3 March 2026
KEK Cosmo

Bell's Inequality and Some Ramifications

— 2. Development of Bell —

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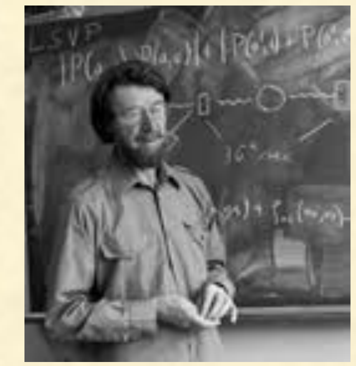
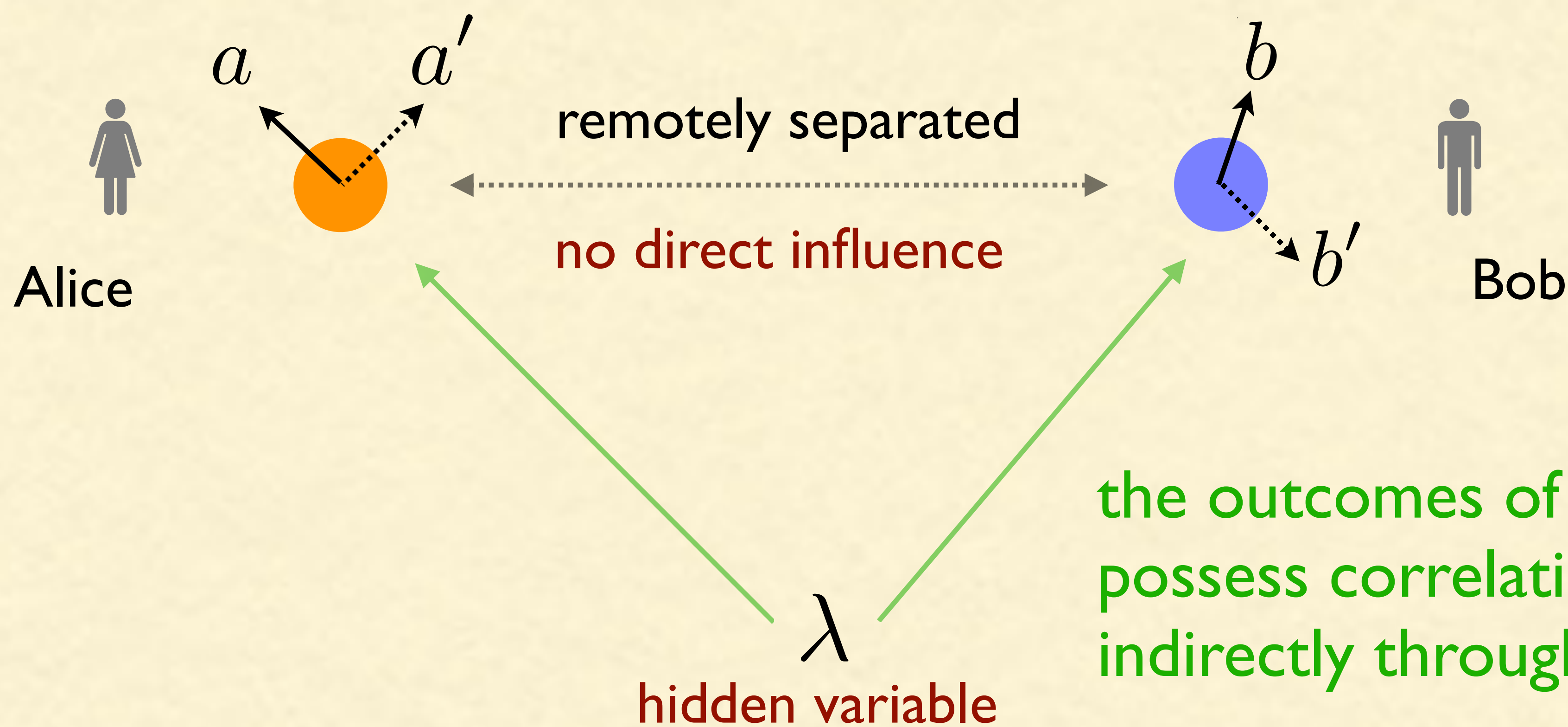
III. Bell Test



J. Bell

(1928 - 1990)

Local realistic HVT



the outcomes of measurement may possess correlation generated indirectly through λ

Bell's inequality (1964)

$$|\langle A(a)B(b)\rangle + \langle A(a)B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a')B(b')\rangle| \leq 2$$

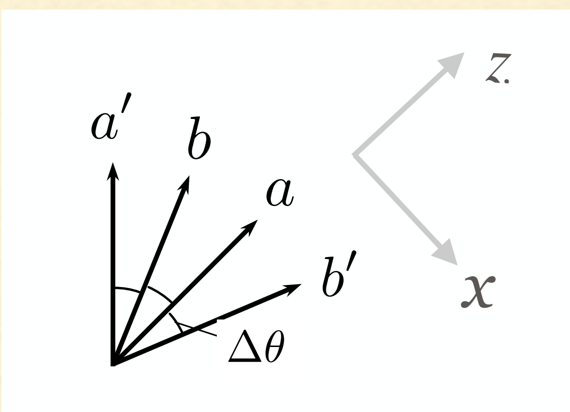
Bell test: experimental verification on the validity of Bell's inequality

QM

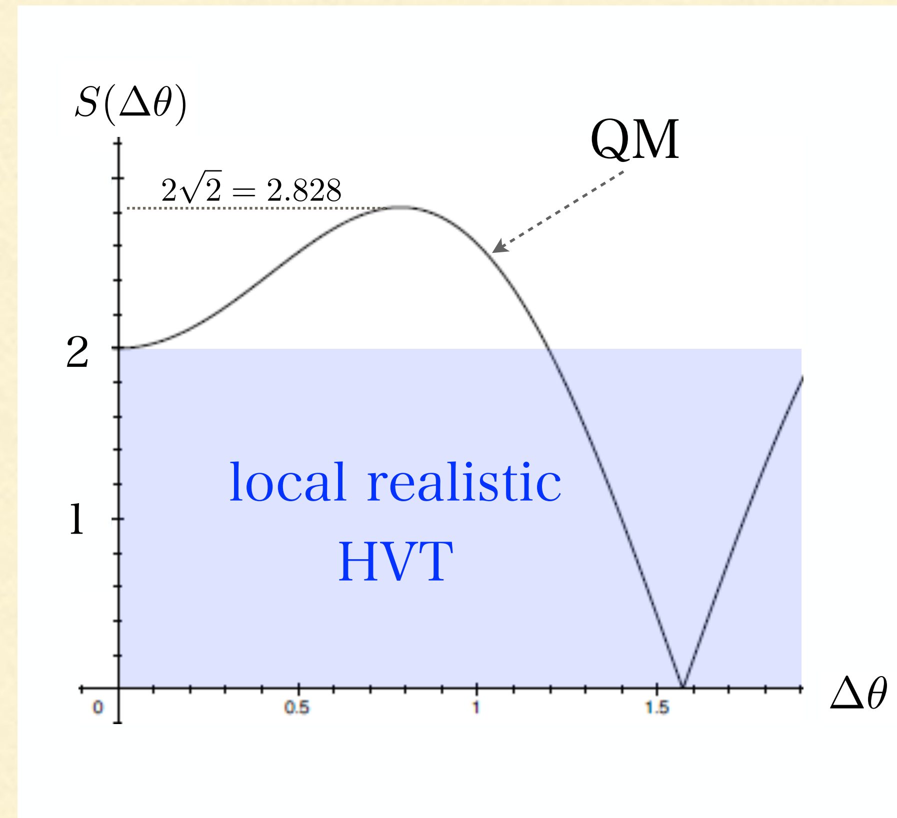
entangled state (singlet state)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\right)$$

if we choose



$$\rightarrow \begin{cases} \langle A(a)B(b)\rangle = -\cos\Delta\theta \\ \langle A(a)B(b')\rangle = -\cos\Delta\theta \\ \langle A(a')B(b)\rangle = -\cos\Delta\theta \\ \langle A(a')B(b')\rangle = -\underline{\cos 3\Delta\theta} \end{cases}$$



$$S(\Delta\theta) = |\langle A(a)B(b)\rangle + \langle A(a)B(b')\rangle + \langle A(a')B(b)\rangle - \langle A(a')B(b')\rangle|$$

$$= |3\cos(\Delta\theta) - \cos(3\Delta\theta)| \xrightarrow[\Delta\theta = \frac{\pi}{4}]{} \left|3 \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right| = 2\sqrt{2} > 2$$

QM does not respect local realism

History of Bell tests (Wikipedia)

Aspect → 4.2

B meson → 4.7

claimed to 'overcome' the three loopholes → 4.12, 4.13

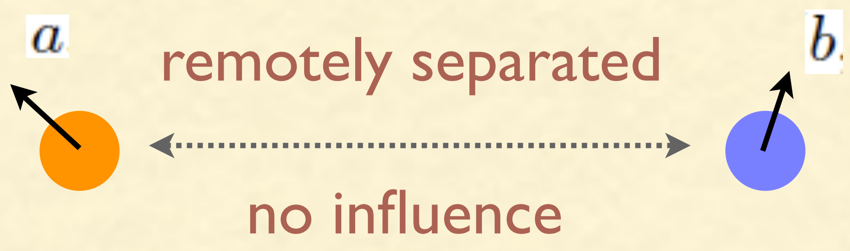
distant quasars → 4.18

Contents [hide]

Three loopholes in Bell Test

- Locality loophole

  The distance between the two particles is not sufficient to ensure the no influence condition between the two.



- Detection loophole

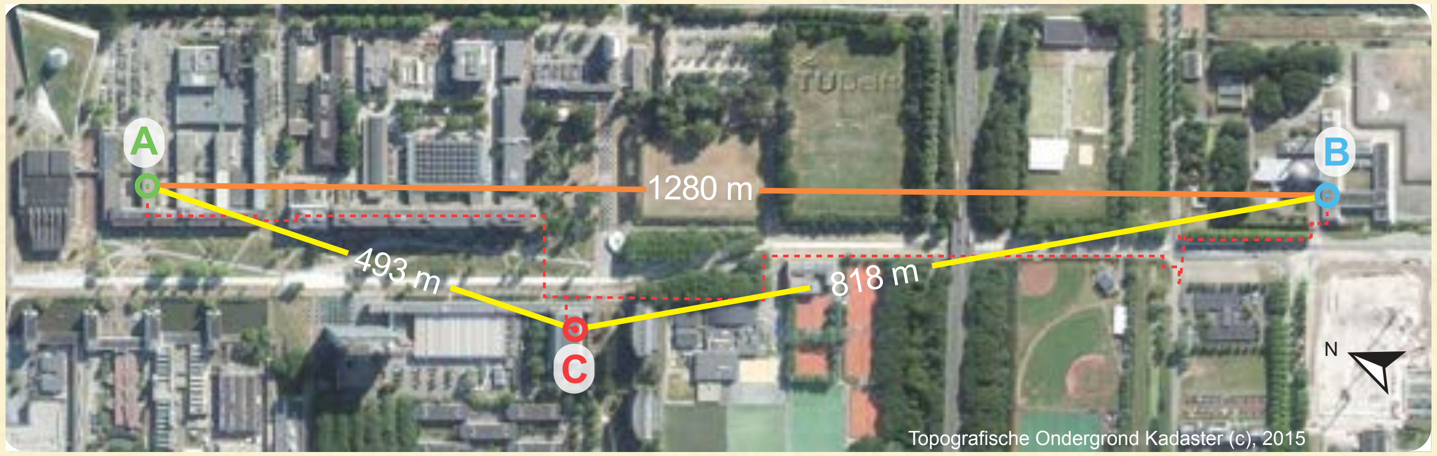
  The efficiency of detection is not sufficient to ensure that events of no detection cannot overturn the conclusion of the Bell test.

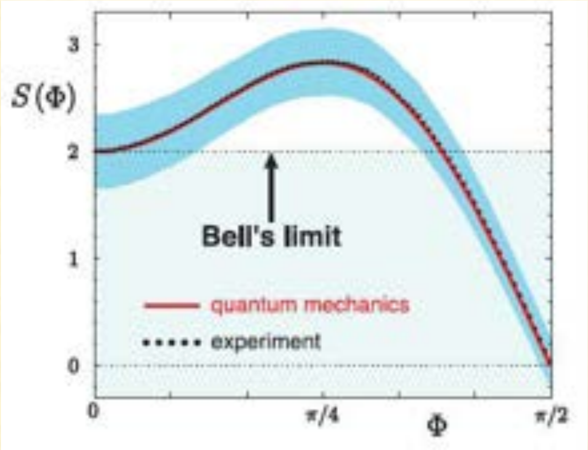
- Freedom of choice loophole

  The choice of measurement (such as the spin axes) may be determined from the physical states of the system/measurement apparatus distorting the correlation enough to overturn the conclusion of the Bell test.

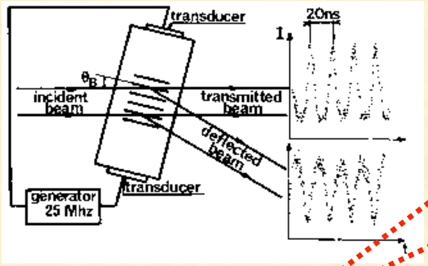
recent tests vs. loopholes

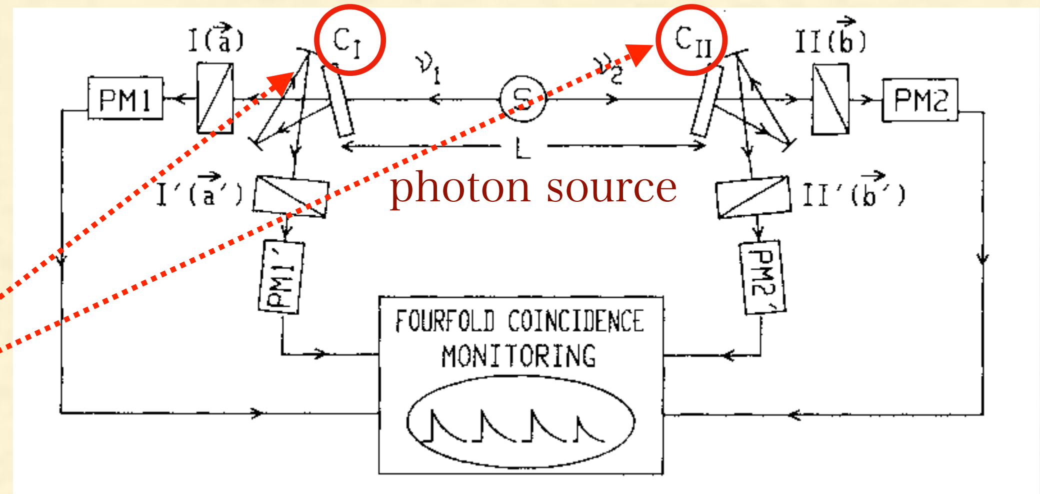
		locality	detection
Aspect et al. (1982)	photon: 12 m	△	×
Weihs et al. (1998)	photon: 400 m	○	×
Rowe et al. (2001)	ion	×	○
Sakai et al. (2006)	proton	△	△
Hensen et al. (2015)	electron	○	○
Giustina et al. (2015)	proton	○	○

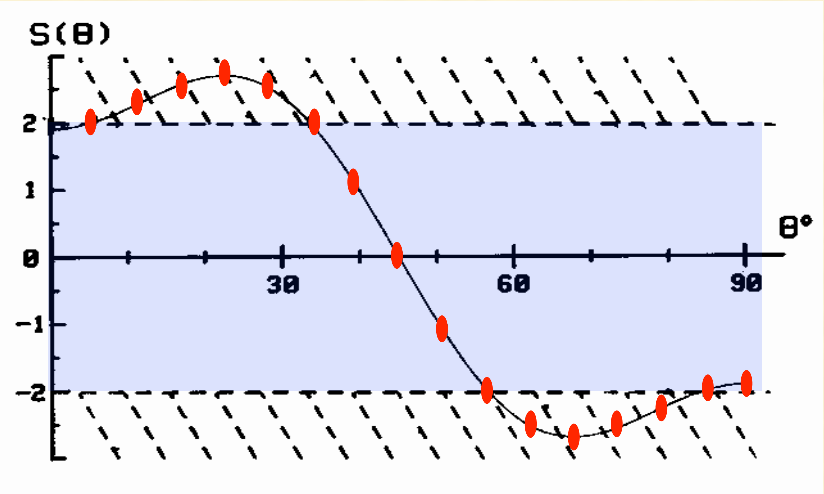




Aspect (1982)







C_I, C_{II}

optical switch

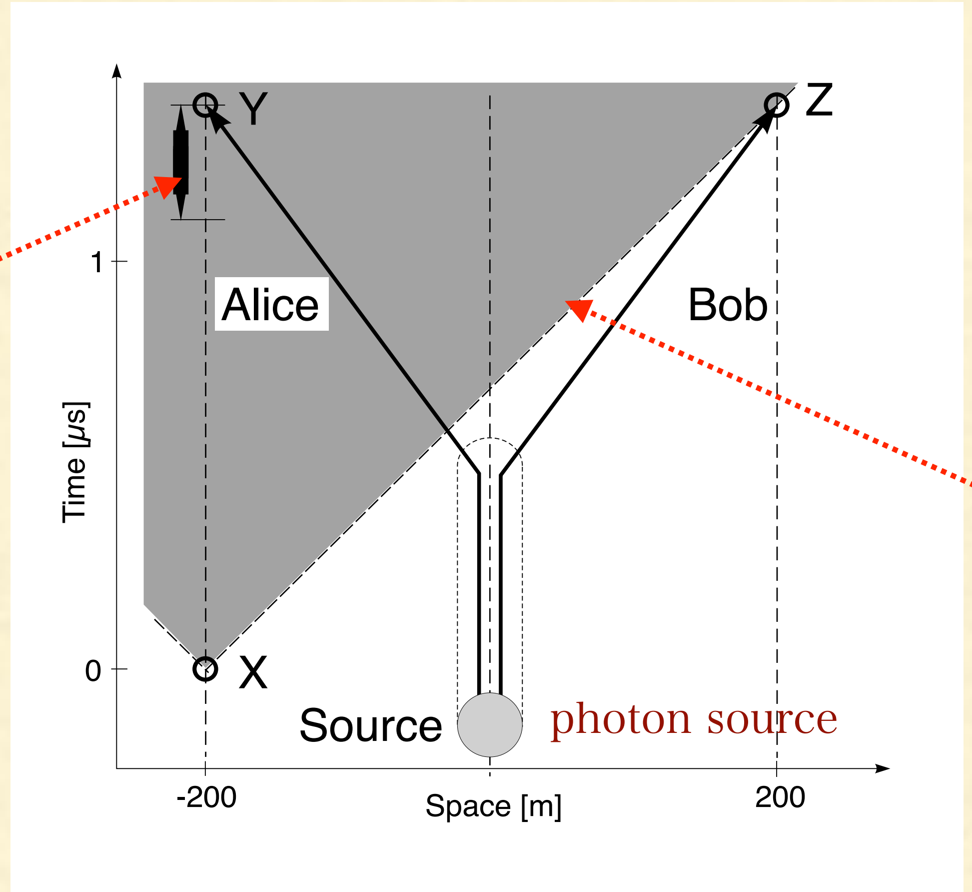
50 MHz

'random' choice

→ Freedom of choice loophole

$S = 2.697$

Weihs et al. (1998)



beem splitter

'random' choice

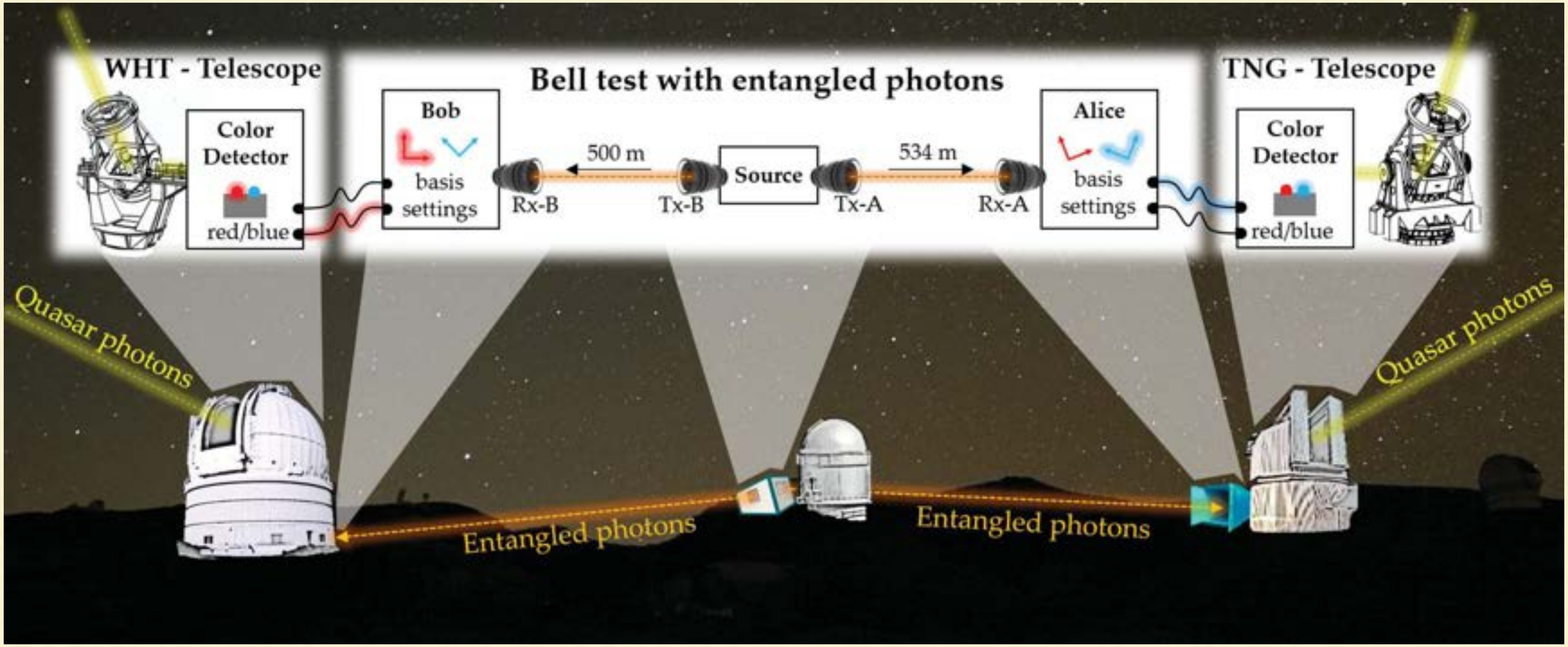
→ Freedom of choice loophole

light cone

→ Locality loophole

$S = 2.73$

Rauch et al. (2018): cosmic Bell test

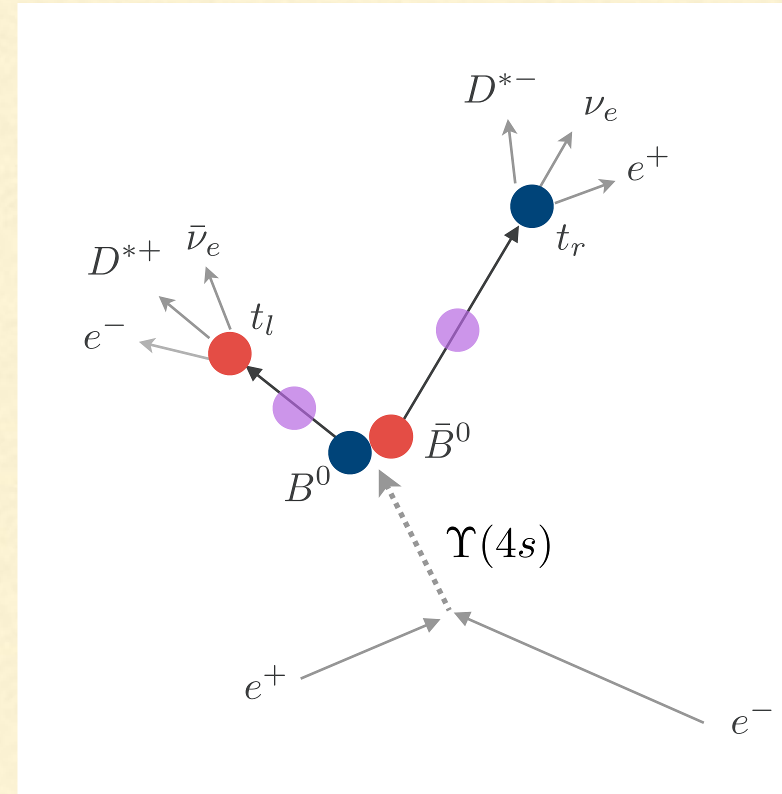


'random' choice of measurement settings is performed using the frequency of photons arriving from distant quasars, 7 and 3 Gyr ago at Alice and 12 Gyr ago at Bob.

→ ~~Freedom of choice loophole~~

Bell test with mesons

1999	K meson	CPLEAR (CERN)
2006	K meson	KLOE (DAΦNE)
2007	B meson	Belle (KEK)



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|B^0\rangle|\overline{B}^0\rangle - |\overline{B}^0\rangle|B^0\rangle\right)$$

energy eigenstates

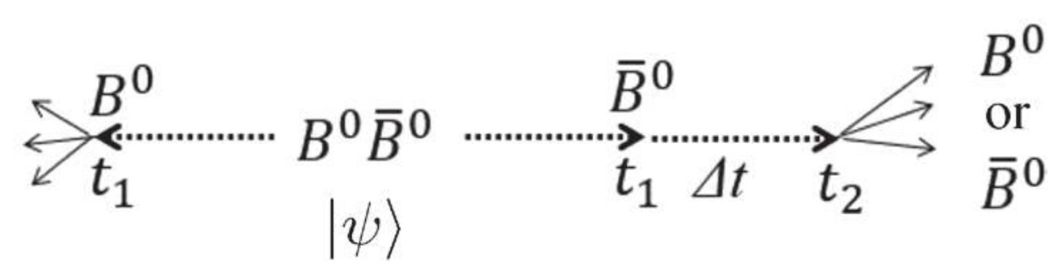
$$\begin{cases} |B_\mathrm{H}\rangle = (|B^0\rangle + |\overline{B}^0\rangle)/\sqrt{2} \longrightarrow |B_\mathrm{H}(t)\rangle = \mathrm{e}^{-\mathrm{i}\lambda_\mathrm{H}t}|B_\mathrm{H}\rangle \\ |B_\mathrm{L}\rangle = (|B^0\rangle - |\overline{B}^0\rangle)/\sqrt{2} \longrightarrow |B_\mathrm{L}(t)\rangle = \mathrm{e}^{-\mathrm{i}\lambda_\mathrm{L}t}|B_\mathrm{L}\rangle \end{cases}$$

after the entangled meson pair state $|\psi\rangle$ is generated, each of $|B^0\rangle$ and $|\overline{B}^0\rangle$ oscillates between them

flavor oscillation

$$\lambda_\mathrm{H} = M_\mathrm{H} - \mathrm{i}\Gamma_\mathrm{H}/2, \quad \lambda_\mathrm{L} = M_\mathrm{L} - \mathrm{i}\Gamma_\mathrm{L}/2$$

$$\Delta M = M_\mathrm{H} - M_\mathrm{L} = 3.334 \times 10^{-10}\ \mathrm{MeV}$$



state at the decay of the two mesons

$$|\psi(t_1, t_2)\rangle = \frac{1}{\sqrt{2}}\left(|B_{\mathrm{H}}(t_1)\rangle|B_{\mathrm{L}}(t_2)\rangle - |B_{\mathrm{L}}(t_1)\rangle|B_{\mathrm{H}}(t_2)\rangle\right)$$

decay probability

decay factor

$$P^{\mathrm{Q}}(A, B, t_1, t_2) = \frac{\mathrm{e}^{-\Gamma(t_1+t_2)}}{4}\left(1 - AB\cos(\Delta M \Delta t)\right)$$

dichotomic value assignment for A, B

B^0 → $+1$

or

$\bar{B}^0$ → -1

normalized probability

$$P_{t_1,t_2}(A, B) = \frac{P(A, B, t_1, t_2)}{\sum_{A,B} P(A, B, t_1, t_2)}$$

correlation $\quad C^{\mathrm{Q}}(t_1, t_2) = \sum_{A,B} AB\, P^{\mathrm{Q}}_{t_1,t_2}(A, B) = -\cos(\Delta M \Delta t).$

$$S^Q(\Delta t) = C^Q(t_1, t_2) + C^Q(t_1', t_2) + C^Q(t_1, t_2') - C^Q(t_1', t_2')$$

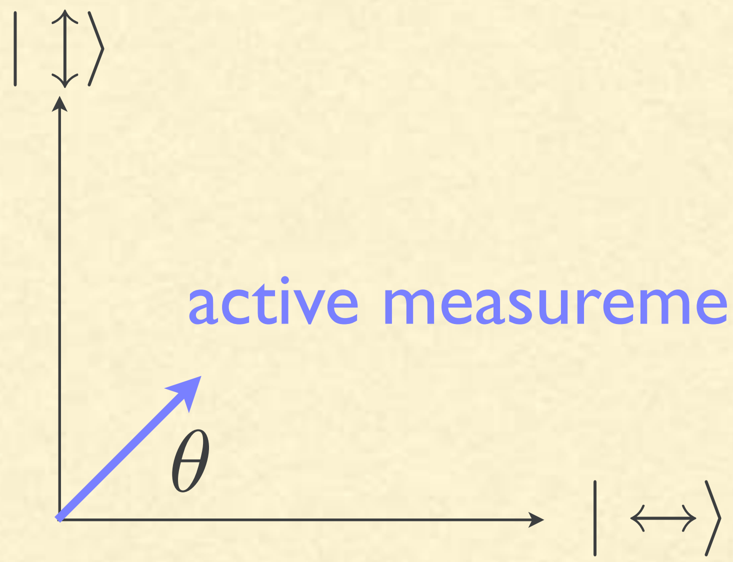
$$= -3\cos(\Delta M \Delta t) + \cos(3\Delta M \Delta t)$$

$$t_2 - t_1' = t_1 - t_2 = t_2' - t_1 = \Delta t$$

analogy with the spin system

photon pairs vs. meson pairs

polarization states



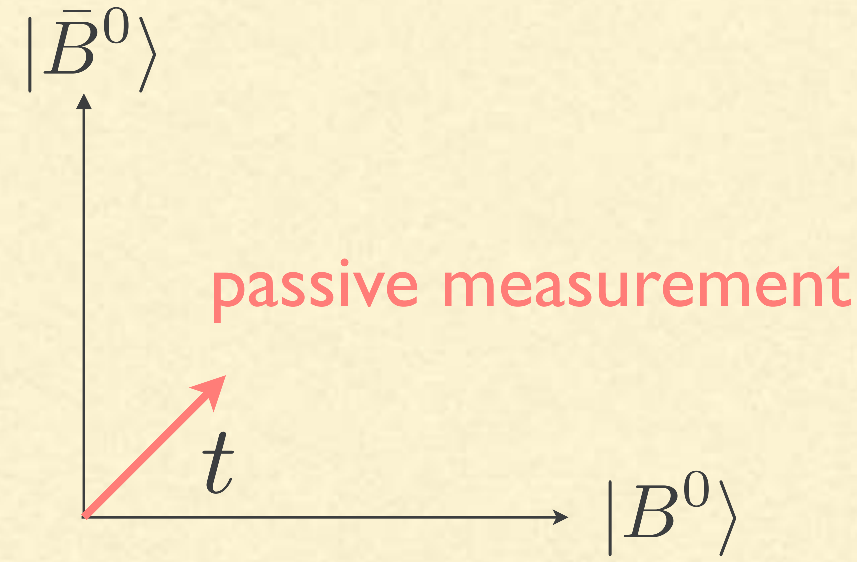
$$\frac{|\leftrightarrow\rangle|\updownarrow\rangle - |\updownarrow\rangle|\leftrightarrow\rangle}{\sqrt{2}}$$

同時確率

$$\mathrm{Pr}_{\leftrightarrow\updownarrow}(\theta_l,\theta_r) = \mathrm{Pr}_{\updownarrow\leftrightarrow}(\theta_l,\theta_r)$$

$$= \quad [1+\cos(\Delta\theta)]/4.$$

B meson states



$$\frac{|B^0\rangle|\bar{B}^0\rangle - |\bar{B}^0\rangle|B^0\rangle}{\sqrt{2}}$$

同時確率

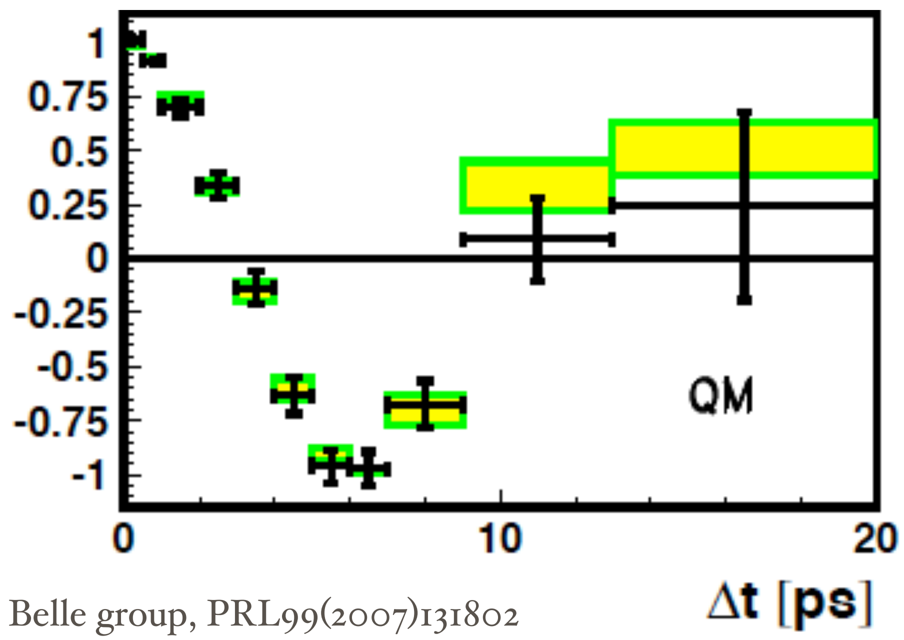
$$\mathrm{Pr}_{B^0\bar{B}^0}(t_l,t_r) = \mathrm{Pr}_{\bar{B}^0 B^0}(t_l,t_r)$$

$$= e^{-\frac{t_l+t_r}{\tau_B}}\left[1+\cos(\Delta M \Delta t)\right]/4$$

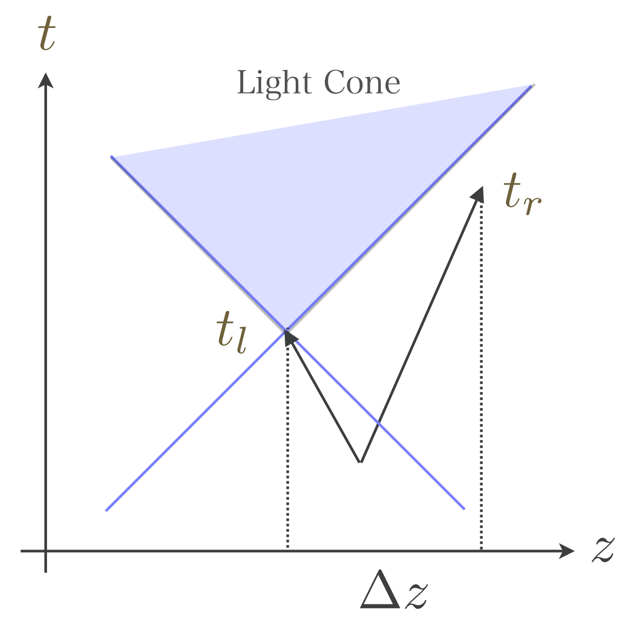
decay factor

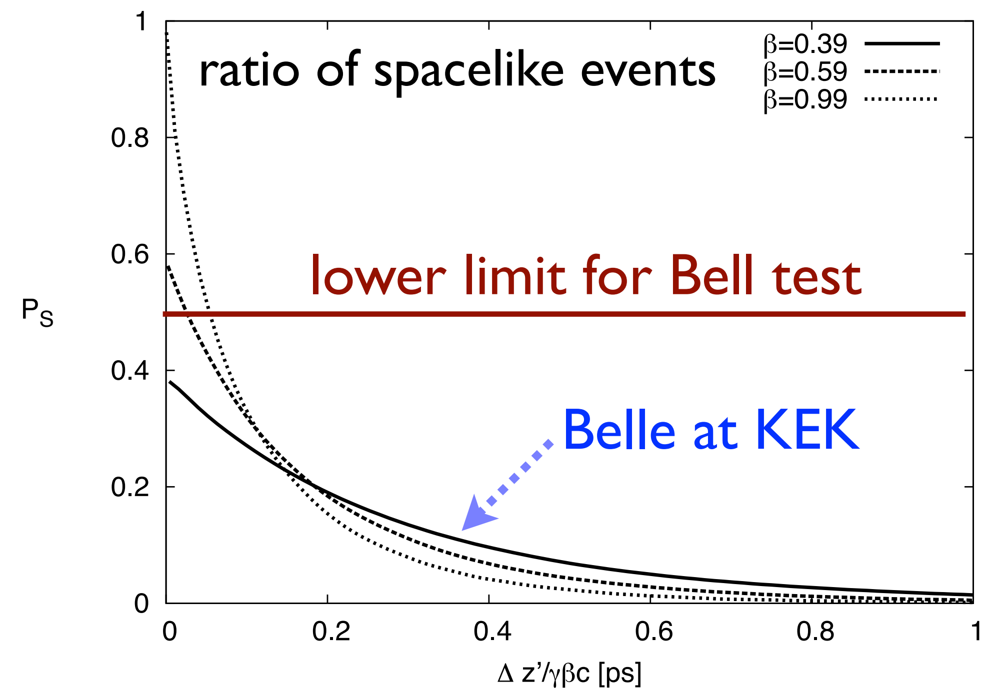
Bell test at Belle (2007) : first attempt of Bell test in high energy physics

$$e^+ e^- \longrightarrow \Upsilon(4s) \longrightarrow \frac{|B^0\rangle|\bar{B}^0\rangle - |\bar{B}^0\rangle|B^0\rangle}{\sqrt{2}}$$



Belle group, PRL99(2007)131802





T. Ichikawa et al., PLA 373 (2008) 39

incomplete due to
the Locality loophole

significance of Bell test with B meson

- massive particle
- high energy regime
- flavor entanglement
- connection to CP violation

B^0	5,279.4 MeV

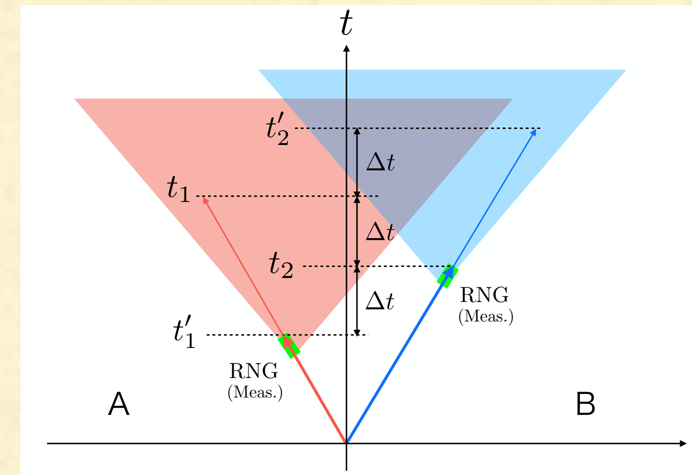
plus

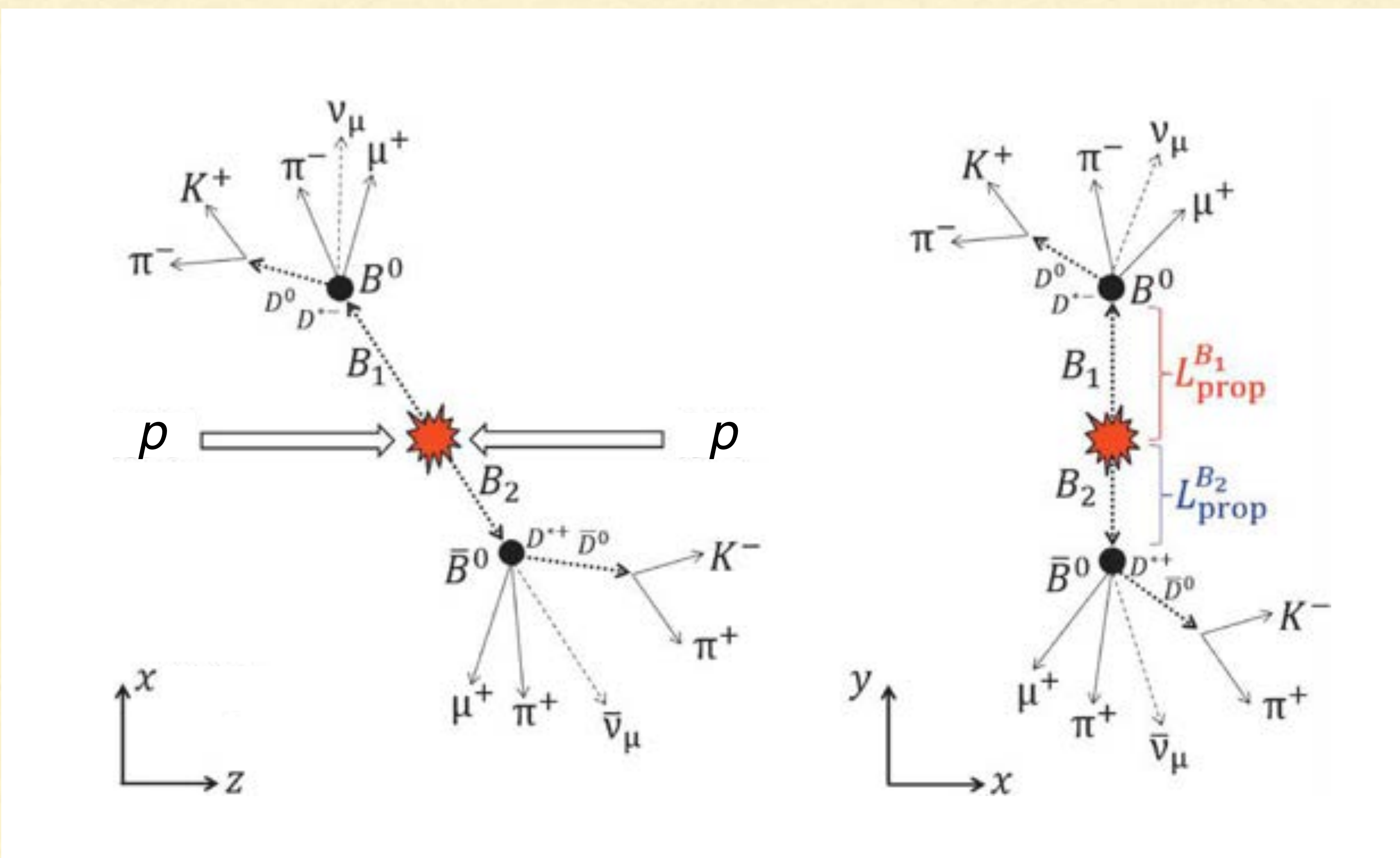
decay timing considered truly random → choice of measurement → ~~Freedom of choice loophole~~

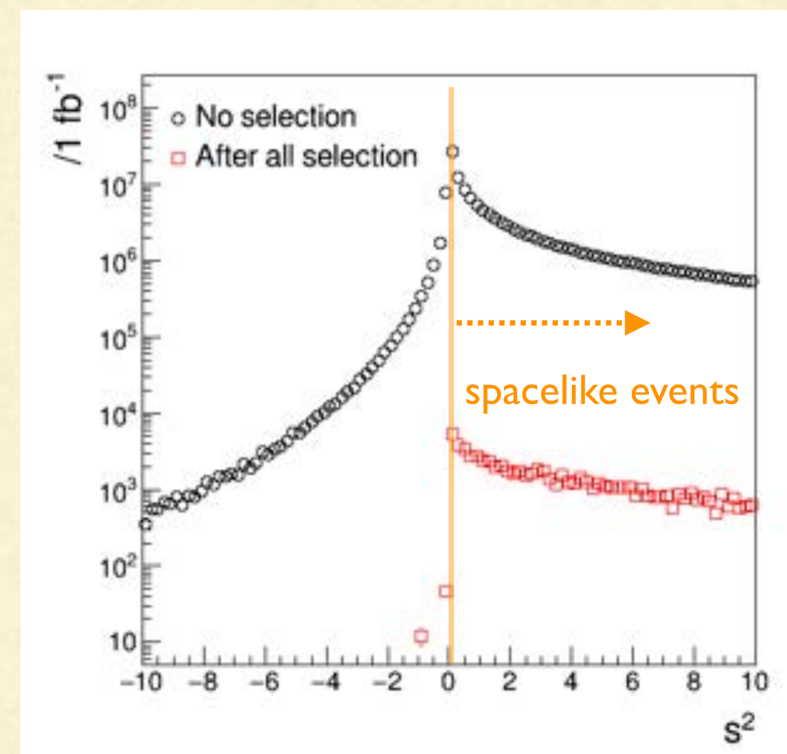
toward a better Bell test ATLAS (CERN)

- ATLAS : Unlike Belle, B mesons move in all directions, allowing us to determine the point of the pair creation as well as the decay point.

selection of spacelike events ⋯⋯▶ ~~Locality loophole~~



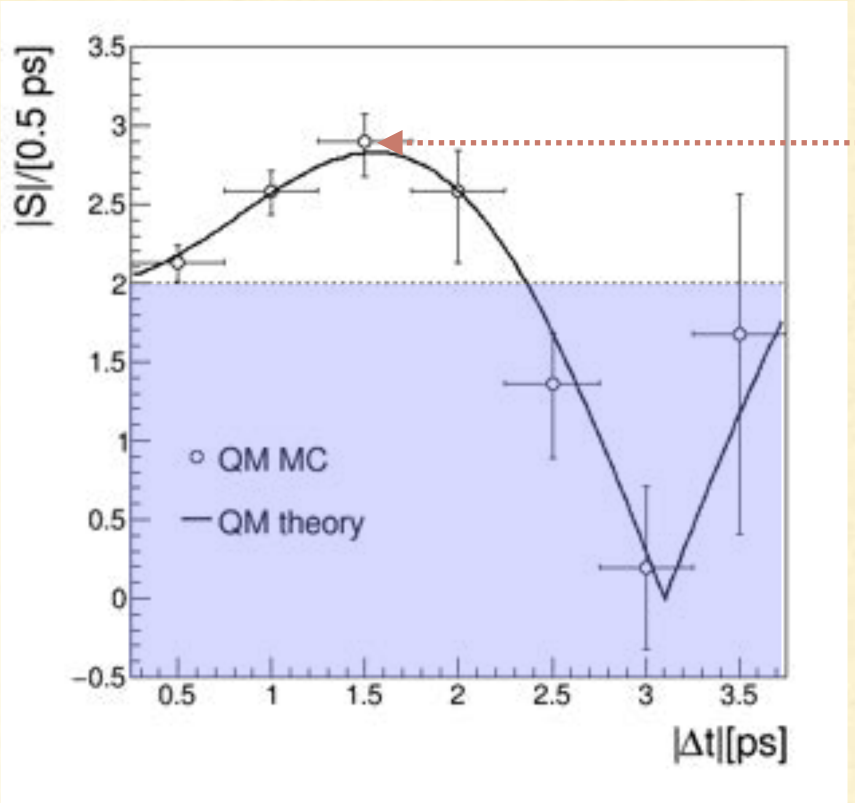




resolution:

- Belle： σ(Δz) ~ 100 μm, σ(Δt) ~ 1.2 ps
- ATLAs： σ(L = ct) ~ 34 μm, σ(t) = 0.11 ps → Detection loophole

MC simulation



Takubo, et al., (2021)

$$\Delta t = \pi/4\Delta M \approx 1.55 \text{ ps}$$

$$(\Delta M = 3.334 \times 10^{-10} \text{ MeV})$$

expected largest breakdown of Bell's inequality

Probabilistic description of measurement outcomes

$P(A|a)$ probability of finding the value A for the measurement setting a

$P(B|b)$ probability of finding the value B for the measurement setting b

$P(A,B|a,b)$ joint probability

Probabilistic (stochastic) HVT

$$P(A|a) \to P(A|a;\lambda), \quad P(B|b) \to P(B|b;\lambda)$$

joint probability

$$P(A,B|a,b) = \int_\Lambda d\lambda\, \rho(\lambda)\, \underline{P(A|a;\lambda)P(B|b;\lambda)}$$

Bell locality

correlation

Bell's inequality

$$\langle A(a)B(b)\rangle = \sum_{A,B} AB\, P(A,B|a,b) \quad \Rightarrow \quad |S| \leq 2$$

Fine's Theorem

A. Fine, *PRL* 48 (1982) 291

All of (i) - (iv) below are equivalent:

(i) Deterministic local HVT

(ii) Probabilistic HVT + Bell's locality

(iii) Bell (CHSH) inequalities (all combinations)

(iv) Existence of a four-point joint probability distribution possessing two-point joint probability distributions as marginal distributions

$$P(A, B|a, b) = \sum_{A', B'} P(A, A', B, B'|a, a', b, b')$$

→ Instead of considering Bell's inequality, one may consider the consistency between four-point correlations and two-point correlations.

IV. Contextuality: Kochen-Specker Theorem

What conditions should be imposed on hidden variable theory (HVT)?

value assignment $v(A) := A(\lambda)$ etc., to match the actual measurement outcome

In QM, $\hat{C} = \hat{A} + \hat{B}$ → $\langle C \rangle = \langle A \rangle + \langle B \rangle$ (QM) expectation value is additive

then in HVT $v(C) = v(A) + v(B)$?

This is impossible: (Counterexample)

$$\hat{A} = \sigma_x\,, \quad \hat{B} = \sigma_y \qquad \hat{A} + \hat{B} = \sigma_x + \sigma_y$$

$v(A) = \pm 1, \quad v(B) = \pm 1$ but $v(A+B) \overset{\text{QM}}{=} \pm\sqrt{2}$

Based on this, von Neumann 'disproved' HVT (no-go theorem: 1932)

meaningless: unnecessary requirement for physical quantities that cannot be measured simultaneously (Bell 1964) simultaneously measurable $A, B, C, \ldots$

In QM, if they satisfy the condition

$$f(\hat{A}, \hat{B}, \hat{C}, \ldots) = 0 \quad \Rightarrow \quad f(v(A), v(B), v(C), \ldots) = 0$$

then in HVT

(Proof)

simultaneously measurable $\longrightarrow$ $\hat{A}, \hat{B}, \hat{C}, \ldots$ are mutually exchangeable

Simultaneous eigenstates $\hat{A}|\psi\rangle = a|\psi\rangle, \quad \hat{B}|\psi\rangle = b|\psi\rangle, \quad \hat{C}|\psi\rangle = c|\psi\rangle, \quad \ldots$

HVT must also present these eigenvalues = measured values

$$v(A) = a, \quad v(B) = b, \quad v(C) = c, \quad \ldots$$

Therefore

$$0 = f(\hat{A}, \hat{B}, \hat{C}, \ldots)|\psi\rangle = f(a, b, c, \ldots)|\psi\rangle$$
$$= f(v(A), v(B), v(C), \ldots)|\psi\rangle$$

KS (Kochen-Specker) Theorem (1967) $\dim \mathcal{H} \geq 3$

In HVT, it is impossible to assign a value to all physical quantities independently of the measurement context. → contextuality

Assigning values to mutually commutable (simultaneously measurable) pairs of physical quantities

$$A, B, C, \ldots \longrightarrow v(A), v(B), v(C), \ldots$$

measured values (eigenvalues)

if we they fulfill the condition

$$f(\hat{A}, \hat{B}, \hat{C}, \ldots) = 0 \implies f(v(A), v(B), v(C), \ldots) = 0$$

then, in general, the assignment of $v(A)$ depends on the choice of $B, C, \ldots$

Proof of KS Theorem Lite ($\dim \mathcal{H} \geq 4$) Mermin's Magic Square (1990)

$1 \otimes \sigma_z$	$\sigma_z \otimes 1$	$\sigma_z \otimes \sigma_z$	+1
$\sigma_x \otimes 1$	$1 \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$	+1
$-\sigma_x \otimes \sigma_z$	$-\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$	+1
-1	-1	-1	

Relational Conditions

Value Assignment

$$1 \otimes \sigma_z \longrightarrow v(1 \otimes \sigma_z) = \pm 1$$

$$-\sigma_x \otimes \sigma_z \longrightarrow v(-\sigma_x \otimes \sigma_z) = \pm 1$$

etc.

(Failed Example)

-1	$+1$	-1
$+1$	-1	-1
$+1$	$+1$	$+1$
		+1

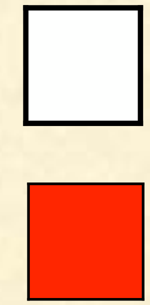
For example, the third column

$$(\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x)(\sigma_y \otimes \sigma_y) = \sigma_z \sigma_x \sigma_y \otimes \sigma_z \sigma_x \sigma_y$$

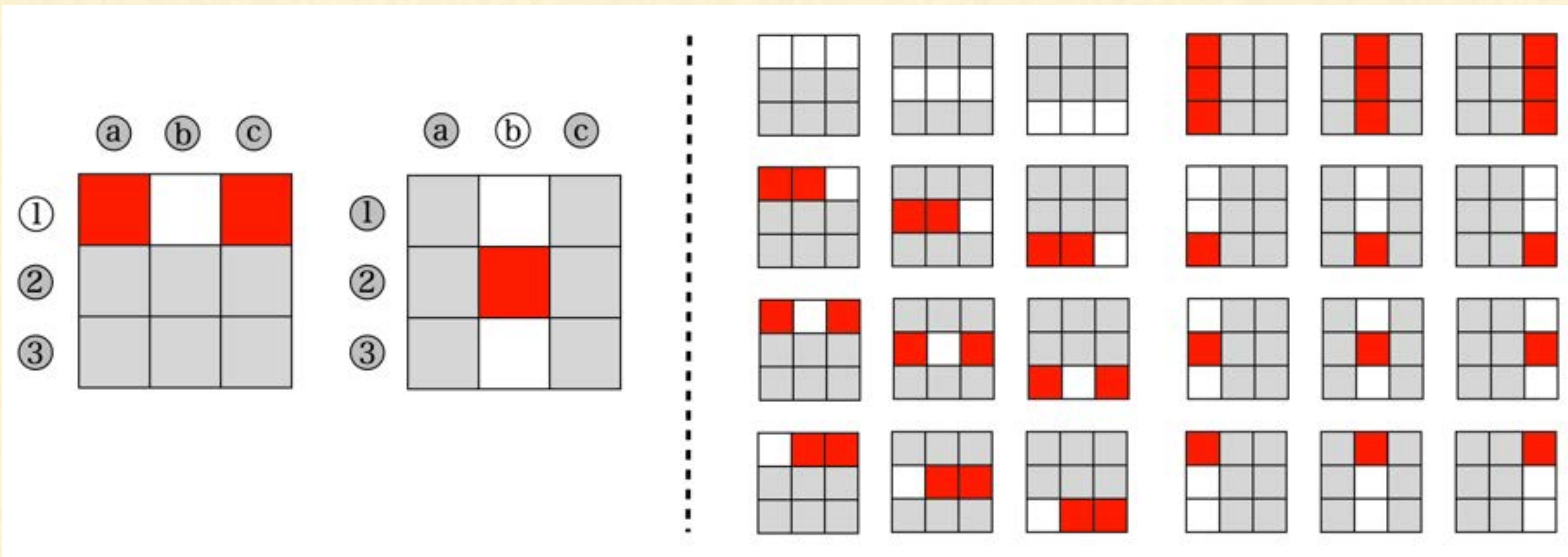
$$= i\sigma_z \sigma_z \otimes i\sigma_z \sigma_z = i^2 1 \otimes 1 = -1$$

Properties of Mermin's magic square

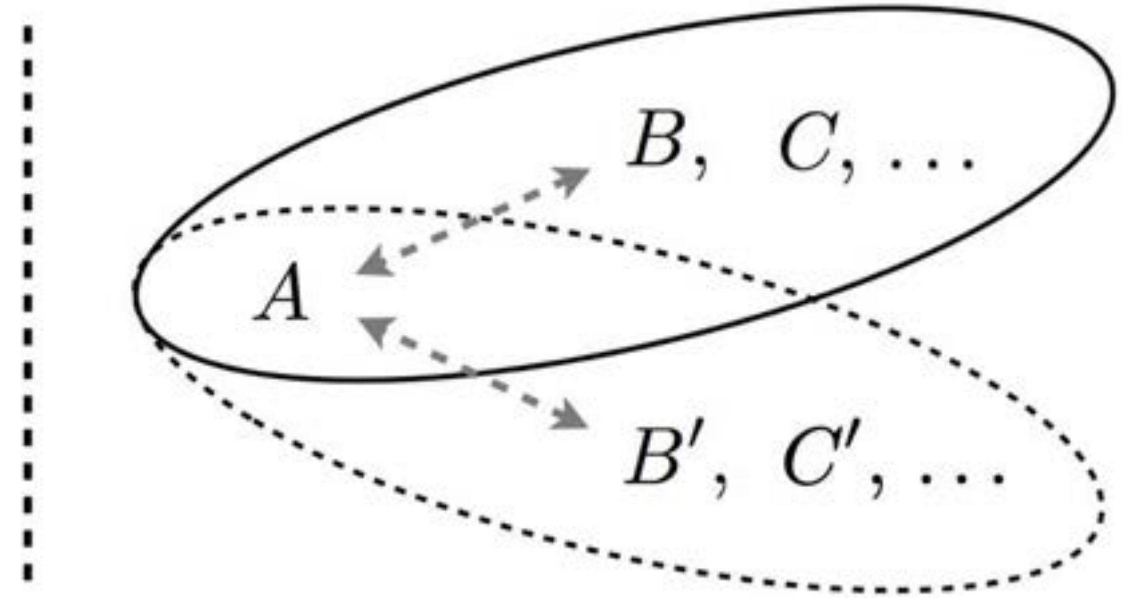
Simultaneous easurement of a set of three physical quantities in either the vertical column or the horizontal row is possible.


= +1

= -1

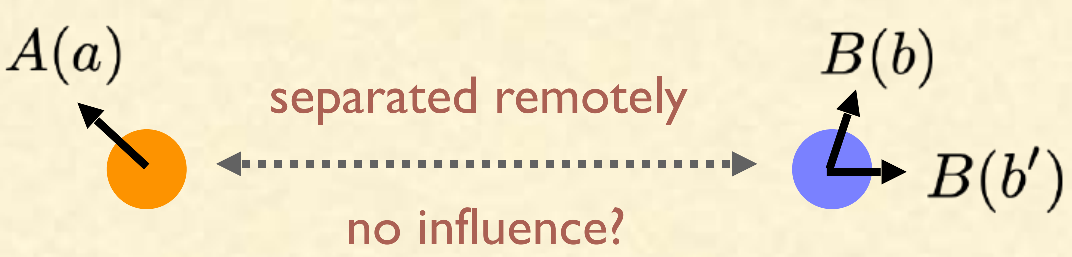


Implication of the KS theorem



Physical quantities do not exist independently of the context

Locally, it is possible for the measurement of one physical quantity to influence the measurement of another. However, in the case of non-local measurements at a distance, it is expected that such influence cannot be possible.

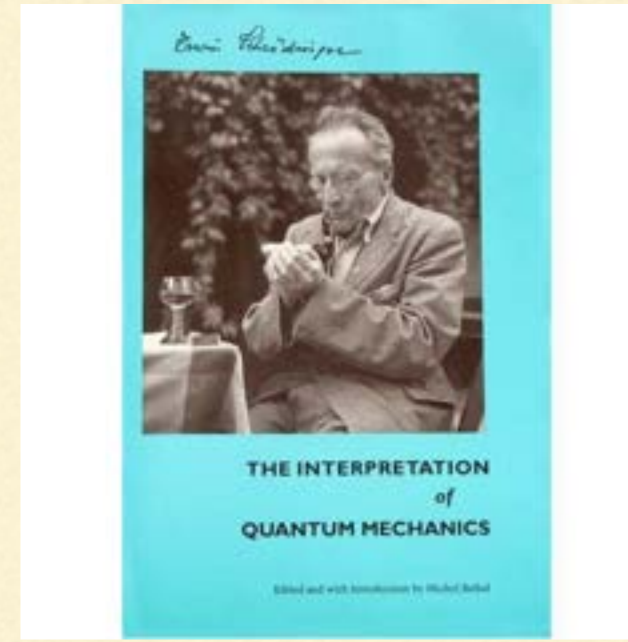


Breakdown of Bell's inequality reject this expectation

→ nonlocal contextuality

On Entanglement   Schrödinger, 1935

When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or ψ-functions) have become entangled.



Schrödinger's residence in Dublin

Maximal knowledge of a total system does not necessarily include total knowledge of all its parts, not even when these are fully separated from each other and at the moment are not influencing each other at all.