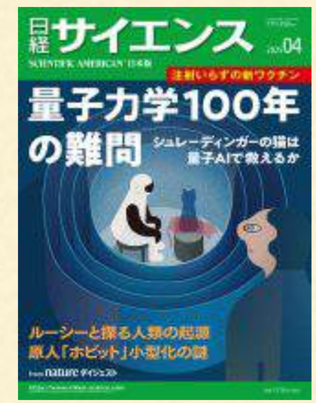
Quantum mechanics at 100: an unfinished revolution

A century ago, physics underwent a change in perspective that was as consequential for the physical sciences as the theory of evolution by natural selection was for biology.

"The revolution did not begin with physicists throwing away the laws of classical mechanics."

A quantum year





Road toward the birth of QM

1900	Planck	Energy quanta	←	BB radiation
1905	Einstein	Light quanta	←	PE effect
1913	Bohr	Model of H-atom	←	Atomic spectrum
1915	Sommerfeld, Wilson, Ishiwara	Quantization cond.	→	Canonical Commutation Relation
1916	Einstein	Induced emission	→	Transition probability
1922	Stern・Gerlach	Space quantization	→	'Spin'
1923	de Broglie	Matter wave	→	Wave nature of matter
1924	Bohr・Krammers・Slater	Dispersion and radiation	→	Dispersion and atom
1925	Heisenberg・Born・Jordan	Matrix Mechanics	}	**Birth of QM**
1926	Schrödinger	Wave mechanics		
1926	Born	Probability interpretation		

Papers around the time of QM birth

1923	9.10	de Broglie：matter wave
1924	7.2	Bose：photon gas
	7.10	Einstein：ideal gas & BE statistics I
	11.25	de Broglie：matter wave（thesis）
1925	1.8	Einstein：ideal gas & BE statistics II
	1.26	Pauli：exclusion principle
	7.29	Heisenberg：quantum transition and matrix
	9.27	Born-Jordan：Matrix Mechanics I
	11.7	Dirac：Poisson bracket and quantization
	11.16	Born-Heisenberg-Jordan：Matrix Mechanics II
1926	1.27	Schrödinger：Wave Mechanics I
	2.23	Schrödinger：Wave Mechanics II
	3.16	Schrödinger：equivalence between MM & WM
	5.10	Schrödinger：Wave Mechanics III
	6.21	Schrödinger：Wave Mechanics IV
	7.21	Born：probability interpretation of wave function
	8.26	Dirac：FD statistics
	12.2	Dirac：general transformation
	12.18	Jordan：transformation theory
1927	3.23	Heisenberg：Uncertainty Principle

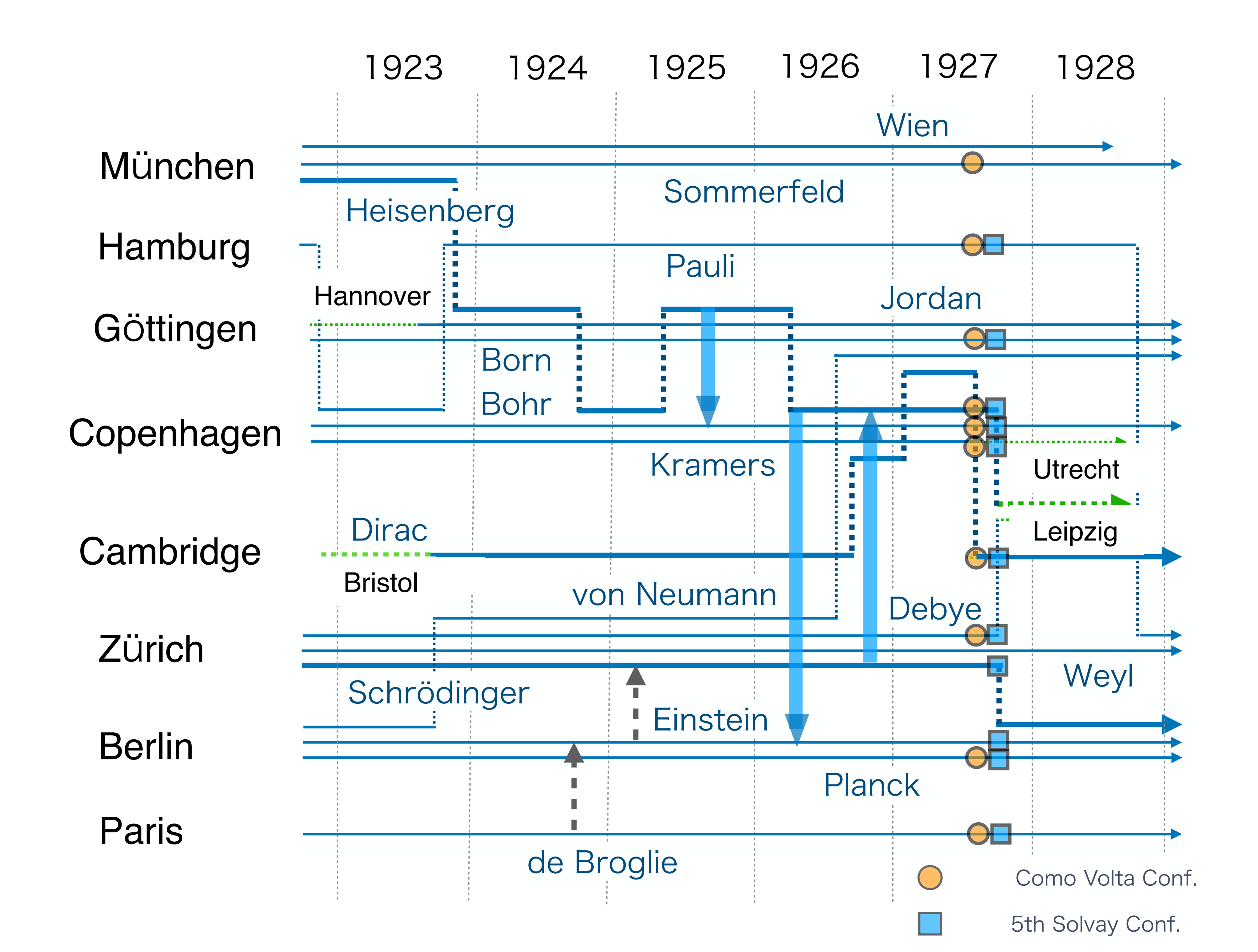
Werner Heisenberg
(1901 – 1976)

Max Born
(1882 – 1970)

Pascual Jordan
(1902 – 1980)

Erwin Schrödinger
(1887 – 1961)

Exchanges around QM birth



Niels Bohr
(1885 – 1962)

Louis de Broglie
(1892 – 1987)

Wolfgang Pauli
(1900 – 1958)

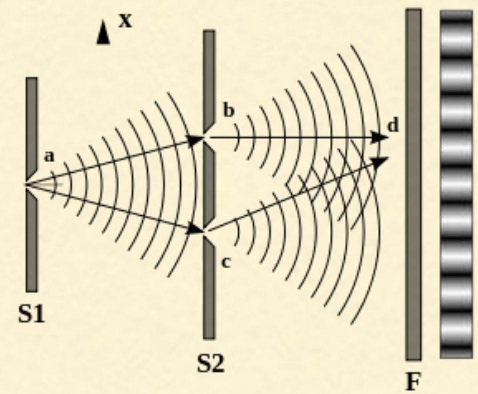
Paul Dirac
(1902 – 1982)

I. EPR: is QM incomplete?

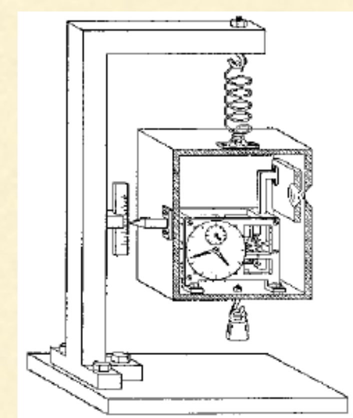
Einstein-Bohr debate

Is the uncertainty principle valid?

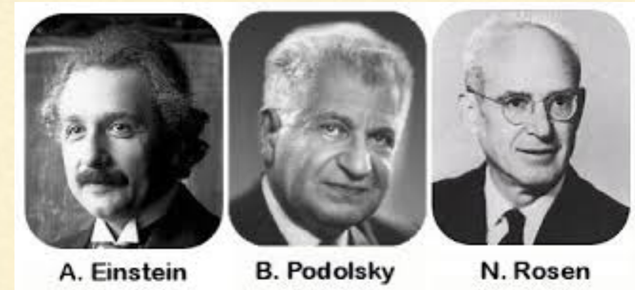
5th Solvay conference
(1927)



6th Solvay conference
(1930)



Einstein-Podolsky-Rosen (EPR) paper (1935)



DESCRIPTION OF PHYSICAL REALITY 777

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

This investigation was carried out under the supervision of Professor G. Breit, and I wish to thank him for the invaluable advice and assistance so freely given. I also take this opportunity to acknowledge the award of a Fellowship by the Royal Society of Canada, and to thank the University of Wisconsin and the Department of Physics for the privilege of working here.

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

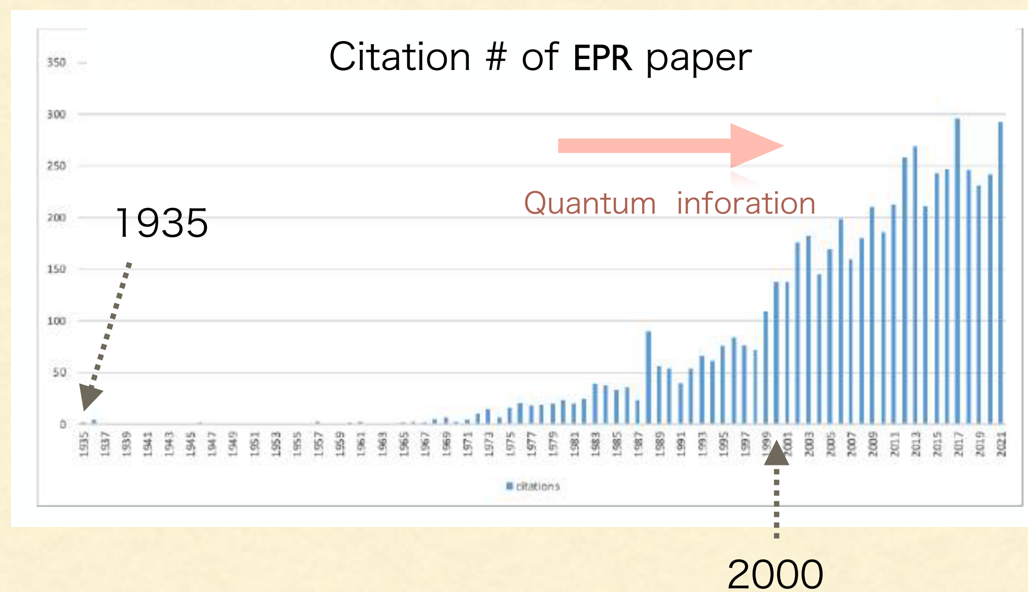
Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

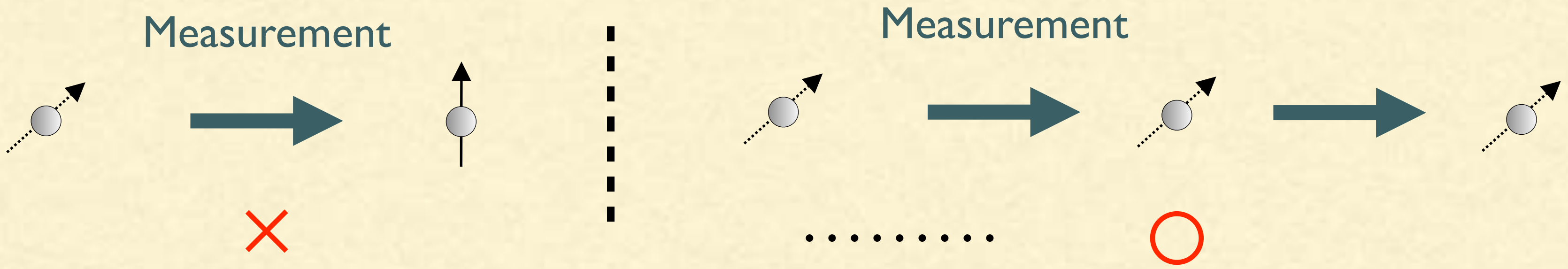
QM must be incomplete because it admits entangled states, which appear to invoke unreasonable 'spooky action at a distance'.



Premises of EPR

- elements of physical reality

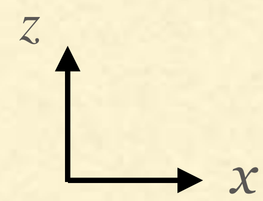
If we can predict the value of a physical quantity G with certainty (100%) without disturbing the system, then there exists an element of physical reality corresponding to G.



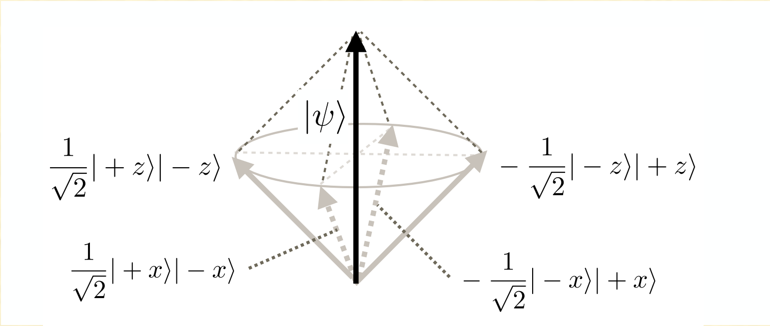
- complete theory

Every element of a physical reality has a counterpart in the theory.

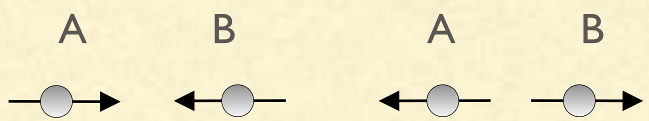
Entangled state (ex.) singlet spin state of 2 electrons A and B



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right)$$



which can also be written as



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+x\rangle|-x\rangle - |-x\rangle|+x\rangle\right)$$

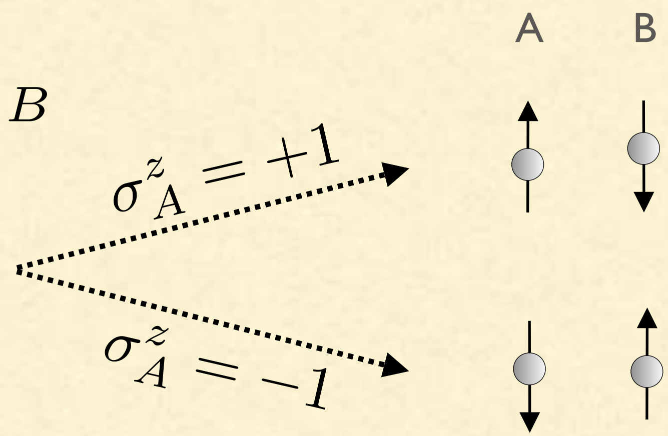
Projection Measurement (von Neumann's projection postulate)

Physical observable		Measurement		
$G\|g_i\rangle = g_i\|g_i\rangle$, $i = 1, 2, 3, \ldots$	$\|\psi\rangle$	$\xrightarrow[G = g_i]{}$	$\|g_i\rangle$	Collapse of quantum state (wave function)

case 1)

measurement for σ_A^z of A: $G^z = \sigma_A^z \otimes 1_B$

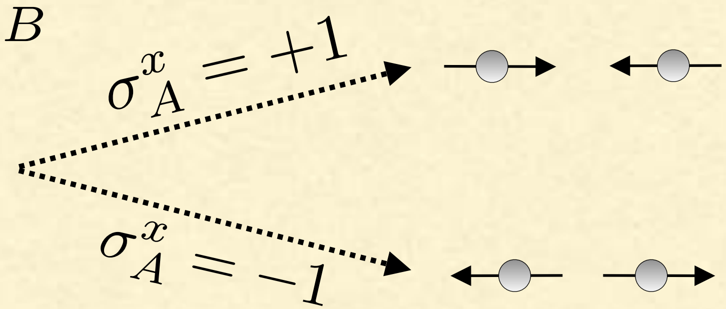
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right)$$



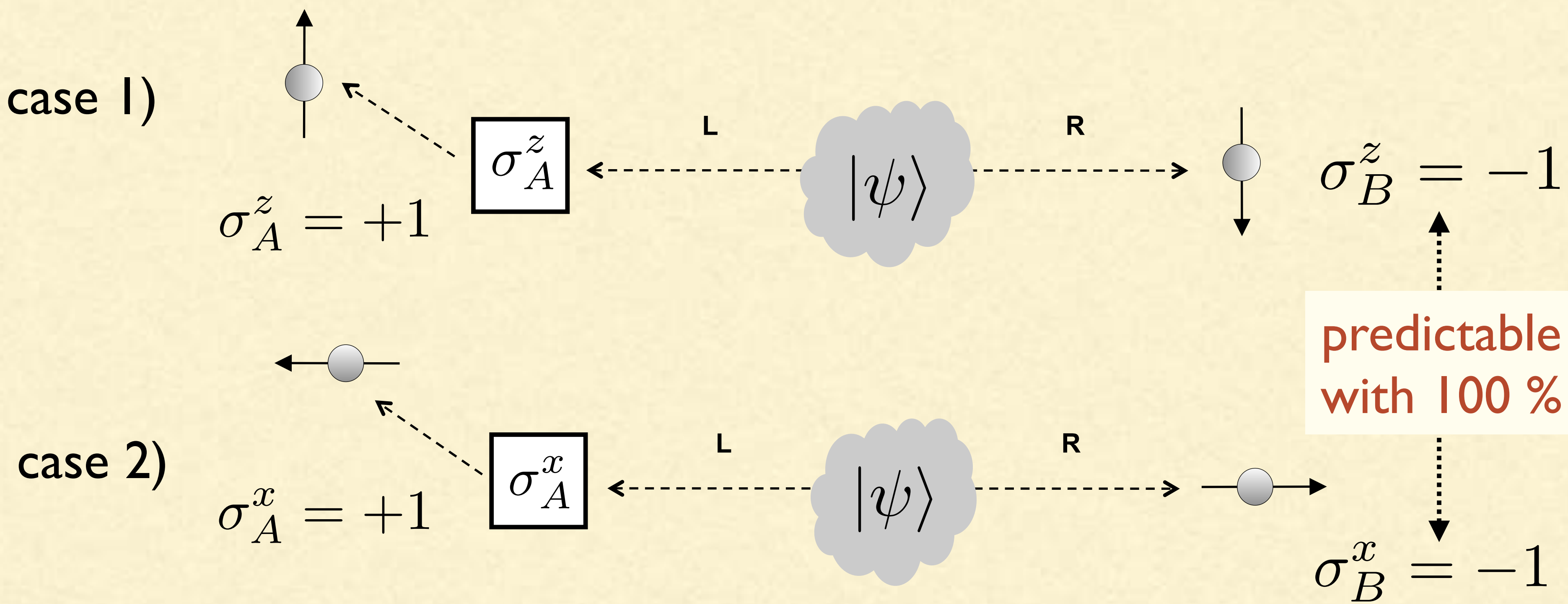
case 2)

measurement for σ_A^x of A: $G^x = \sigma_A^x \otimes 1_B$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+x\rangle|-x\rangle - |-x\rangle|+x\rangle\right)$$



After putting the particle A and B at a remote distance, if, e.g., we find



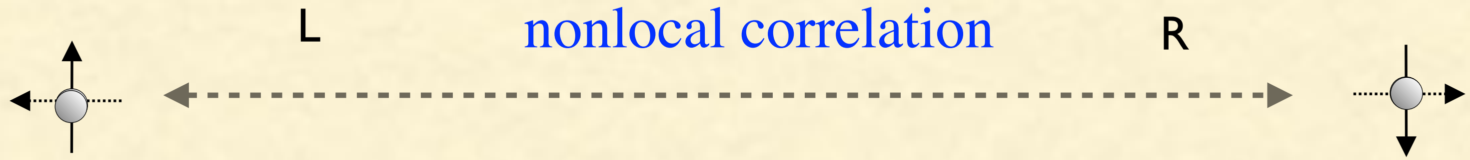
measurement of A cannot disturb B

one can choose either case 1 and 2 freely

} both σ_B^z and σ_B^x are elements of physical reality

→ QM, in which σ_B^z and σ_B^x cannot be assigned a definite value simultaneously, must be incomplete!

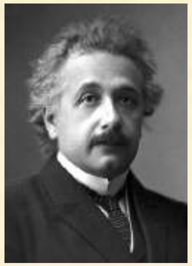
Or what?



Are particles exchanging information of measurement？

The spin values of the two particles have not been determined until the actual measurement is carried out, that is, they are dependent on the choice (context) of measurement.

→ nonlocal correlation exist!



The spin values of the two particles must have been predetermined before the measurement, and there must be some (unknown/hidden) element which determines those values.

→ hidden variables exist!

II. Bell's inequality and nonlocal realism

Does the hidden variable, desired by Einstein, really exist？

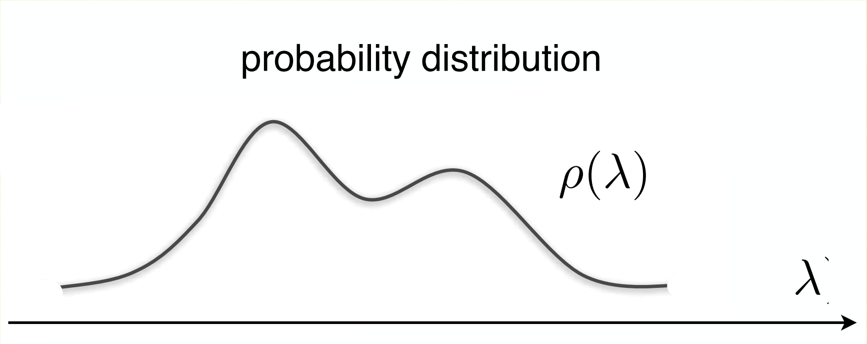
Hidden Variable Theory (HVT)

- hidden variable： $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$

… determines the values of physical quantities obtained in measurement

- values of physical observable： $A = A(\lambda)$ $\longrightarrow$ reproduce measurement outcomes

- distribution of HV： $\rho(\lambda) \geq 0$ s.t. $\int d\lambda\, \rho(\lambda) = 1$

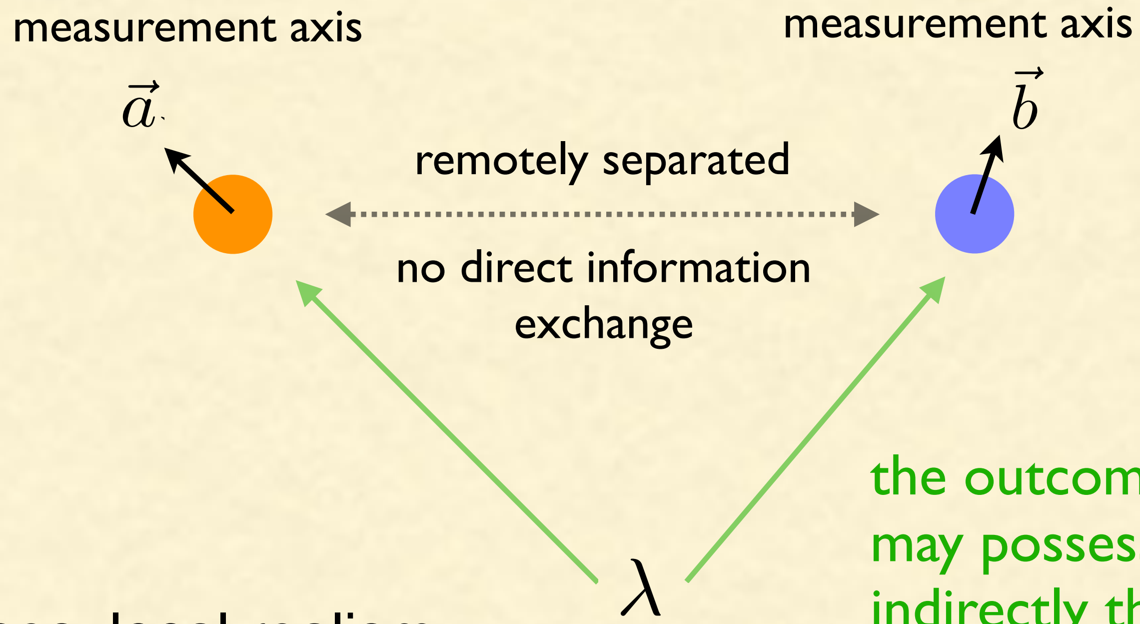


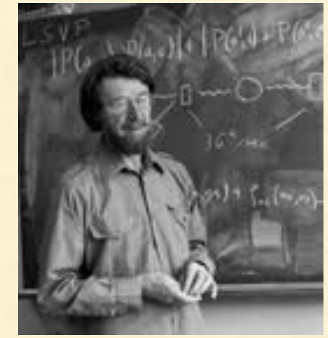
QM vs. HVT

	deterministic factor	measured value of A	state	expectation value $\langle A \rangle$
QM	none	one of the eigenvalues $a_1, a_2, \ldots$ of the self-adjoint operator $\hat{A}$	ψ	$\langle \psi \vert \hat{A} \vert \psi \rangle$
HVT	λ	$A(\lambda)$	$\rho(\lambda) = \rho_\psi(\lambda)$	$\int d\lambda \, \rho_\psi(\lambda) \, A(\lambda)$

Given a state, λ may take different values each time measurement is performed with the distribution $\rho(\lambda)$.

Bell's inequality (1964)





J. Bell
(1928 - 1990)

the outcomes of measurement may possess correlation generated indirectly through λ

Premises: local realism

reality

$$A = A(\vec{a}, \vec{b}, B, \lambda)$$

$$B = B(\vec{a}, \vec{b}, A, \lambda)$$

locality

$$A(\vec{a}, \vec{b}, B, \lambda) = A(\vec{a}, \lambda)$$

$$B(\vec{a}, \vec{b}, B, \lambda) = B(\vec{b}, \lambda)$$

Correlation in HVT   dichotomic (2-valued) case

measured values $A(\vec{a}, \lambda) = \pm 1$ $B(\vec{b}, \lambda) = \pm 1$

correlation $\langle A(\vec{a}) B(\vec{b}) \rangle = \int d\lambda \, \rho_\psi(\lambda) \, A(\vec{a}, \lambda) B(\vec{b}, \lambda)$

Since each measurement will have different λ, we write

$$A_1(a) = A(a, \lambda_1), \quad A_2(a) = A(a, \lambda_2), \quad A_3(a) = A(a, \lambda_3), \ldots$$

$$B_1(b) = B(b, \lambda_1), \quad B_2(a) = B(b, \lambda_2), \quad B_3(a) = B(b, \lambda_3), \ldots$$

---

simplify : $\vec{a} \rightarrow a \qquad \vec{b} \rightarrow b$

Correlation from repeated measurements: *N* times

ex.) outcomes $$A_1(a) = +1,\ A_2(a) = -1,\ A_3(a) = -1, \ldots$$

$$B_1(b) = -1,\ B_2(b) = -1,\ B_3(b) = +1, \ldots$$

correlation

$$\langle A(a)B(b) \rangle = \frac{A_1(a)B_1(b) + A_2(a)B_2(b) + \cdots + A_N(a)B_N(b)}{N}$$

$$\rightarrow \quad -1 \leq \langle A(a)B(b) \rangle \leq 1$$

considering two choices measurement axes for each of the particles,

we have

$$-1 \leq \langle A(a)B(b) \rangle \leq 1, \qquad -1 \leq \langle A(a)B(b') \rangle \leq 1,$$

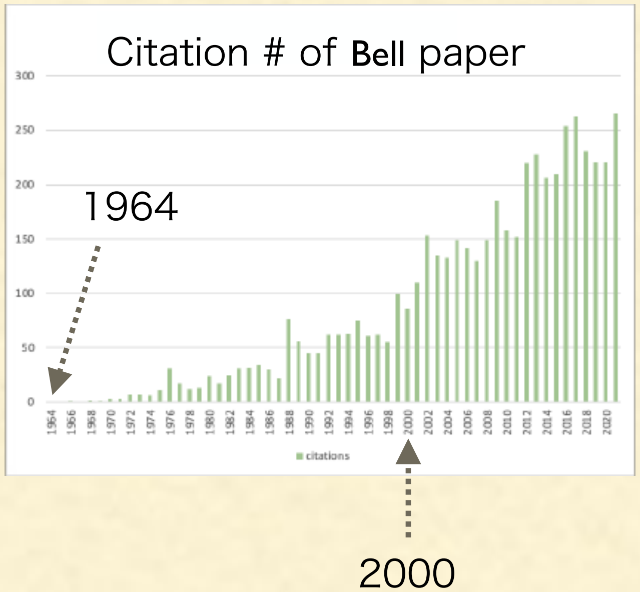
$$-1 \leq \langle A(a')B(b) \rangle \leq 1, \qquad -1 \leq \langle A(a')B(b') \rangle \leq 1$$

The four correlations in locally realistic HVT satisfy Bell’s inequality

$$|\langle A(a)B(b)\rangle + \langle A(a)B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a')B(b')\rangle| \leq 2$$

also known as the CHSH (Clauser–Horne–Shimony–Holt) inequality according to the authors who proposed in 1969 the improved version of the original inequality of Bell

Note: there are a number of variations for inequalities applied to local realistic HVT and it is customary to call them under the name ‘Bell’s inequality’.



Proof:

Since all of the values are dichotomic,

$$A_i(a) = \pm 1, \quad B_i(b) = \pm 1, \quad A_i(a') = \pm 1, \quad B_i(b') = \pm 1$$

we have

$$\begin{aligned} &A_i(a)B_i(b) + A_i(a)B_i(b') + A_i(a')B_i(b) - A_i(a')B_i(b') \\ &= A_i(a)\underbrace{\left(B_i(b) + B_i(b')\right)}_{\pm 2 \text{ or } 0} + A_i(a')\underbrace{\left(B_i(b) - B_i(b')\right)}_{0 \text{ or } \pm 2} \\ &= \pm 2A_i(a) \text{ or } \pm 2A_i(a') \\ &= \pm 2 \end{aligned}$$

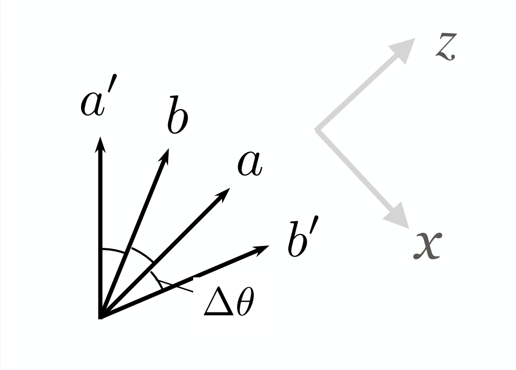
Averaging over $i = 1, 2, 3, \ldots N$ gives a value between -2 and +2, and hence

$$|\langle A(a)B(b)\rangle + \langle A(a)B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a')B(b')\rangle| \leq 2$$

QED.

Remark: note the *counterfactual* aspect in forming the correlation!

Bell's Theorem



entangled state (singlet state)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right)$$

if we choose the measurement axes on a mutual plane as above

$$\hat{A}(a) = \sigma_z \qquad \hat{B}(b) = \sigma_z \cos(\Delta\theta) + \sigma_x \sin(\Delta\theta)$$

quantum correlation

$$\langle A(a)B(b)\rangle = \langle\psi|\hat{A}(a)\otimes\hat{B}(b)|\psi\rangle$$

$\longrightarrow$ $\langle A(a)B(b)\rangle = -\cos\Delta\theta$

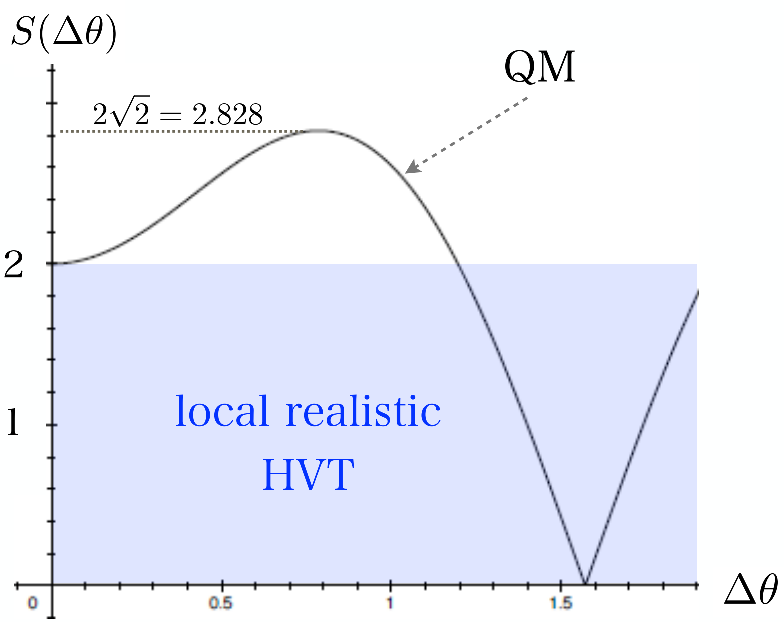
which implies

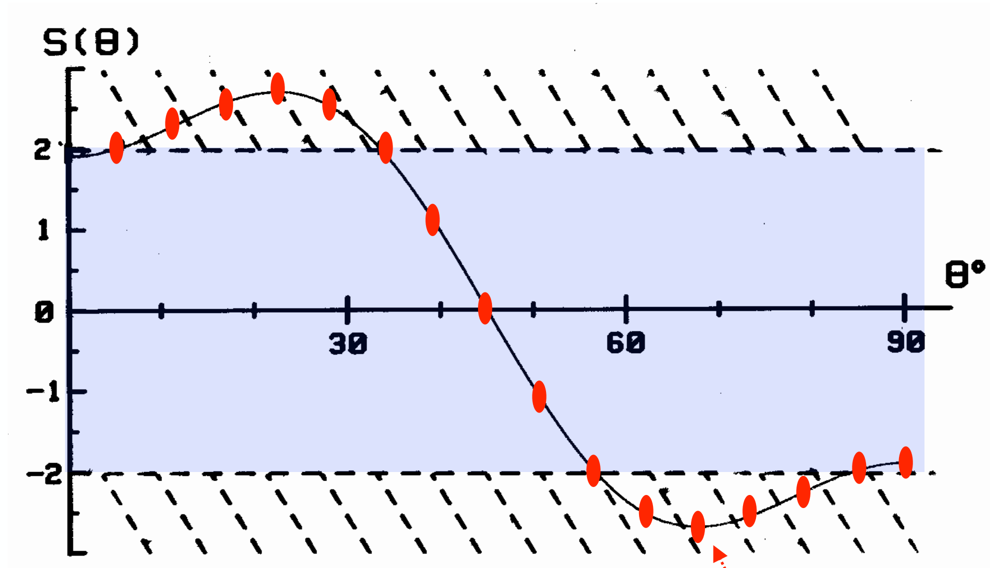
$$S(\Delta\theta) = |\langle A(a)B(b)\rangle + \langle A(a)B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a')B(b')\rangle|$$

$$= |3\cos(\Delta\theta) - \cos(3\Delta\theta)| \xrightarrow{\Delta\theta = \frac{\pi}{4}} \left|3\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right| = 2\sqrt{2} > 2$$

quantum correlation breaks Bell's inequality $\longrightarrow$ QM does not respect local realism

Bell test: experimental verification on the validity of Bell's inequality





experiment (Aspect)

Nature is not locally realistic!

Nobel Prize in Physics 2022

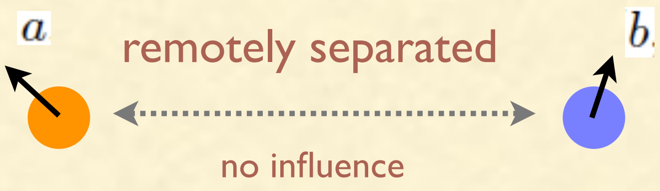
<u>early history of Bell test</u>

realistic proposal	Clauser-Horne-Shimony-Holt (1969)	
photon (atomic radiative cascades)	Freedman-Clauser (1972)	~~LRT~~
	Holt-Pipkin (1978)	~~QM~~
	Clauser (1976)	~~LRT~~
photon (positronium decay)	Kasday-Ullman-Wu (1975)	~~LRT~~
photon (atomic radiative cascades - improved)	Aspect et al. (1980 - 1985)	~~LRT~~
photon (parametric down conversion)	Brendel et al. (1992)	energy & time
	Tapster et al. (1994)	optical fiber 4km
	Tittel et al. (1998)	more than 10km

Three loopholes in Bell Test

- Locality loophole

The distance between the two particles is not sufficient to ensure the no influence condition between the two.



- Detection loophole

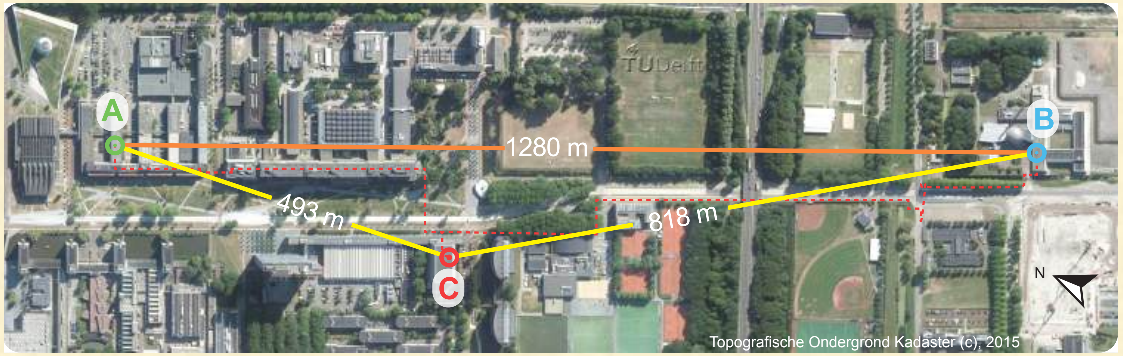
The efficiency of detection is not sufficient to ensure that events of no detection cannot overturn the conclusion of the Bell test.

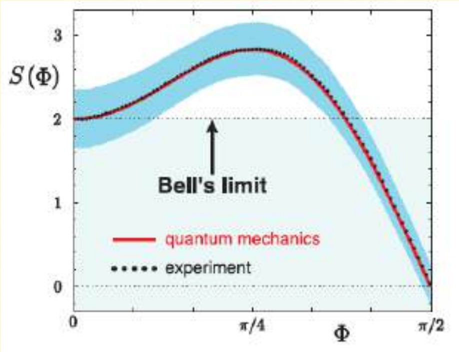
- Freedom of choice loophole

The choice of measurement (such as the spin axes) may be determined from the physical states of the system/measurement apparatus distorting the correlation enough to overturn the conclusion of the Bell test.

recent tests vs. loopholes

		locality	detection
Aspect et al. (1982)	photon: 12 m	△	×
Weihs et al. (1998)	photon: 400 m	○	×
Rowe et al. (2001)	ion	×	○
Sakai et al. (2006)	proton	△	△
Hensen et al. (2015)	electron	○	○
Giustina et al. (2015)	proton	○	○





History of Bell tests (Wikipedia)

Aspect → 4.2
B meson → 4.7
'clear' three loopholes → 4.12, 4.13
distant quasars → 4.18

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