



KEK-Cosmo 2025

The Imprints of Primordial Non-Gaussianity on the Scalar-Induced Gravitational-Wave Background

Based on arXiv:2505.16820

by JL, Sai Wang, Zhi-Chao Zhao, and Kazunori Kohri

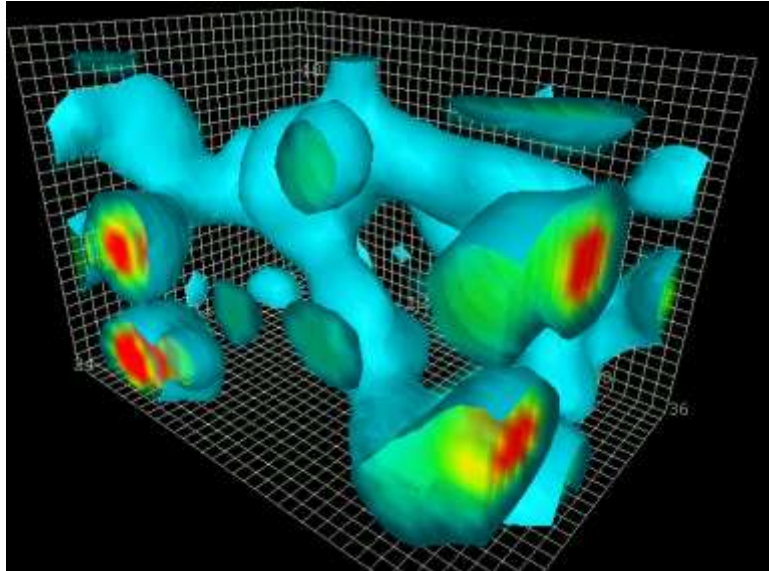
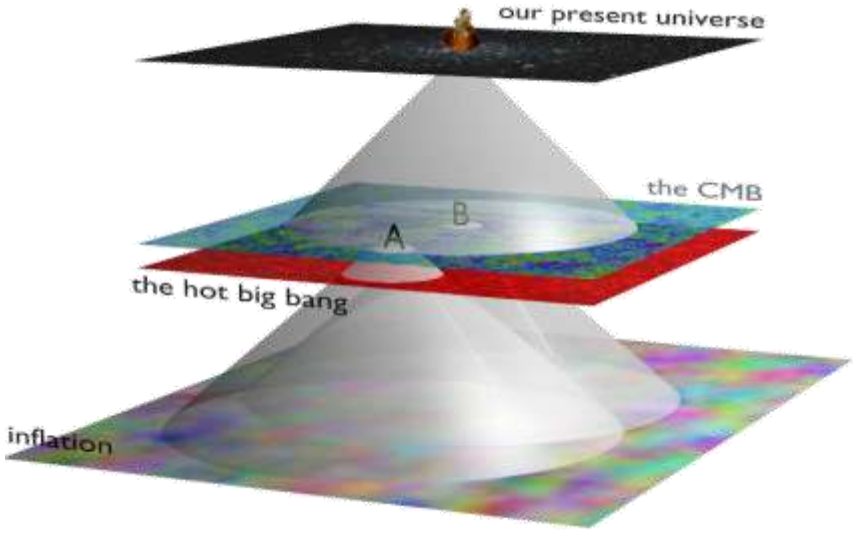
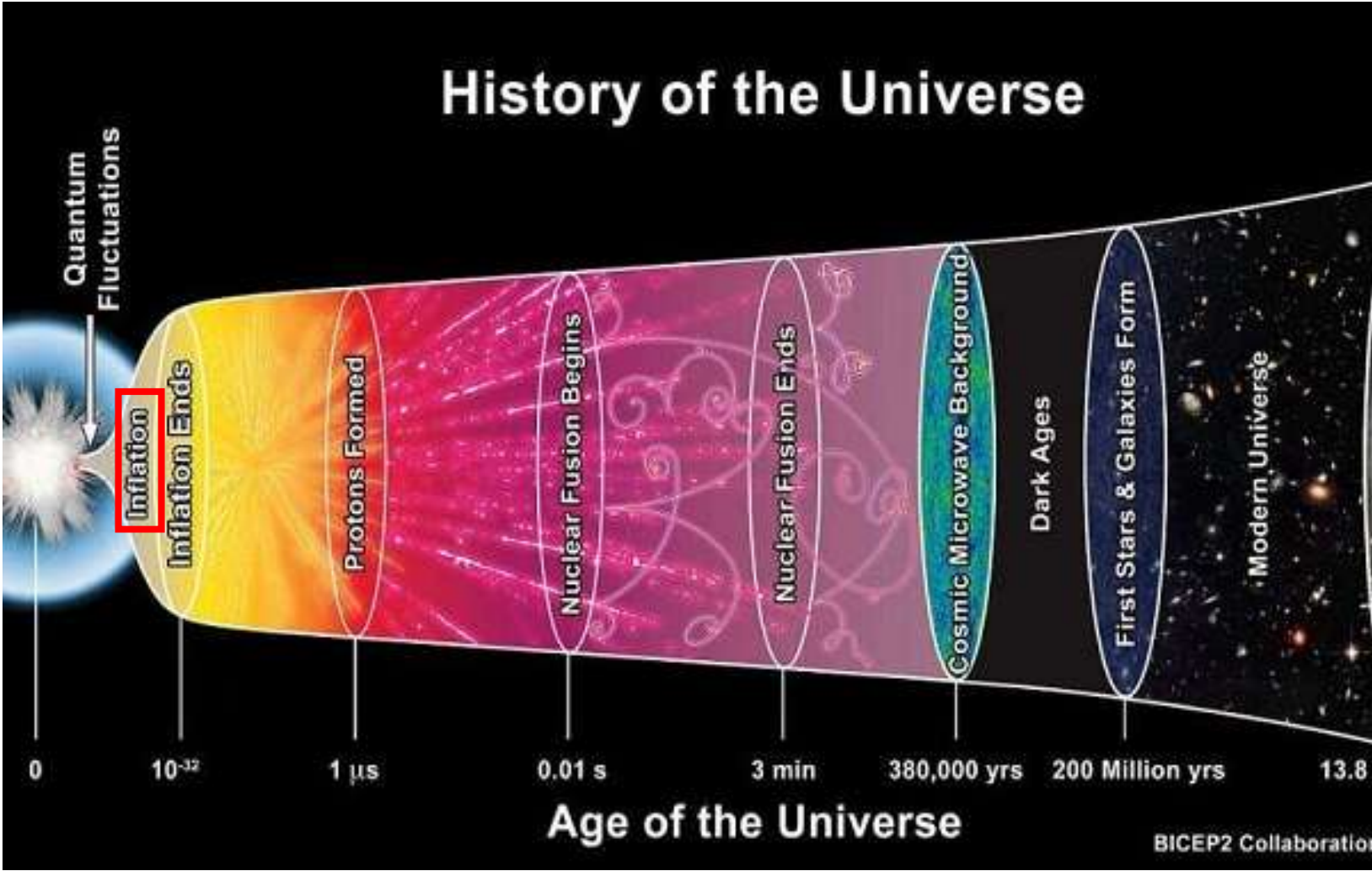
Speaker: Jun-Peng Li (李俊蓬)
National Astronomical Observatory of Japan

KEK, March 4th, 2026

OUTLINE

- 01.** Introduction
- 02.** Scalar-Induced Gravitational Waves
- 03.** Isotropic Component
- 04.** Anisotropies and Non-Gaussianity
- 05.** Summary

Primordial Curvature Perturbations



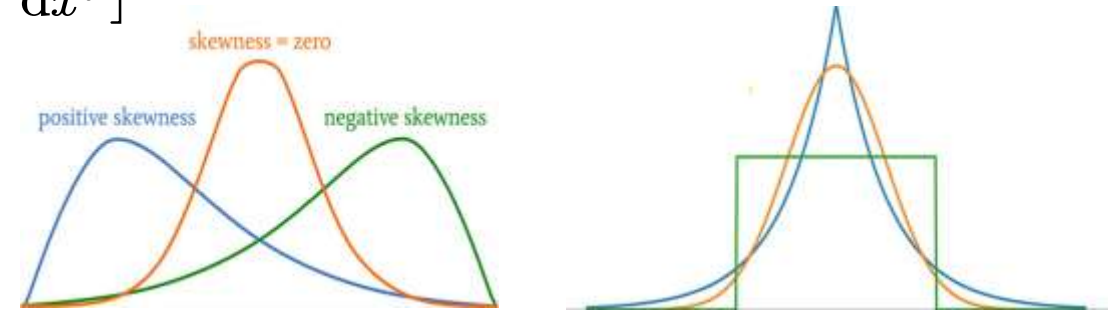
primordial curvature perturbations \longleftrightarrow quantum fluctuations



Primordial Non-Gaussianity

Metric during inflation: $ds^2 = a^2(\eta) [-e^{2\zeta} d\eta^2 + e^{-2\zeta} \delta_{ij} dx^i dx^j]$

Statistics: $D(X) = E[(X - E(X))^2]$
 $\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \rangle = \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2) P_\zeta(k_1)$



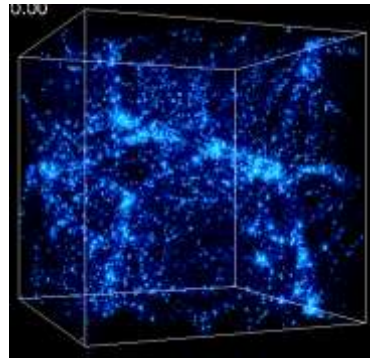
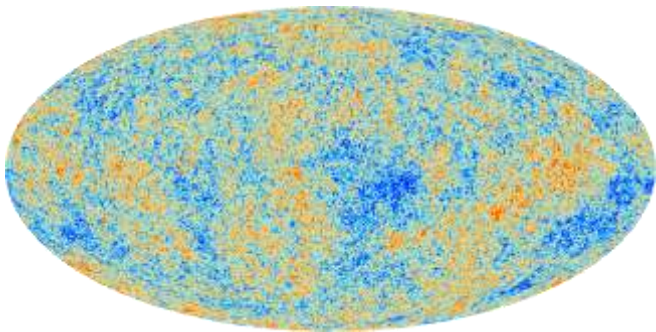
$$\text{Skew}(X) = E\left[\left(\frac{X - E(X)}{\sqrt{D(X)}}\right)^3\right]$$

$$\text{Kurt}(X) = E\left[\left(\frac{X - E(X)}{\sqrt{D(X)}}\right)^4\right]$$

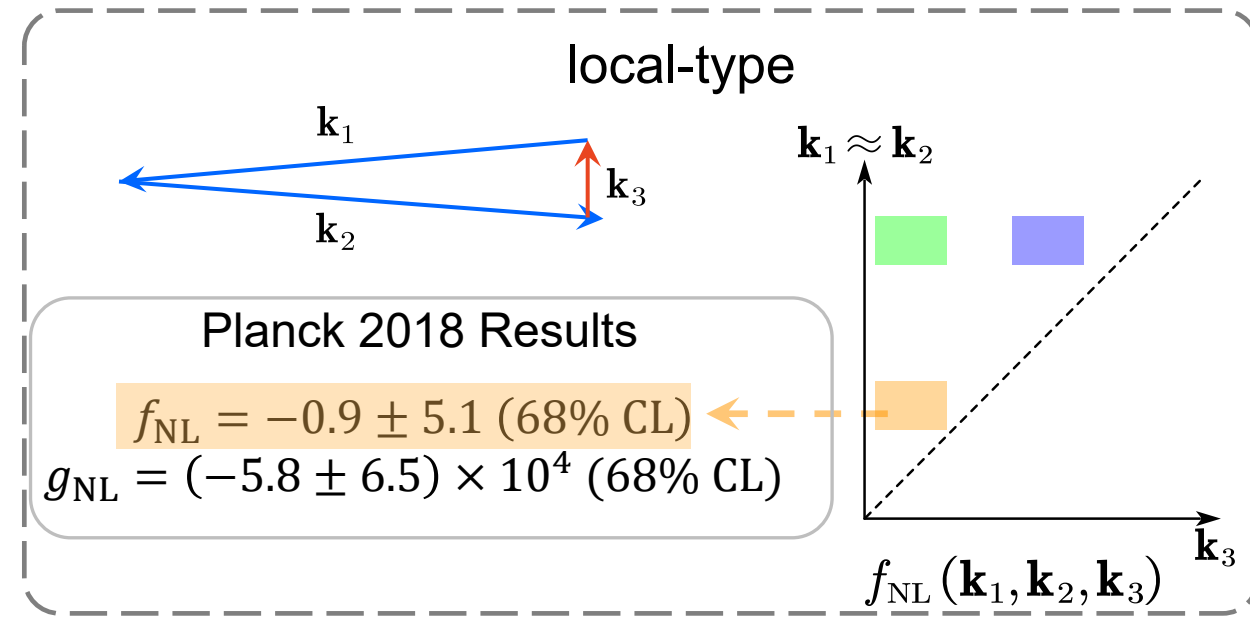
$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle = \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_\zeta(k_1)$$

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \zeta(\mathbf{k}_4) \rangle_c = \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T_\zeta(k_1)$$

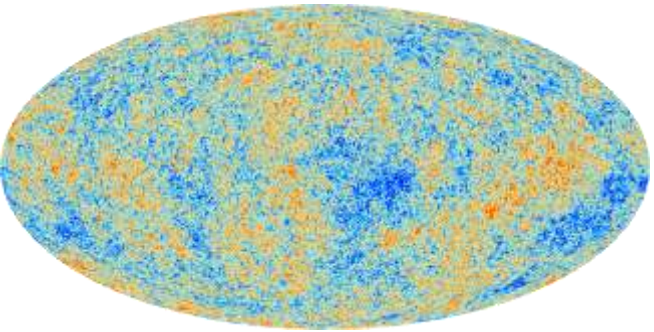
$$\zeta = \zeta_g + \frac{3}{5} f_{\text{NL}} (\zeta_g^2 - \langle \zeta_g \rangle^2) + \frac{9}{25} g_{\text{NL}} \zeta_g^3 + \dots$$



CMB & LSS: $\sim 10^{-16}$ Hz

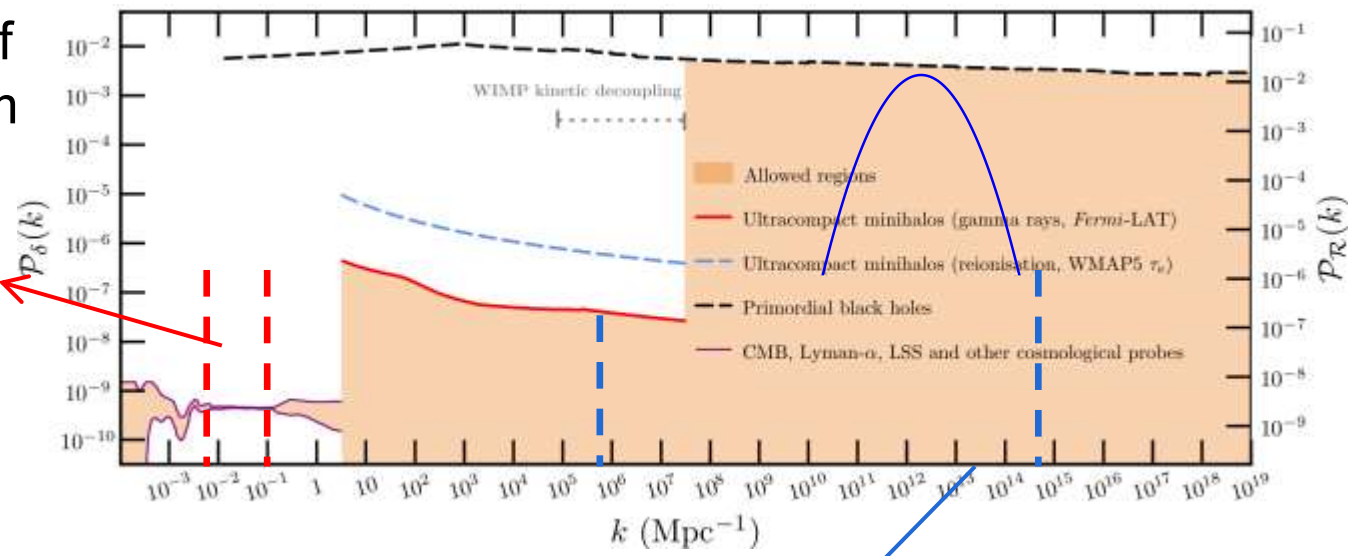


Small- & Large-Scale Decomposition $\zeta = \zeta_S + \zeta_L$

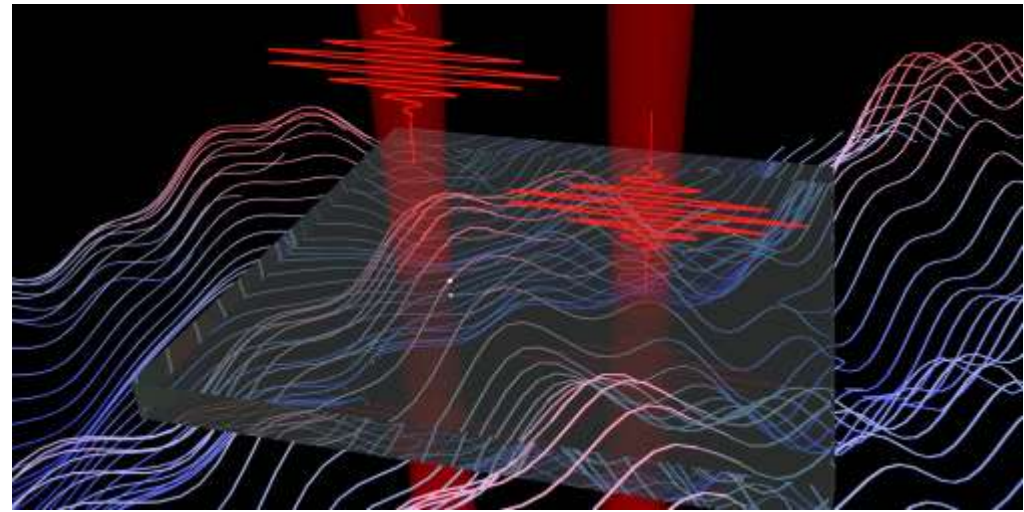
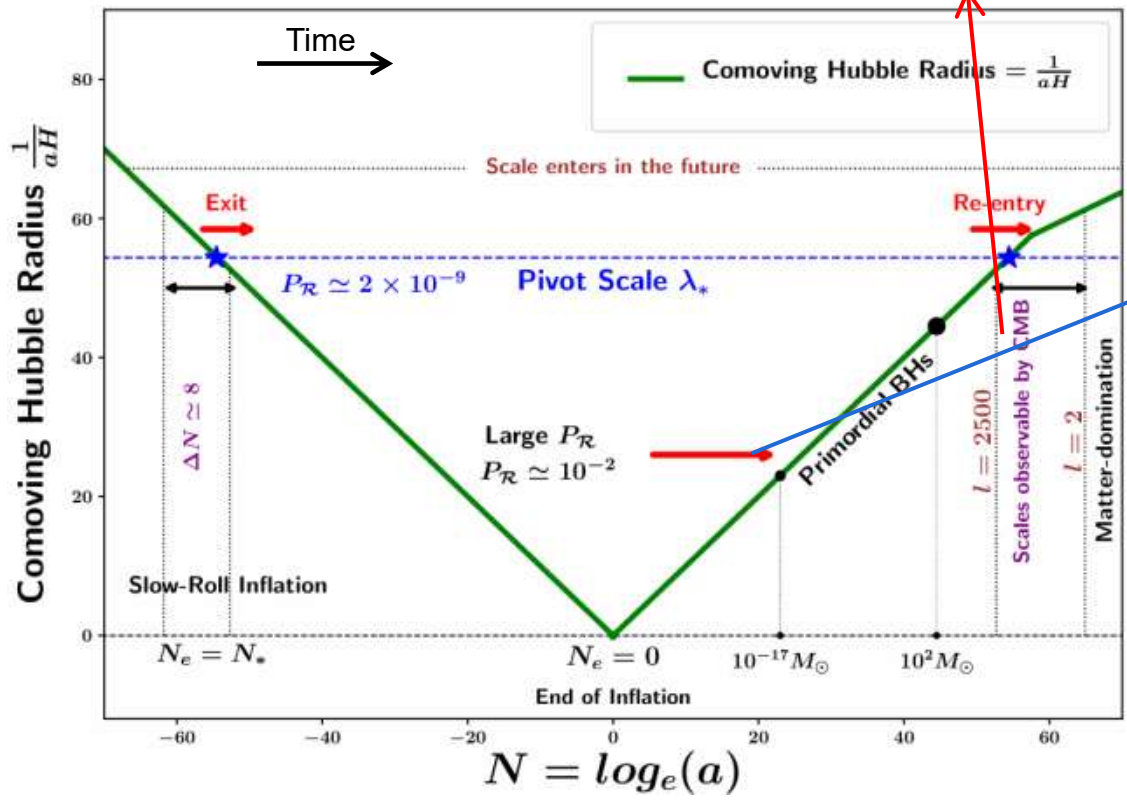


A : Amplitude of power spectrum

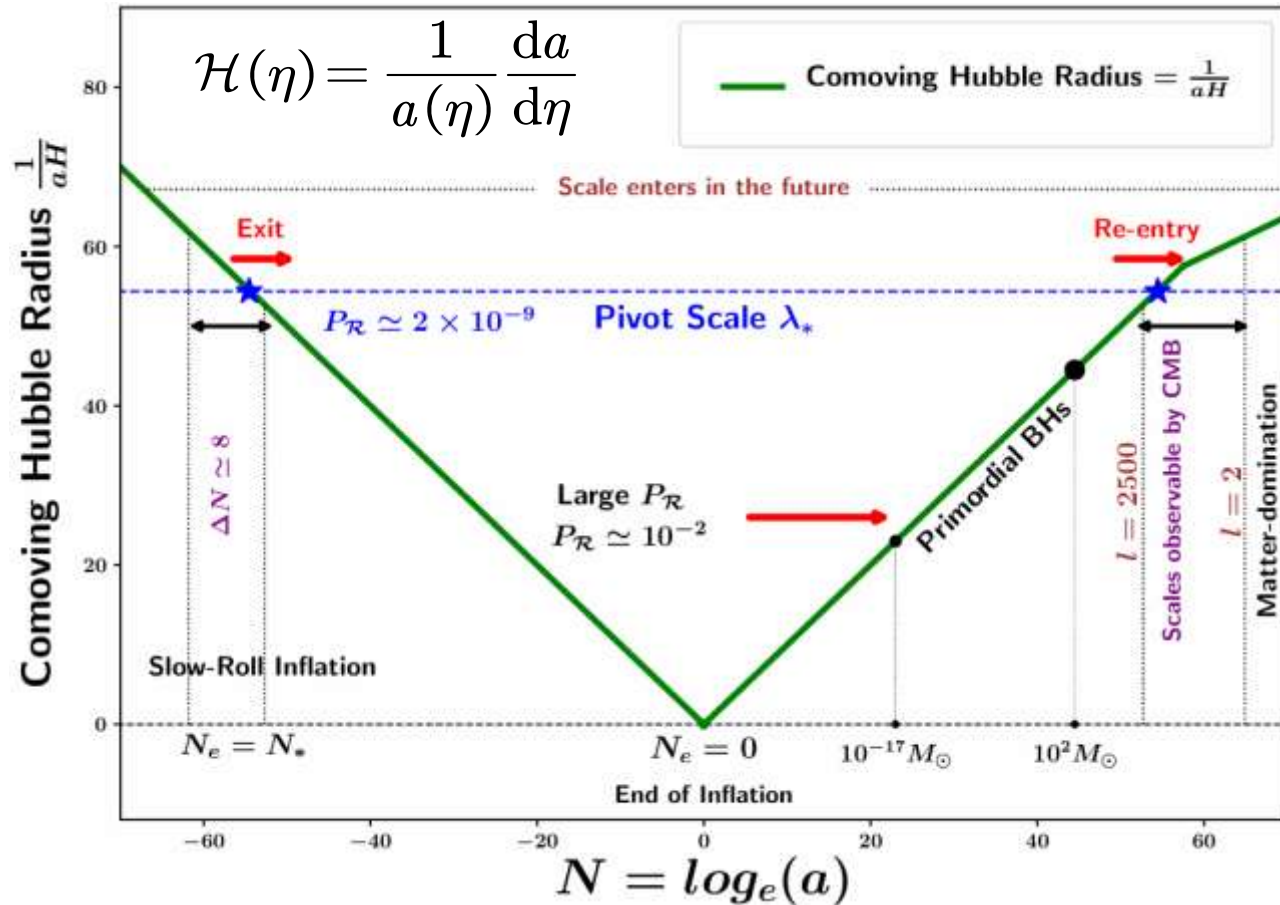
Large-Scale:
 $\sim 10^{-16}$ Hz
 $A_L \sim 10^{-9}$



Small-Scale: $10^{-9} \sim 1$ Hz
 Assum peak: $A_S \lesssim 10^{-2}$



Scalar-Induced Gravitational Waves



$$ds^2 = a^2 \left[-e^{2\Phi} d\eta^2 + \left(e^{-2\Phi} \delta_{ij} + \frac{h_{ij}}{2} \right) dx^i dx^j \right]$$

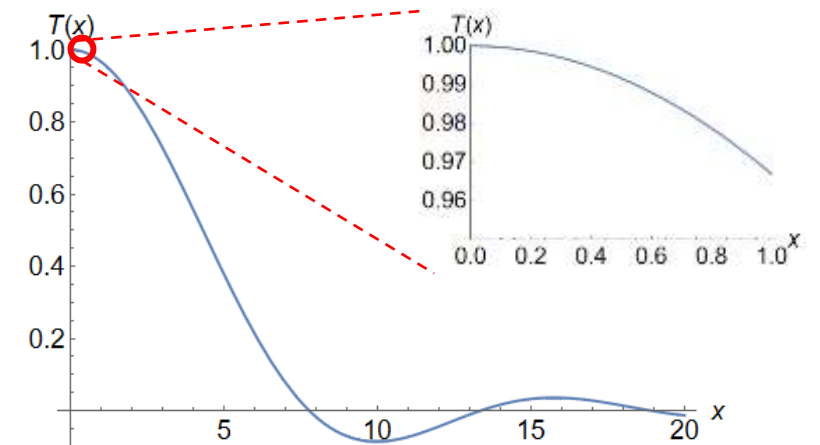
Perfect Fluid $T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + Pg_{\mu\nu}$

SIGW Motion Equation

$$\partial_{\eta}^2 h_{\lambda}(\eta, \mathbf{q}) + 2\mathcal{H}\partial_{\eta} h_{\lambda}(\eta, \mathbf{q}) + q^2 h_{\lambda}(\eta, \mathbf{q}) = 4S_{\lambda}(\eta, \mathbf{q})$$

$$S_{\lambda} \sim Q_{\lambda} \Phi \Phi$$

$$\Phi(\eta, \mathbf{q}) = \left(\frac{3 + 3w}{5 + 3w} \right) T(q\eta) \zeta(\mathbf{q})$$



$$h_{\lambda}(\eta, \mathbf{q}) = 4 \int \frac{d^3 \mathbf{q}_a}{(2\pi)^{3/2}} \zeta(\mathbf{q}_a) \zeta(\mathbf{q} - \mathbf{q}_a) Q_{\lambda}(\mathbf{q}, \mathbf{q}_a) \hat{I}(|\mathbf{q} - \mathbf{q}_a|, q, \eta)$$

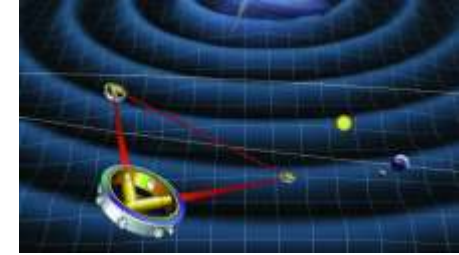
$$\hat{I}(|\mathbf{q} - \mathbf{q}_a|, q, \eta) \sim T(|\mathbf{q} - \mathbf{q}_a| \eta) T(q\eta)$$

arxiv: 1804.08577

Energy-density Full Spectrum

Definition: $\omega_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}) = \frac{1}{\rho_c(\eta)} \frac{d\rho_{\text{gw}}(\eta, \mathbf{x})}{d \ln q d^2 \mathbf{n}} \quad \mathbf{q} = q \mathbf{n}$

Density contrast: $\omega_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}) = \bar{\omega}_{\text{gw}}(\eta, q) (1 + \delta_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}))$



Initial moment (SIGW production) $(\eta_{\text{in}}, \mathbf{x}_{\text{in}})$

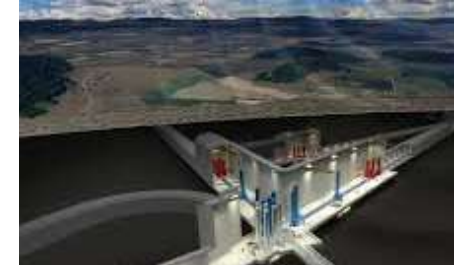
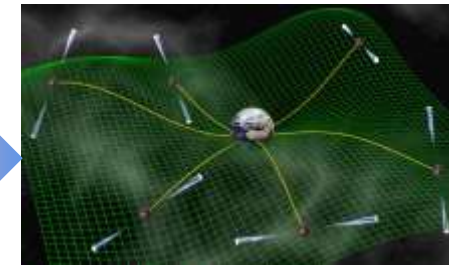
GW energy-density on subhorizon scales

$$\rho_{\text{gw}, \text{in}} = \frac{m_{\text{Pl}}^2}{16a_{\text{in}}} \overline{\partial_l h_{ij} \partial_l h_{ij}}$$

Energy-density full spectrum

$$\bar{\omega}_{\text{gw}, \text{in}} \sim \langle \zeta \zeta \zeta \zeta \rangle$$

$$\delta_{\text{gw}, \text{in}} \sim \langle \zeta \zeta \zeta \zeta \rangle$$



Cosmic
Expansion

Boltzmann
equation
(low multipoles)

Present-day (observation) (η_0, \mathbf{x}_0)

$$q = 2\pi\nu \quad \mathbf{n} = (\mathbf{x}_0 - \mathbf{x}_{\text{in}}) / (\eta_0 - \eta_{\text{in}})$$

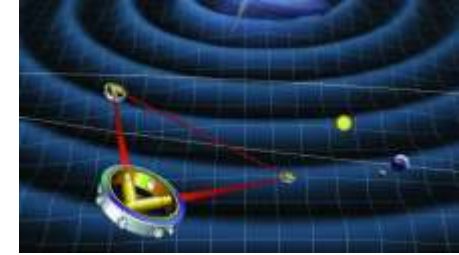
$$\bar{\Omega}_{\text{gw}, 0}(\nu) \simeq 4.2 \times 10^{-5} \times 4\pi \bar{\omega}_{\text{gw}, \text{in}}(q)$$

$$\delta_{\text{gw}, 0}(\mathbf{q}) = \delta_{\text{gw}, \text{in}}(\mathbf{q}) + (4 - n_{\text{gw}}(\nu)) \Phi_{\text{in}}$$

Energy-density Full Spectrum

Definition: $\omega_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}) = \frac{1}{\rho_c(\eta)} \frac{d\rho_{\text{gw}}(\eta, \mathbf{x})}{d \ln q d^2 \mathbf{n}} \quad \mathbf{q} = q \mathbf{n}$

Density contrast: $\omega_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}) = \bar{\omega}_{\text{gw}}(\eta, q) (1 + \delta_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}))$



Initial moment (SIGW production) $(\eta_{\text{in}}, \mathbf{x}_{\text{in}})$

GW energy-density on subhorizon scales
(including NL effects: 2412.12654)

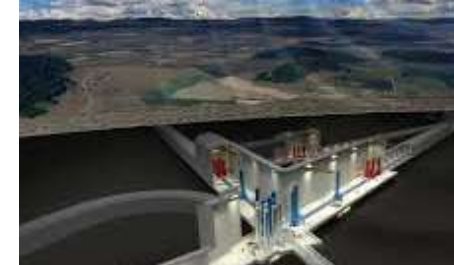
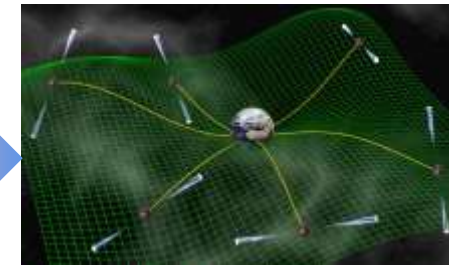
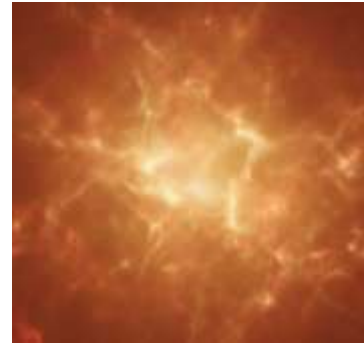
$$\tilde{\rho}_{\text{gw}, \text{in}} = \frac{m_{\text{Pl}}^2 e^{2\Phi_{\text{L}, \text{in}}}}{16a_{\text{in}}} \overline{\partial_l h_{ij} \partial_l h_{ij}}$$

Energy-density full spectrum

$$\bar{\omega}_{\text{gw}, \text{in}} \sim \langle \zeta \zeta \zeta \zeta \rangle$$

$$\tilde{\delta}_{\text{gw}, \text{in}} = \delta_{\text{gw}, \text{in}} + 2\Phi_{\text{L}, \text{in}}$$

$$\delta_{\text{gw}, \text{in}} \sim \langle \zeta \zeta \zeta \zeta \rangle$$



Cosmic
Expansion

Boltzmann
equation
(low multipoles)

Present-day (observation) (η_0, \mathbf{x}_0)

$$q = 2\pi\nu \quad \mathbf{n} = (\mathbf{x}_0 - \mathbf{x}_{\text{in}}) / (\eta_0 - \eta_{\text{in}})$$

$$\bar{\Omega}_{\text{gw}, 0}(\nu) \simeq 4.2 \times 10^{-5} \times 4\pi \bar{\omega}_{\text{gw}, \text{in}}(q)$$

$$\delta_{\text{gw}, 0}(\mathbf{q}) = \delta_{\text{gw}, \text{in}}(\mathbf{q}) + (4 - n_{\text{gw}}(\nu)) \Phi_{\text{in}}$$

$$\delta_{\text{gw}, 0}(\mathbf{q}) = \delta_{\text{gw}, \text{in}}(\mathbf{q}) + (6 - n_{\text{gw}}(\nu)) \Phi_{\text{in}}$$

The SIGW Background

The SIGW Background

$$\bar{\Omega}_{\text{gw},0}(\nu) \propto 4\pi \bar{\omega}_{\text{gw},\text{in}}(q)$$

$$\delta_{\text{gw},0}(\mathbf{q}) = \delta_{\text{gw},\text{in}}(\mathbf{q}) + (6 - n_{\text{gw}}(\nu)) \Phi_{\text{in},L}$$

$$\sim \langle \zeta \zeta \zeta \zeta \rangle$$

Isotropic Component

Anisotropies

$$\bar{\omega}_{\text{gw}}(\eta, q) = \langle \omega_{\text{gw}}(\eta, \mathbf{x}, \mathbf{q}) \rangle$$

$$\bar{\Omega}_{\text{gw},0}(\nu) \propto 4\pi \bar{\omega}_{\text{gw},\text{in}}(q)$$

Energy density
fraction spectrum
1-pt

$$\tilde{C}_\ell(\nu) \sim \left\langle \prod_{i=1}^2 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle$$

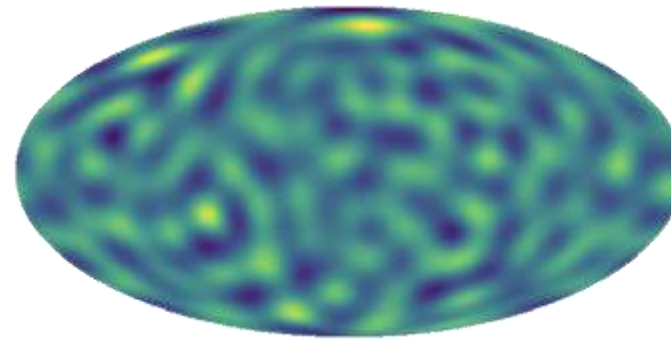
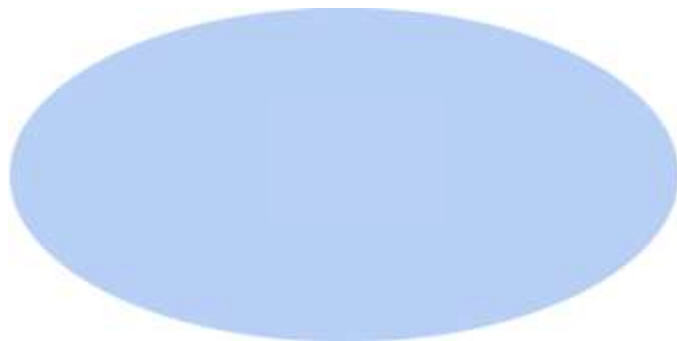
Angular power spectrum
2-pt

Non-Gaussianity

$$B_\ell(\nu) \sim \left\langle \prod_{i=1}^3 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle$$

$$t_{\ell_3 \ell_4}^{\ell_1 \ell_2}(L, \nu) \sim \left\langle \prod_{i=1}^4 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle_c$$

Angular bispectrum
and trispectrum
3-pt, 4-pt, ...



Isotropic Component

The SIGW Background

Isotropic Component

R.-G. Cai, S. Pi, M. Sasaki, PRL 122 (2019) 201101
 P. Adshead, K. D. Lozanov, Z. J. Weiner, JCAP 10 (2021), 080
 K. T. Abe, R. Inui, Y. Tada, S. Yokoyama, JCAP 5 (2023), 044
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056
 C. Yuan, D.-S. Meng, Q.-G. Huang, JCAP 12 (2023), 036
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039
 Y.-H. Yu, S. Wang, PRD 109 (2024) 083501

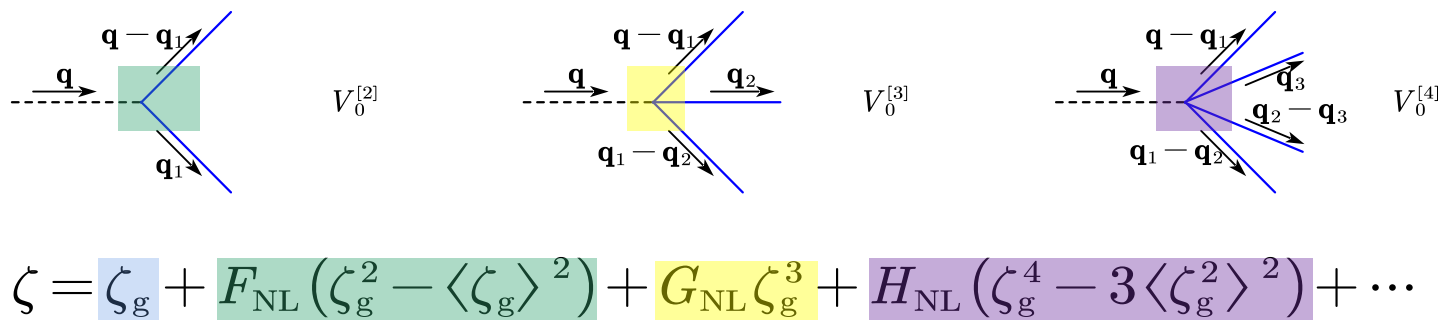
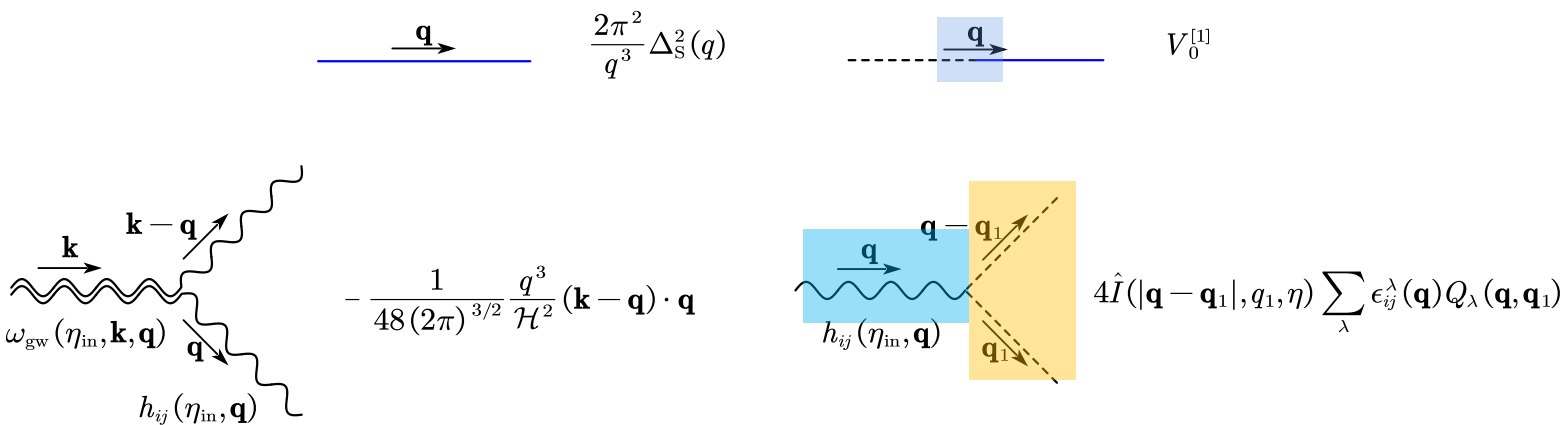
Energy density fraction spectrum

$$\bar{\Omega}_{\text{gw},0}(\nu) = 4\pi h^2 \Omega_{\text{rad},0} \bar{\omega}_{\text{gw},\text{in}}(q) \left(\frac{g_{*,\rho}(T_{\text{in}})}{g_{*,\rho}(T_{\text{eq}})} \right) \left(\frac{g_{*,s}(T_{\text{eq}})}{g_{*,s}(T_{\text{in}})} \right)^{4/3}$$

$$\bar{\omega}_{\text{gw},\text{in}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \langle \zeta_S(\mathbf{q}_1) \zeta_S(\mathbf{q} - \mathbf{q}_1) \zeta_S(\mathbf{q}_2) \zeta_S(\mathbf{q} - \mathbf{q}_2) \rangle'$$

$$\times \sum_{\lambda=+,\times} Q_\lambda(\mathbf{q}, \mathbf{q}_1) Q_\lambda(\mathbf{q}, \mathbf{q}_2) \hat{I}(|\mathbf{q} - \mathbf{q}_1|, q_1, \eta) \hat{I}(|\mathbf{q} - \mathbf{q}_2|, q_2, \eta)$$

Basic Diagrammatic Approach



$$\zeta = \zeta_g + F_{\text{NL}} (\zeta_g^2 - \langle \zeta_g \rangle^2) + G_{\text{NL}} \zeta_g^3 + H_{\text{NL}} (\zeta_g^4 - 3 \langle \zeta_g \rangle^2) + \dots$$

$$h_\lambda(\eta_{\text{in}}, \mathbf{q}) = 4 \int \frac{d^3 \mathbf{q}_a}{(2\pi)^{3/2}} \zeta(\mathbf{q}_a) \zeta(\mathbf{q} - \mathbf{q}_a) Q_\lambda(\mathbf{q}, \mathbf{q}_a) \hat{I}(|\mathbf{q} - \mathbf{q}_a|, q_a, \eta_{\text{in}})$$

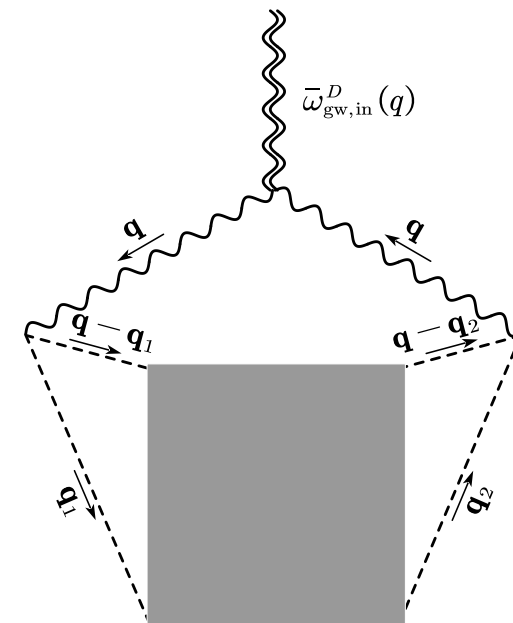
$$\bar{\omega}_{\text{gw, in}}^D(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+, \times} Q_\lambda Q_\lambda \hat{\bar{I}} \right) \mathcal{P}^D$$

P. Adshead, K. D. Lozanov, Z. J. Weiner, JCAP 10 (2021), 080

K. T. Abe, R. Inui, Y. Tada, S. Yokoyama, JCAP 5 (2023), 044

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056

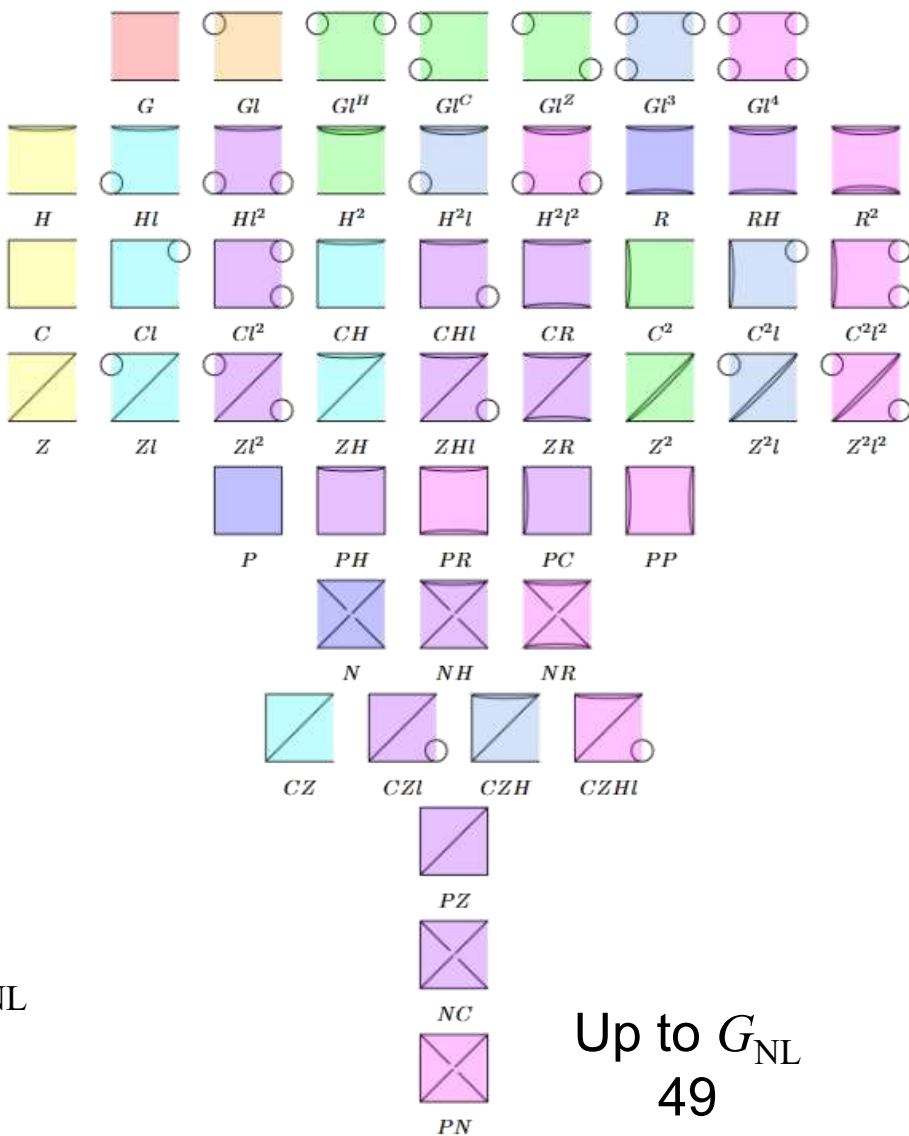
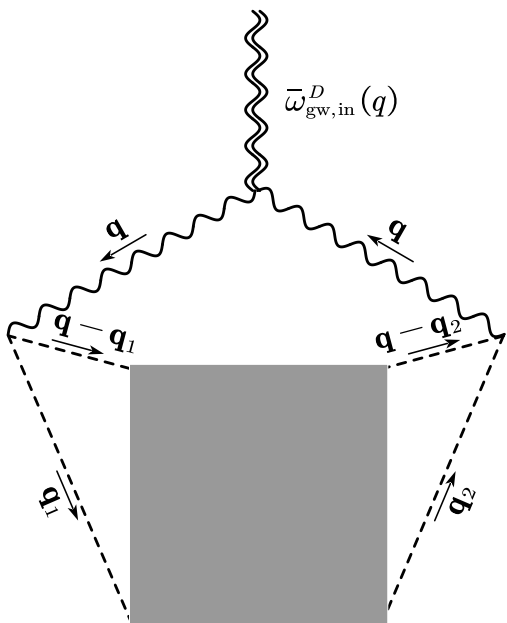
JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 06 (2024) 039



$$\bar{\omega}_{\text{gw, in}}(q) = \sum_D \bar{\omega}_{\text{gw, in}}^D(q)$$

Basic Diagrammatic Approach: Diagrams

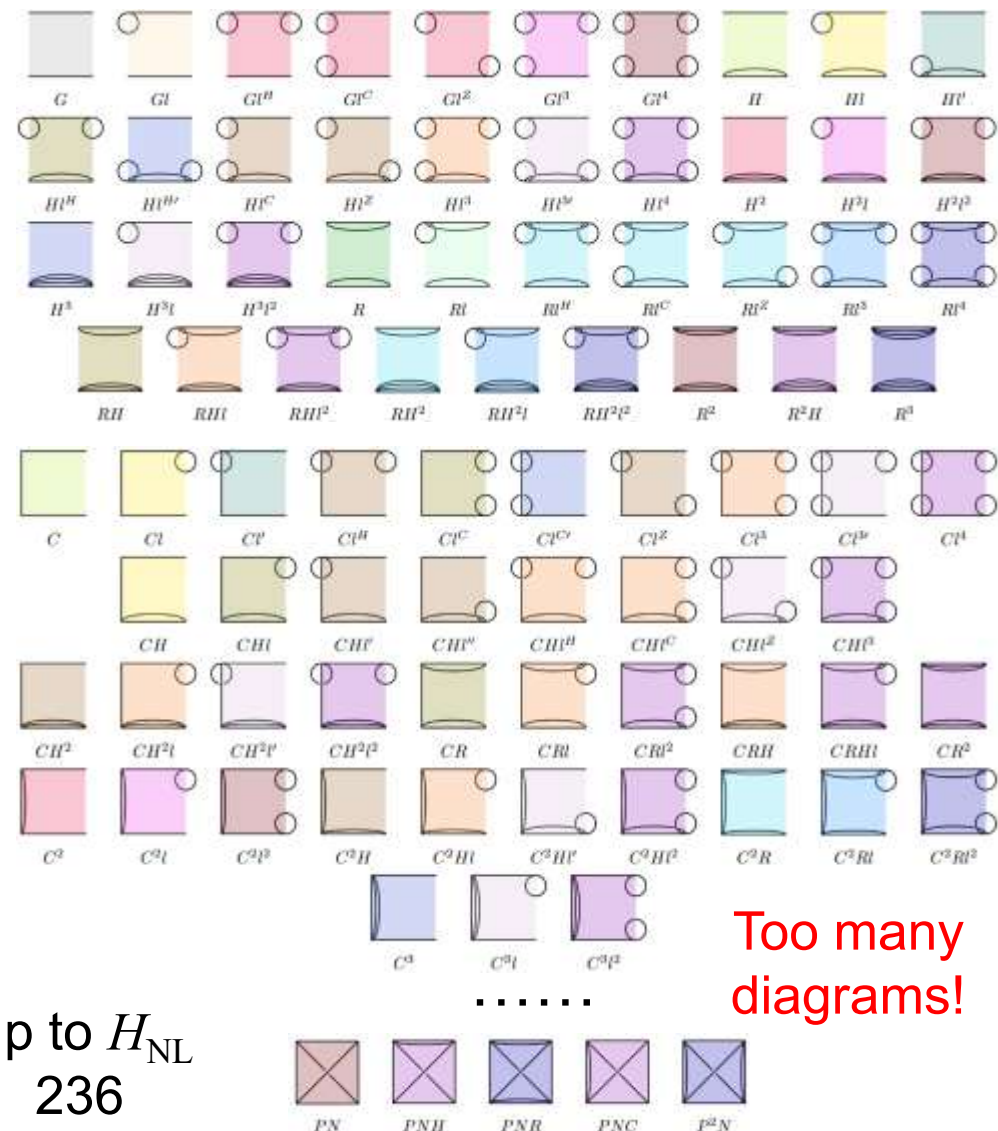
JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792



Up to F_{NL}
7

Up to G_{NL}
49

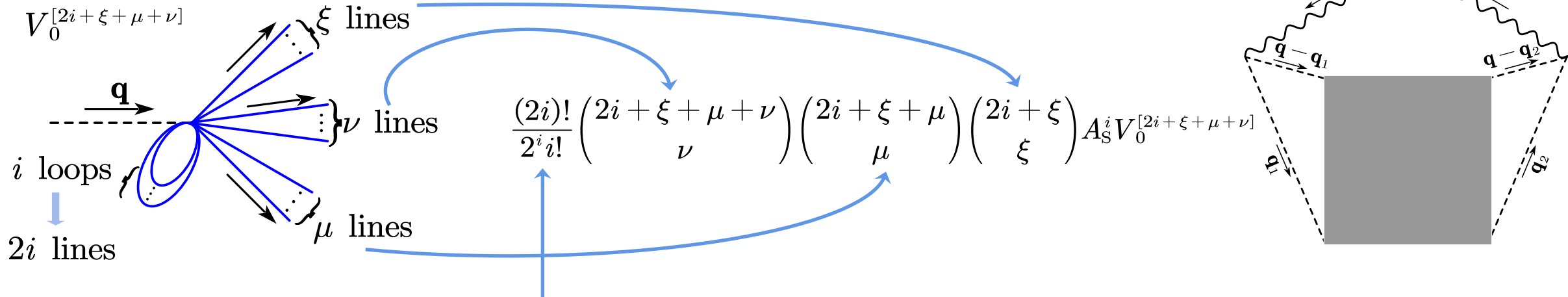
Up to H_{NL}
236



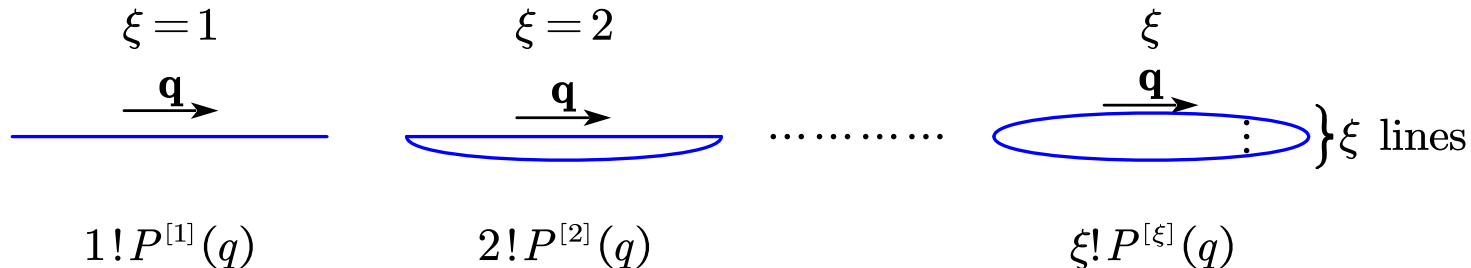
Too many diagrams!

“Renormalized” Diagrammatic Approach

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$



All loops are interchangeable, and likewise the two lines in every loop.



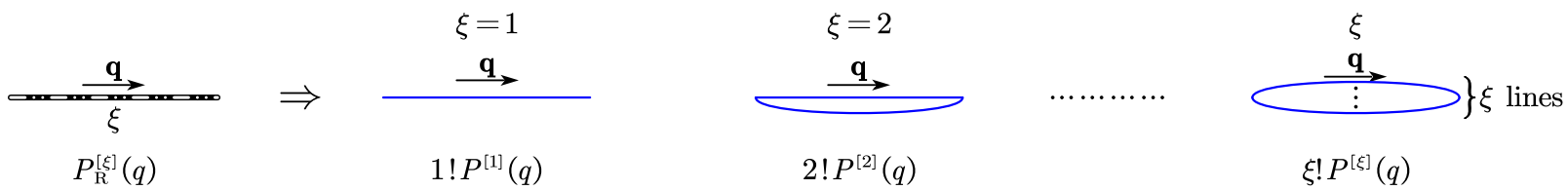
$$P^{[1]}(q) = \frac{2\pi^2}{q^3} \Delta_S^2(q)$$

$$P^{[\xi]}(q) = \int \frac{d^3 \tilde{\mathbf{q}}}{(2\pi)^3} P^{[\xi-1]}(\tilde{q}) P^{[1]}(|\mathbf{q} - \tilde{\mathbf{q}}|),$$

where $\xi \geq 2, \xi \in \mathbb{N}_+$

$$P_{R,a}^{[\xi]} = \xi! P_a^{[\xi]}$$

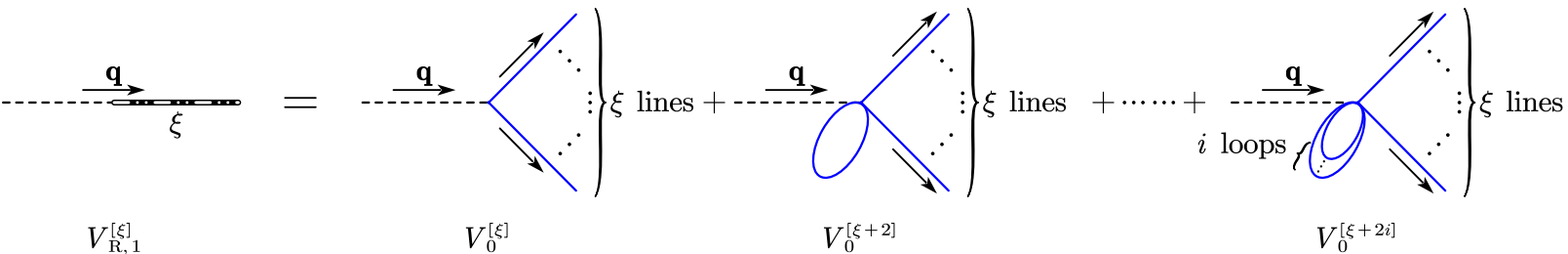
“Renormalized” Diagrammatic Approach



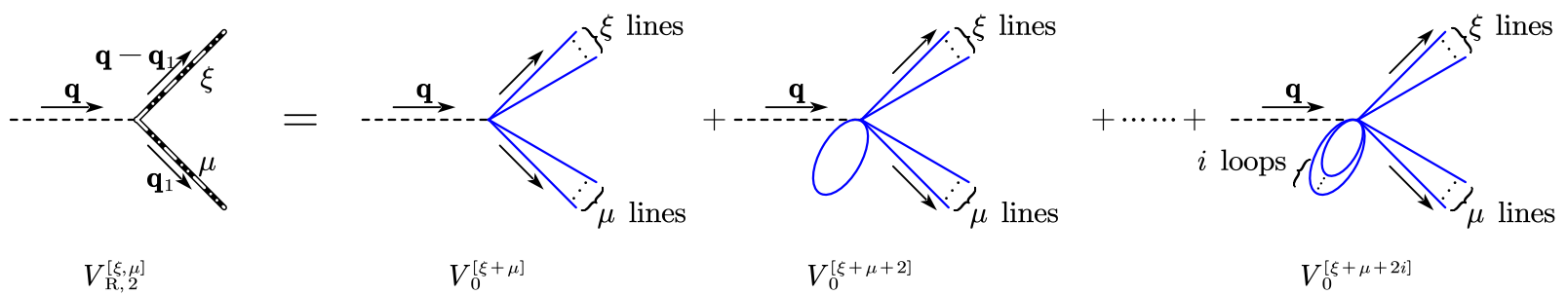
$$P^{[1]}(q) = \frac{2\pi^2}{q^3} \Delta_S^2(q)$$

$$P^{[\xi]}(q) = \int \frac{d^3 \tilde{\mathbf{q}}}{(2\pi)^3} P^{[\xi-1]}(\tilde{q}) P^{[1]}(|\mathbf{q} - \tilde{\mathbf{q}}|),$$

where $\xi \geq 2, \xi \in \mathbb{N}_+$

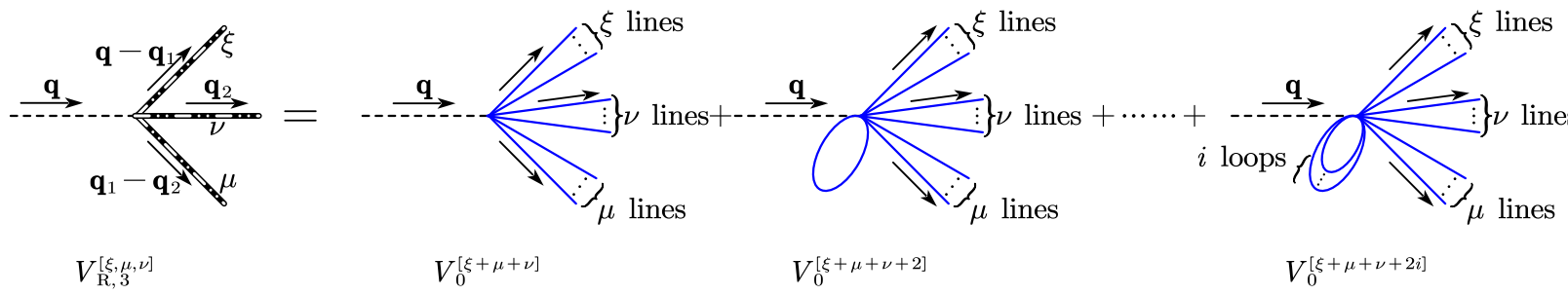


$$P_{R,a}^{[\xi]} = \xi! P_a^{[\xi]}$$



$$V_{R,3}^{[\xi]} = \sum_{i=0}^{\lfloor (o-\xi)/2 \rfloor} \frac{(2i)!}{2^i i!} \binom{2i+\xi}{\xi} A_S^i V_0^{[2i+\xi+\mu+\nu]}$$

$$V_{R,2}^{[\xi, \mu]} = \sum_{i=0}^{\lfloor (o-\xi-\mu)/2 \rfloor} \frac{(2i)!}{2^i i!} \binom{2i+\xi+\mu}{\mu} \times \binom{2i+\xi}{\xi} A_S^i V_0^{[2i+\xi+\mu+\nu]}$$

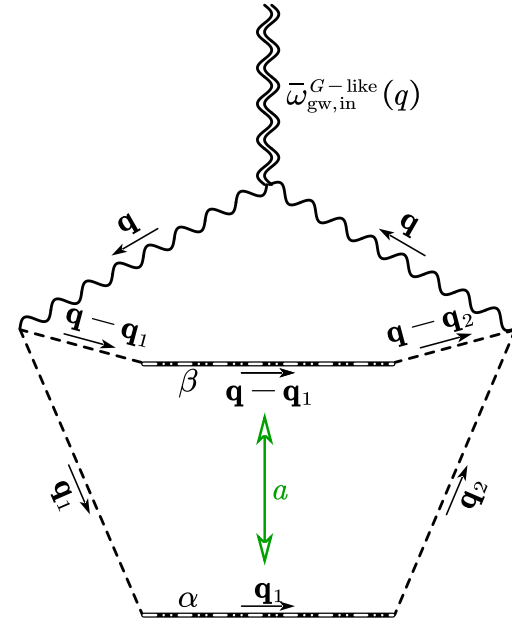


$$V_{R,3}^{[\xi, \mu, \nu]} = \sum_{i=0}^{\lfloor (o-\xi-\mu-\nu)/2 \rfloor} \frac{(2i)!}{2^i i!} \binom{2i+\xi+\mu+\nu}{\nu} \times \binom{2i+\xi+\mu}{\mu} \binom{2i+\xi}{\xi} A_S^i V_0^{[2i+\xi+\mu+\nu]}$$

“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{G\text{-like}} = 2 \left(\sum_{\alpha=1}^o V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} \right) \left(\sum_{\beta=1}^o V_{\text{R},1}^{[\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]} \right)$$

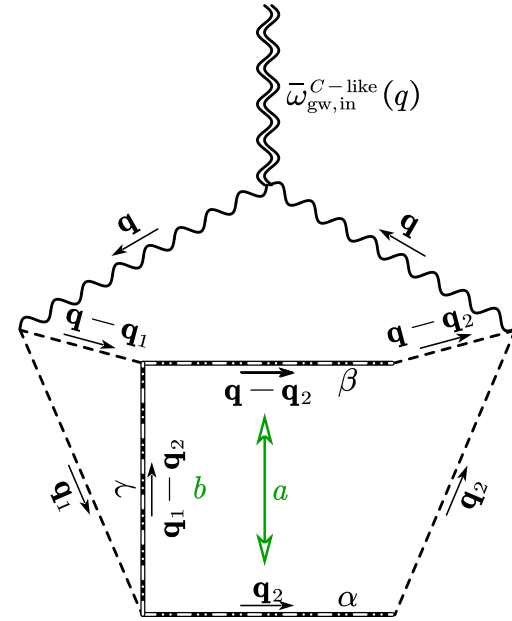


“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{G\text{-like}} = 2 \left(\sum_{\alpha=1}^o V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} \right) \left(\sum_{\beta=1}^o V_{\text{R},1}^{[\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]} \right)$$

$$\mathcal{P}^{C\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$



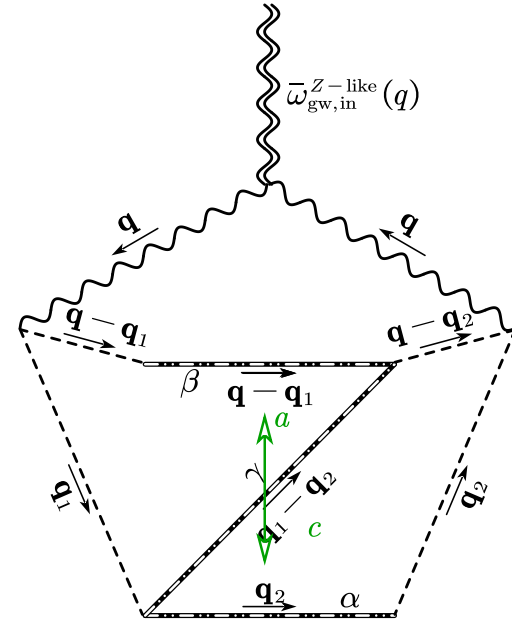
“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{G\text{-like}} = 2 \left(\sum_{\alpha=1}^o V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} \right) \left(\sum_{\beta=1}^o V_{\text{R},1}^{[\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]} \right)$$

$$\mathcal{P}^{C\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$

$$\mathcal{P}^{Z\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$



“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

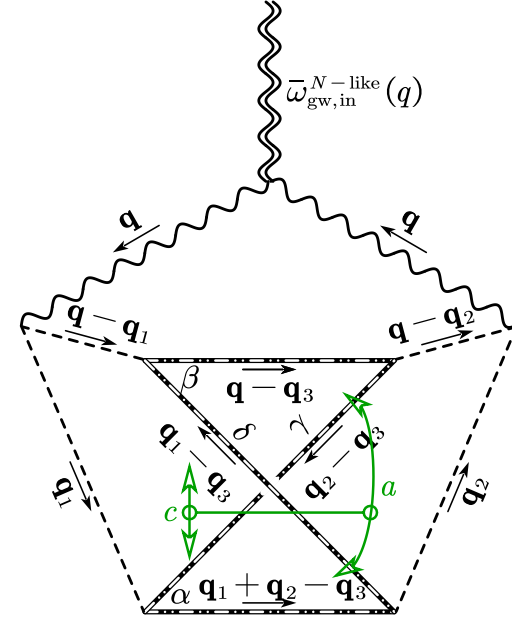
$$\mathcal{P}^{G\text{-like}} = 2 \left(\sum_{\alpha=1}^o V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} \right) \left(\sum_{\beta=1}^o V_{\text{R},1}^{[\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]} \right)$$

$$\mathcal{P}^{C\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$

$$\mathcal{P}^{Z\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$

$$\mathcal{P}^{P\text{-like}} = 2 \sum_{\delta=1}^{o-1} \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta)} \sum_{\alpha=1}^{\min(o-\gamma, o-\delta)} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},2}^{[\beta,\delta]} P_{\text{R},b}^{[\delta]} V_{\text{R},2}^{[\delta,\alpha]}$$

$$\mathcal{P}^{N\text{-like}} = \sum_{\delta=1}^{o-1} \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta)} \sum_{\alpha=1}^{\min(o-\gamma, o-\delta)} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},2}^{[\beta,\delta]} P_{\text{R},c}^{[\delta]} V_{\text{R},2}^{[\delta,\alpha]}$$



“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{G\text{-like}} = 2 \left(\sum_{\alpha=1}^o V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} \right) \left(\sum_{\beta=1}^o V_{\text{R},1}^{[\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]} \right)$$

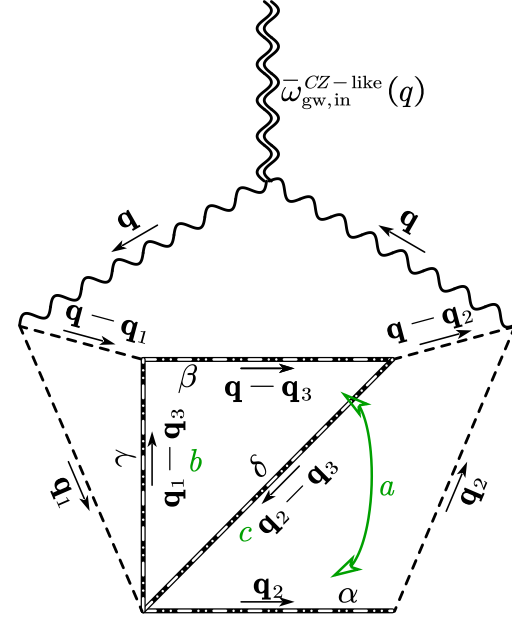
$$\mathcal{P}^{C\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$

$$\mathcal{P}^{Z\text{-like}} = 4 \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{o-\gamma} \sum_{\alpha=1}^{o-\gamma} V_{\text{R},1}^{[\alpha]} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},1}^{[\beta]}$$

$$\mathcal{P}^{P\text{-like}} = 2 \sum_{\delta=1}^{o-1} \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta)} \sum_{\alpha=1}^{\min(o-\gamma, o-\delta)} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},b}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},2}^{[\beta,\delta]} P_{\text{R},b}^{[\delta]} V_{\text{R},2}^{[\delta,\alpha]}$$

$$\mathcal{P}^{N\text{-like}} = \sum_{\delta=1}^{o-1} \sum_{\gamma=1}^{o-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta)} \sum_{\alpha=1}^{\min(o-\gamma, o-\delta)} P_{\text{R},a}^{[\alpha]} V_{\text{R},2}^{[\alpha,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},2}^{[\gamma,\beta]} P_{\text{R},a}^{[\beta]} V_{\text{R},2}^{[\beta,\delta]} P_{\text{R},c}^{[\delta]} V_{\text{R},2}^{[\delta,\alpha]}$$

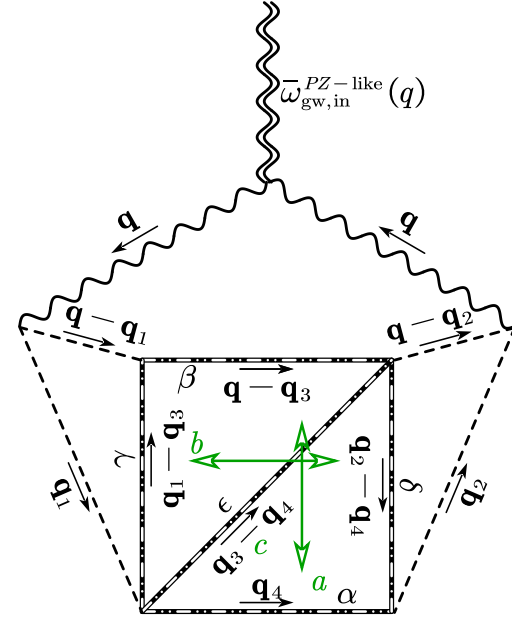
$$\mathcal{P}^{CZ\text{-like}} = 8 \sum_{\delta=1}^{o-2} \sum_{\gamma=1}^{o-2} \sum_{\alpha=1}^{o-\gamma-\delta} \sum_{\beta=1}^{\min(o-\gamma, o-\delta)} P_{\text{R},a}^{[\alpha]} V_{\text{R},1}^{[\alpha]} P_{\text{R},b}^{[\beta]} V_{\text{R},2}^{[\beta,\gamma]} P_{\text{R},c}^{[\gamma]} V_{\text{R},3}^{[\alpha,\gamma,\delta]} P_{\text{R},d}^{[\delta]} V_{\text{R},2}^{[\delta,\beta]}$$



“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{PZ\text{-like}} = 4 \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{o-\epsilon-1} \sum_{\gamma=1}^{o-\epsilon-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta-\epsilon)} \sum_{\alpha=1}^{\min(o-\gamma-\epsilon, o-\delta)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,b}^{[\gamma]} V_{R,2}^{[\beta,\gamma]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,b}^{[\delta]} V_{R,2}^{[\alpha,\delta]} P_{R,c}^{[\epsilon]}$$

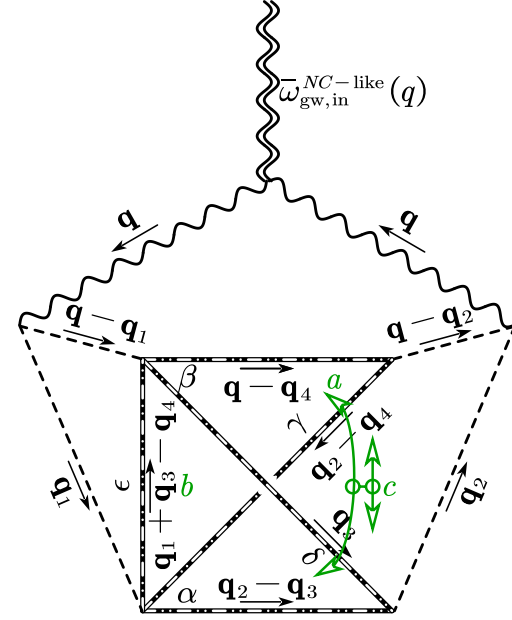


“Renormalized” Diagrammatic Approach: Diagrams

$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{PZ\text{-like}} = 4 \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{o-\epsilon-1} \sum_{\gamma=1}^{o-\epsilon-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta-\epsilon)} \sum_{\alpha=1}^{\min(o-\gamma-\epsilon, o-\delta)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,b}^{[\gamma]} V_{R,2}^{[\beta,\gamma]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,b}^{[\delta]} V_{R,2}^{[\alpha,\delta]} P_{R,c}^{[\epsilon]}$$

$$\mathcal{P}^{Z\text{-like}} = 2 \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{o-\epsilon-1} \sum_{\gamma=1}^{o-\epsilon-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta-\epsilon)} \sum_{\alpha=1}^{\min(o-\gamma-\epsilon, o-\delta)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,c}^{[\gamma]} V_{R,2}^{[\beta,\gamma]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,c}^{[\delta]} V_{R,2}^{[\alpha,\delta]} P_{R,b}^{[\epsilon]}$$



“Renormalized” Diagrammatic Approach: Diagrams

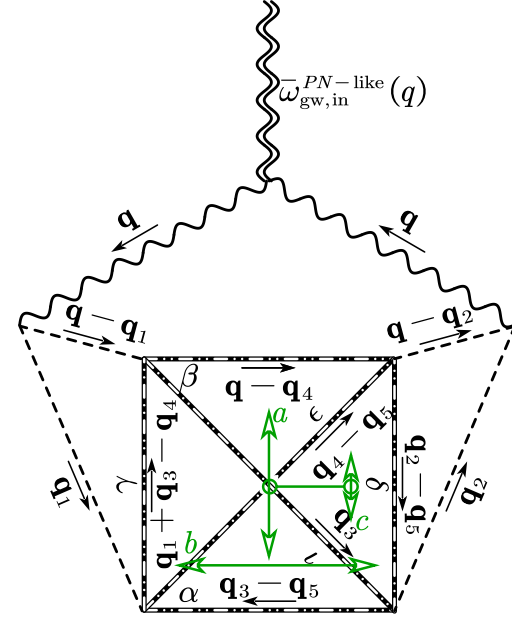
$$\bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q) = \frac{q^5}{3(2\pi)^3 \mathcal{H}^2} \left\{ \prod_{i=1}^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \right\} \left(\sum_{\lambda=+,\times} Q_\lambda Q_\lambda \hat{H} \right) \mathcal{P}^{X\text{-like}}$$

$$\mathcal{P}^{PZ\text{-like}} = 4 \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{o-\epsilon-1} \sum_{\gamma=1}^{o-\epsilon-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta-\epsilon)} \sum_{\alpha=1}^{\min(o-\gamma-\epsilon, o-\delta)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,b}^{[\gamma]} V_{R,2}^{[\beta,\gamma]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,b}^{[\delta]} V_{R,2}^{[\alpha,\delta]} P_{R,c}^{[\epsilon]}$$

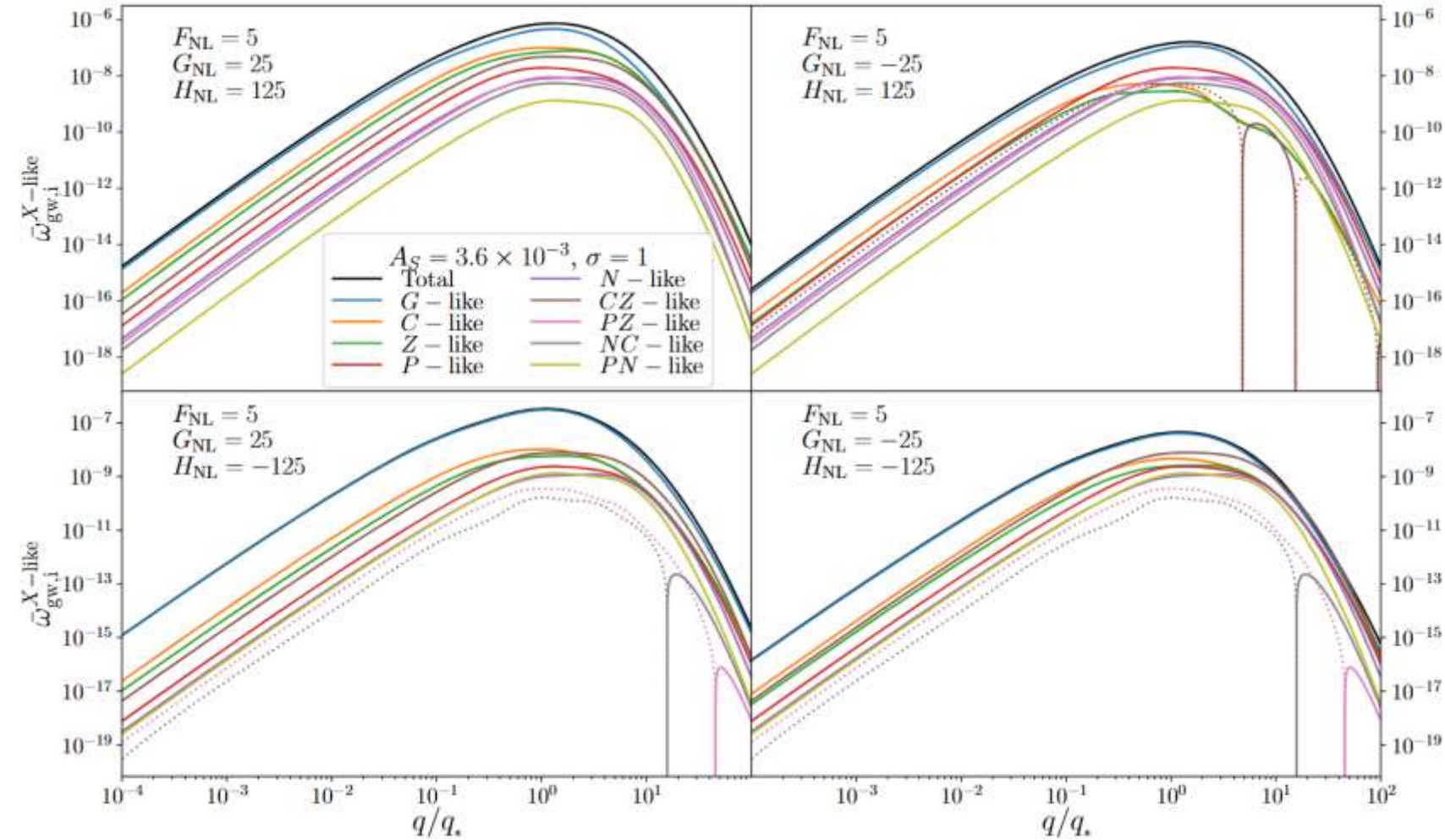
$$\mathcal{P}^{Z\text{-like}} = 2 \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{o-\epsilon-1} \sum_{\gamma=1}^{o-\epsilon-1} \sum_{\beta=1}^{\min(o-\gamma, o-\delta-\epsilon)} \sum_{\alpha=1}^{\min(o-\gamma-\epsilon, o-\delta)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,c}^{[\gamma]} V_{R,2}^{[\beta,\gamma]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,c}^{[\delta]} V_{R,2}^{[\alpha,\delta]} P_{R,b}^{[\epsilon]}$$

$$\mathcal{P}^{PN\text{-like}} = \sum_{\iota=1}^{o-2} \sum_{\epsilon=1}^{o-2} \sum_{\delta=1}^{\min(o-\epsilon, o-\iota)-1} \sum_{\gamma=1}^{\min(o-\epsilon, o-\iota)-1} \sum_{\beta=1}^{\min(o-\gamma-\epsilon, o-\delta-\iota)} \sum_{\alpha=1}^{\min(o-\gamma-\iota, o-\delta-\epsilon)} P_{R,a}^{[\alpha]} V_{R,3}^{[\alpha,\gamma,\epsilon]} P_{R,b}^{[\gamma]} V_{R,3}^{[\beta,\gamma,\iota]} P_{R,a}^{[\beta]} \\ \times V_{R,3}^{[\beta,\delta,\epsilon]} P_{R,b}^{[\delta]} V_{R,3}^{[\alpha,\delta,\iota]} P_{R,c}^{[\epsilon]} P_{R,c}^{[\iota]}$$

$$\bar{\omega}_{\text{gw,in}}(q) = \sum_{X\text{-like}} \bar{\omega}_{\text{gw,in}}^{X\text{-like}}(q)$$



Numerical Results

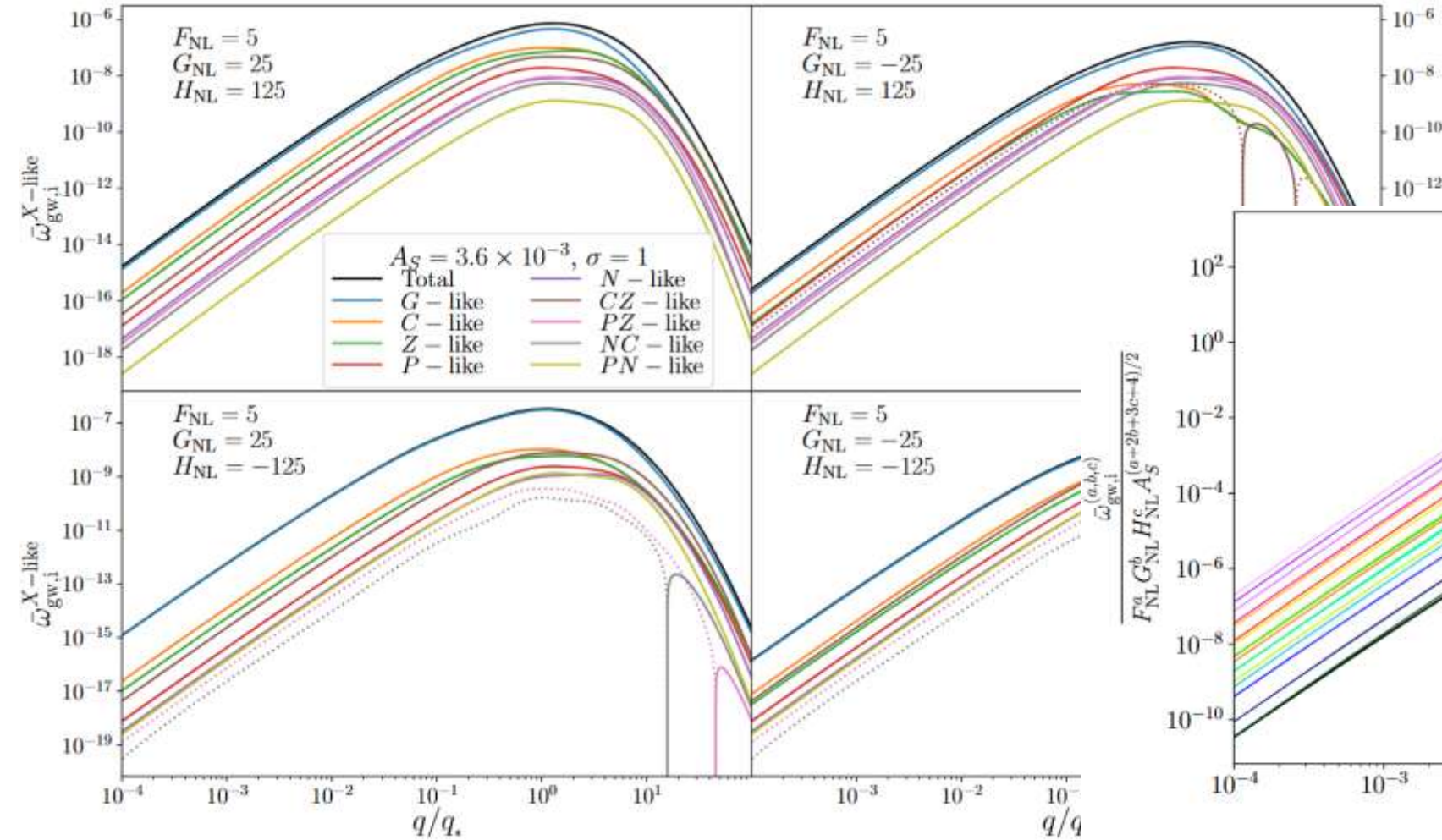


$$\Delta_S^2(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$\sigma = 1$

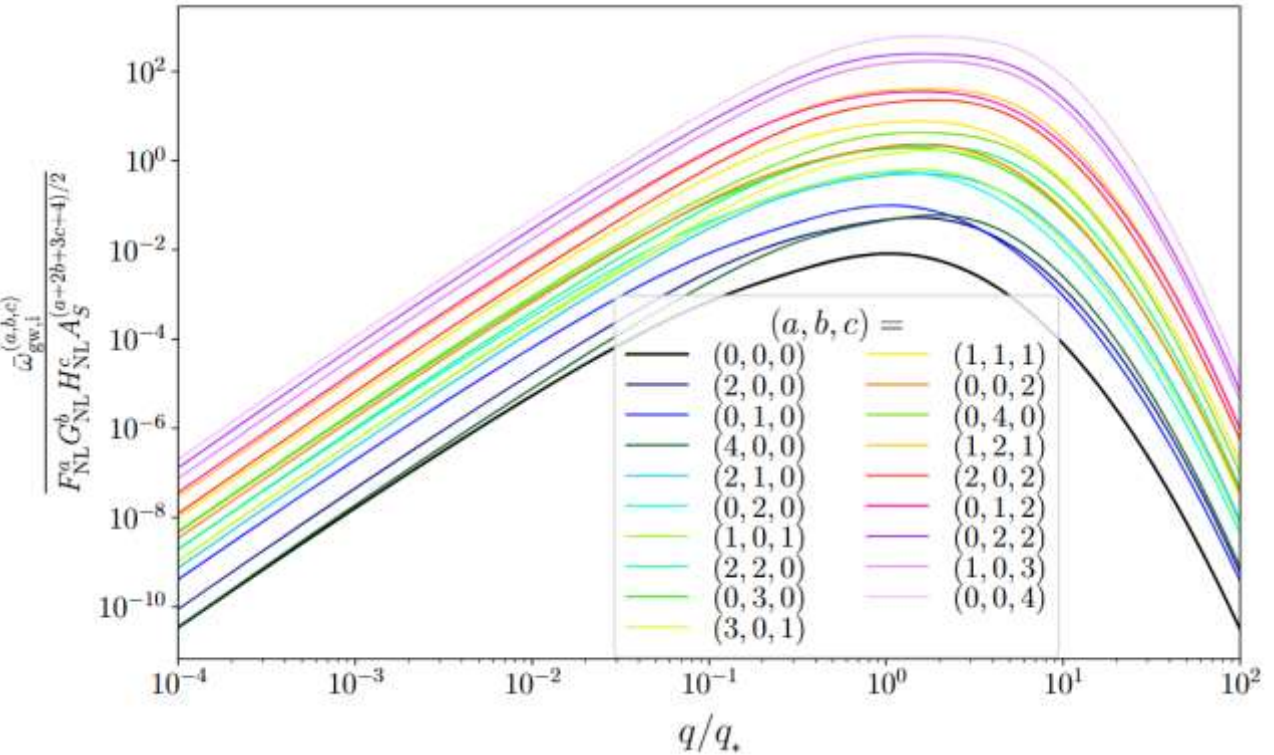
$$\bar{\omega}_{\text{gw},\text{in}}(q) = \sum_{X\text{-like}} \bar{\omega}_{\text{gw},\text{in}}^{X\text{-like}}(q)$$

Numerical Results



$$\Delta_S^2(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

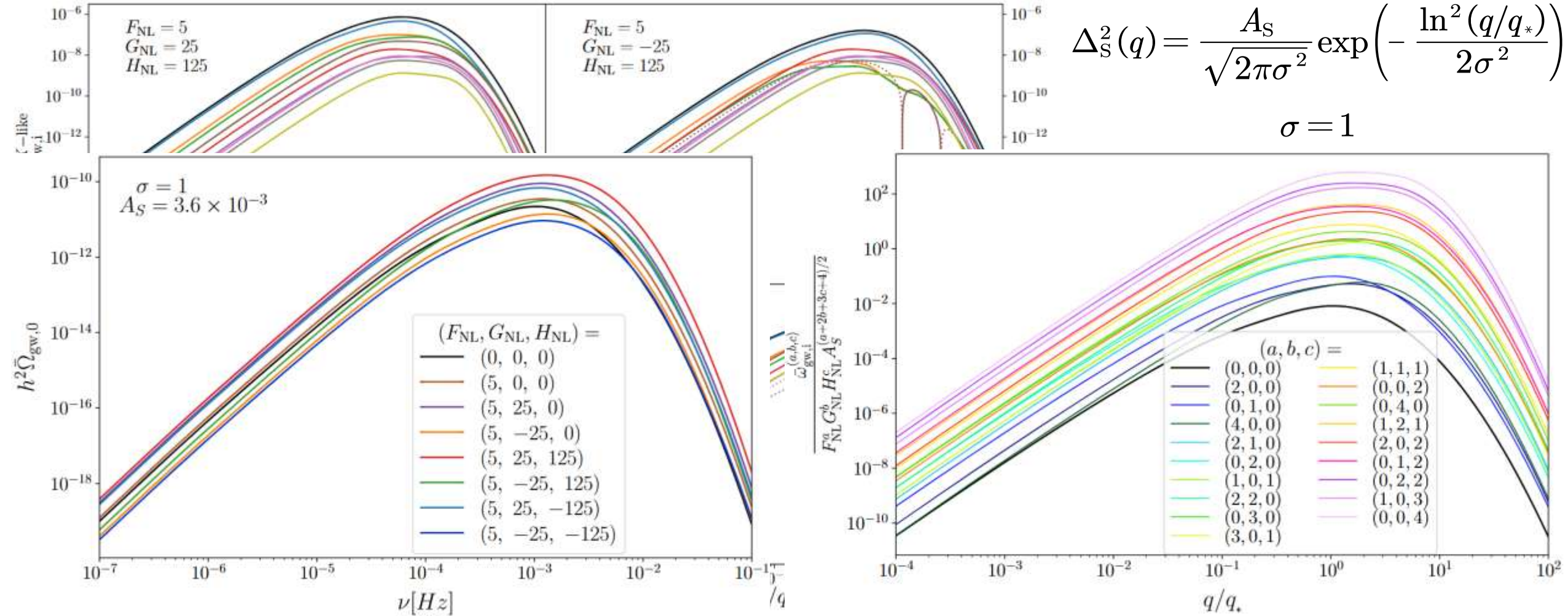
$\sigma = 1$



$$\bar{\omega}_{\text{gw},\text{in}}(q) = \sum_{X\text{-like}} \bar{\omega}_{\text{gw},\text{in}}^{X\text{-like}}(q) = \sum_{(a,b,c)} \bar{\omega}_{\text{gw},\text{in}}^{(a,b,c)}(q)$$

$$\bar{\omega}_{\text{gw},\text{in}}^{(a,b,c)} \propto F_{\text{NL}}^a G_{\text{NL}}^b H_{\text{NL}}^c A^{(a+2b+3c)/2}$$

Numerical Results



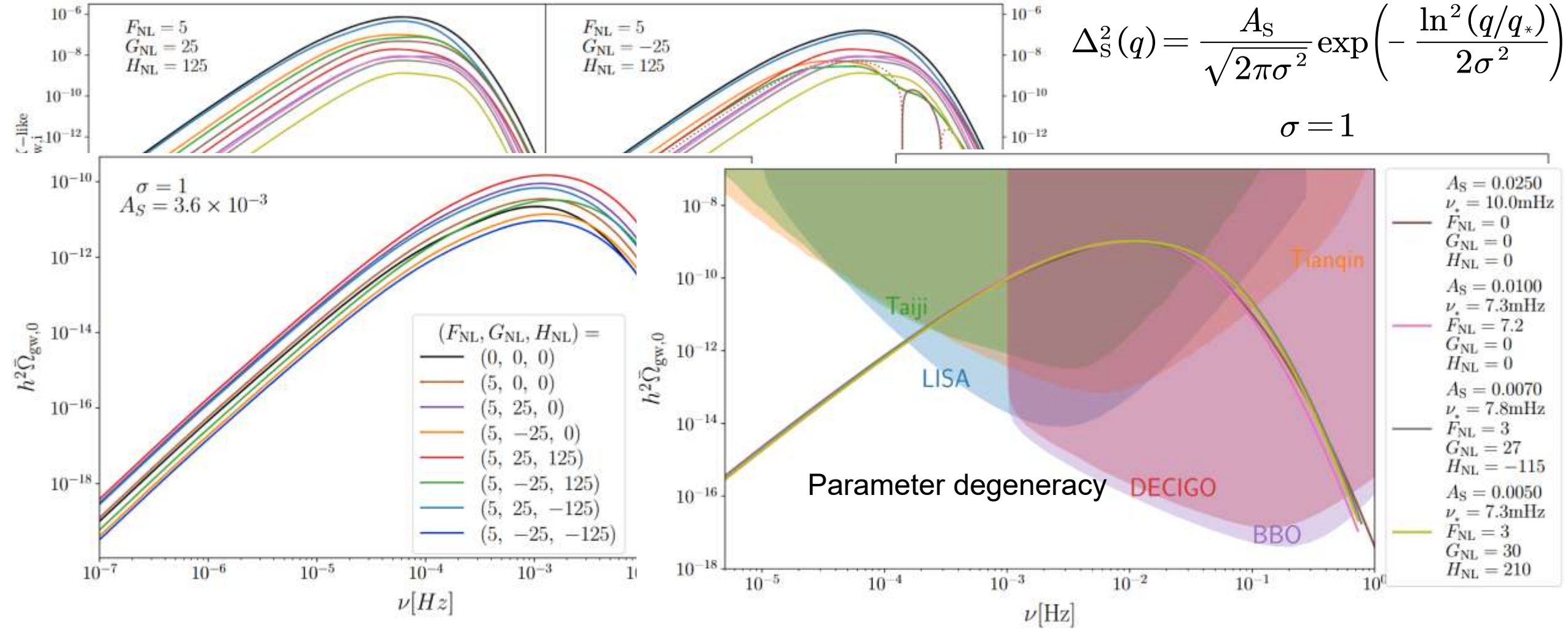
$$\Delta_S^2(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$\sigma = 1$

$$\bar{\omega}_{gw,in}(q) = \sum_{X\text{-like}} \bar{\omega}_{gw,in}^{X\text{-like}}(q) = \sum_{(a,b,c)} \bar{\omega}_{gw,in}^{(a,b,c)}(q)$$

$$\bar{\omega}_{gw,in}^{(a,b,c)} \propto F_{NL}^a G_{NL}^b H_{NL}^c A^{(a+2b+3c)/2}$$

Numerical Results



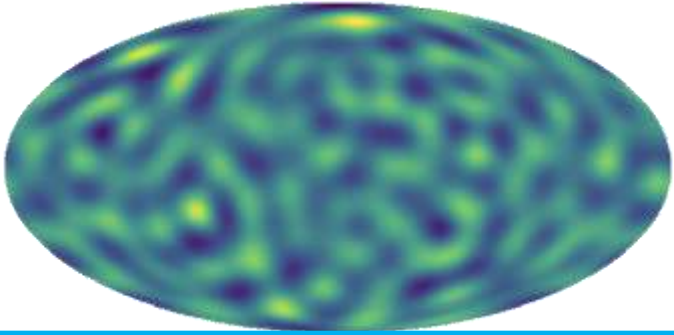
$$\Delta_S^2(q) = \frac{A_S}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$\sigma = 1$

$$\bar{\omega}_{\text{gw},\text{in}}(q) = \sum_{X\text{-like}} \bar{\omega}_{\text{gw},\text{in}}^{X\text{-like}}(q) = \sum_{(a,b,c)} \bar{\omega}_{\text{gw},\text{in}}^{(a,b,c)}(q)$$

$$\bar{\omega}_{\text{gw},\text{in}}^{(a,b,c)} \propto F_{\text{NL}}^a G_{\text{NL}}^b H_{\text{NL}}^c A^{(a+2b+3c)/2}$$

Anisotropic and Non-Gaussian Background



The SIGW Background

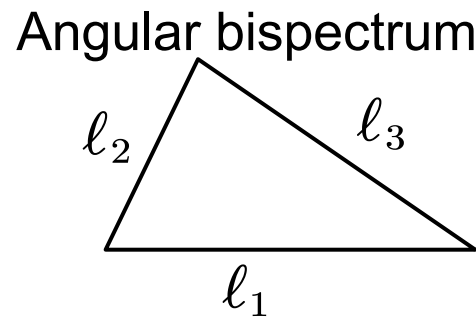
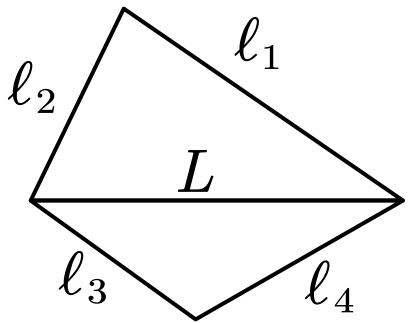
Anisotropies

Non-Gaussianity

Statistical isotropy and parity invariance

$$\delta_{\text{gw},0,\ell m}(2\pi\nu) = \int d^2\mathbf{n} \delta_{\text{gw},0}(\mathbf{q}) Y_{\ell m}^*(\mathbf{n})$$

Angular power spectrum $\left\langle \prod_{i=1}^2 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} (-1)^{m_1} \tilde{C}_\ell(\nu)$



Angular bispectrum $\left\langle \prod_{i=1}^3 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_\ell(\nu)$

Angular trispectrum $\left\langle \prod_{i=1}^4 \delta_{\text{gw},0,\ell_i m_i}(2\pi\nu) \right\rangle_c = \sum_{LM} (-1)^M \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} \begin{pmatrix} \ell_3 & \ell_4 & L \\ m_3 & m_4 & M \end{pmatrix} t_{\ell_3 \ell_4}^{\ell_1 \ell_2}(L, \nu) + 11 \text{ perms}$

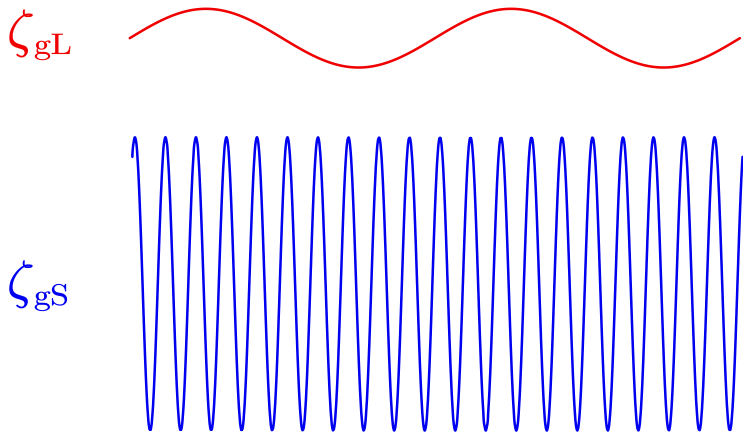
Initial Density Contrasts

Total initial Inhomogeneities SW effect

$$\delta_{\text{gw},0}(\mathbf{q}) = \tilde{\delta}_{\text{gw},\text{in}}(\mathbf{q}) + (4 - n_{\text{gw}}(\nu))\Phi_{\text{L},\text{in}}$$

$$\delta_{\text{gw},0}(\mathbf{q}) = \delta_{\text{gw},\text{in}}(\mathbf{q}) + (6 - n_{\text{gw}}(\nu))\Phi_{\text{L},\text{in}}$$

PNG-induced Inhomogeneity



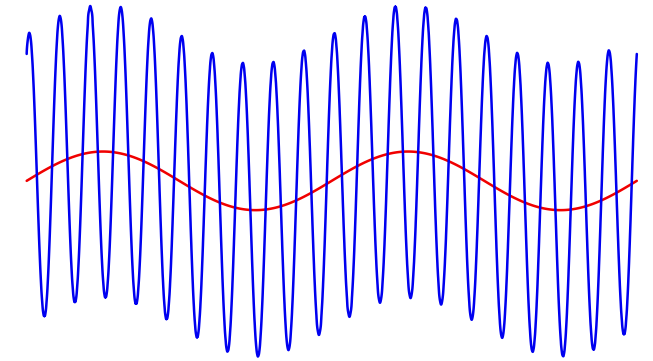
Short modes: SIGW production

Long modes: energy-density modulation

Gaussian scenario:

$$\tilde{\delta}_{\text{gw},\text{in}} = 2\Phi_{\text{L},\text{in}}$$

$$\delta_{\text{gw},\text{in}} = 0$$

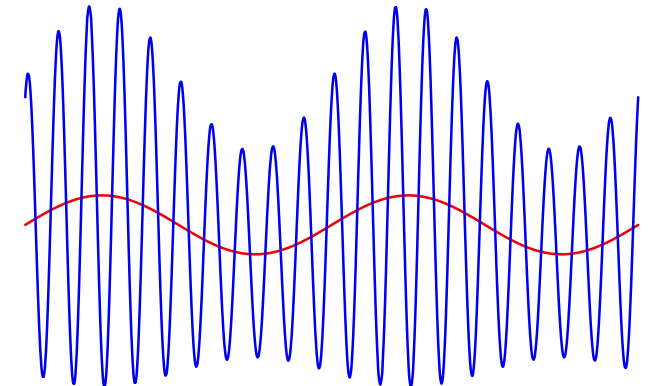


$$\zeta_{\text{g}} = \zeta_{\text{gS}} + \zeta_{\text{gL}}$$

Non-Gaussian scenario:

$$\tilde{\delta}_{\text{gw},\text{in}} = \delta_{\text{gw},\text{in}} + 2\Phi_{\text{L},\text{in}}$$

$$\delta_{\text{gw},\text{in}} \sim \langle \zeta\zeta\zeta\zeta \rangle$$

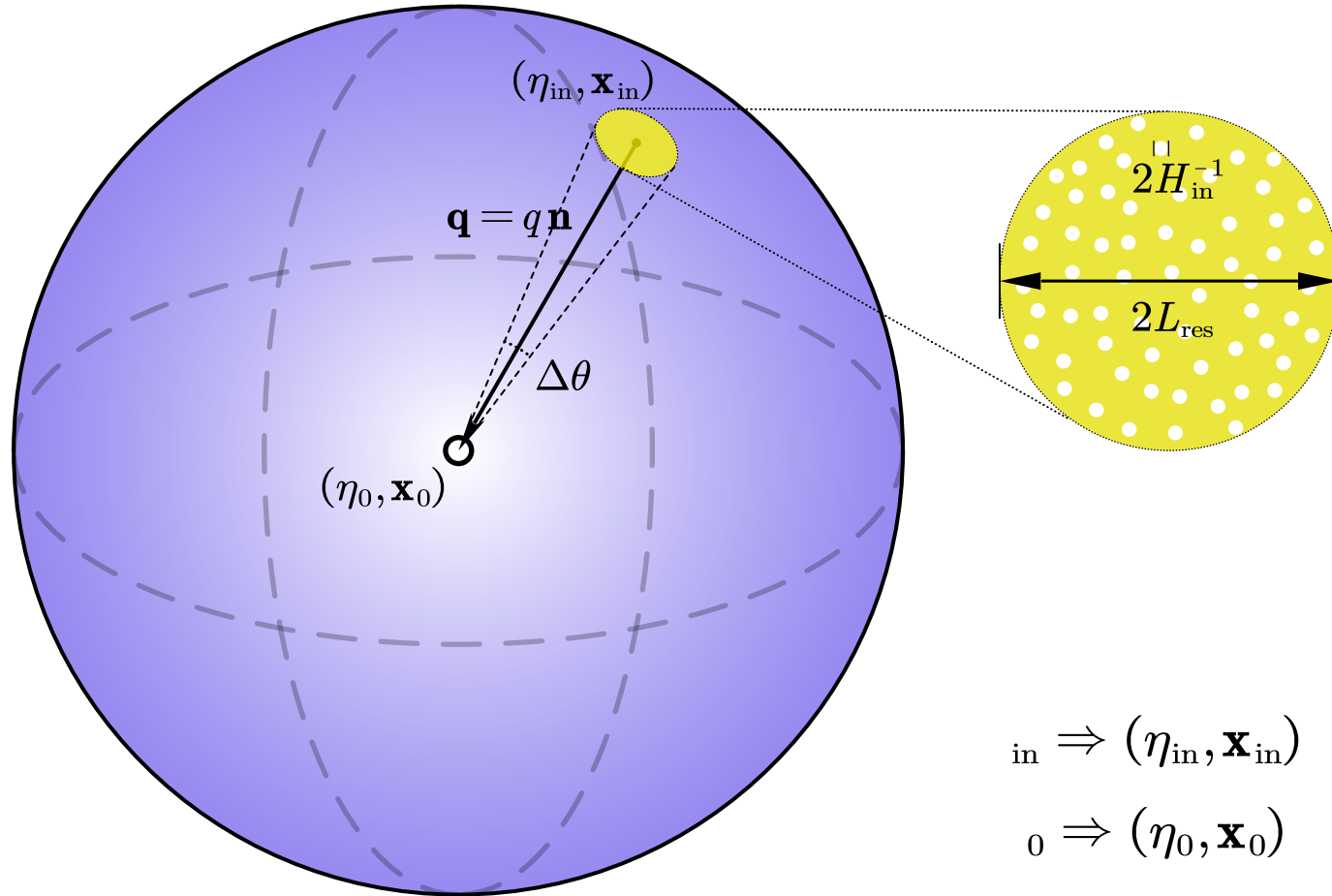


$$\zeta = \zeta_{\text{gS}} + \zeta_{\text{gL}} + f_{\text{NL}}\zeta_{\text{gS}}\zeta_{\text{gL}}$$

Non-Gaussianity: Coupling between short- and long-wavelength modes

Initial Inhomogeneities

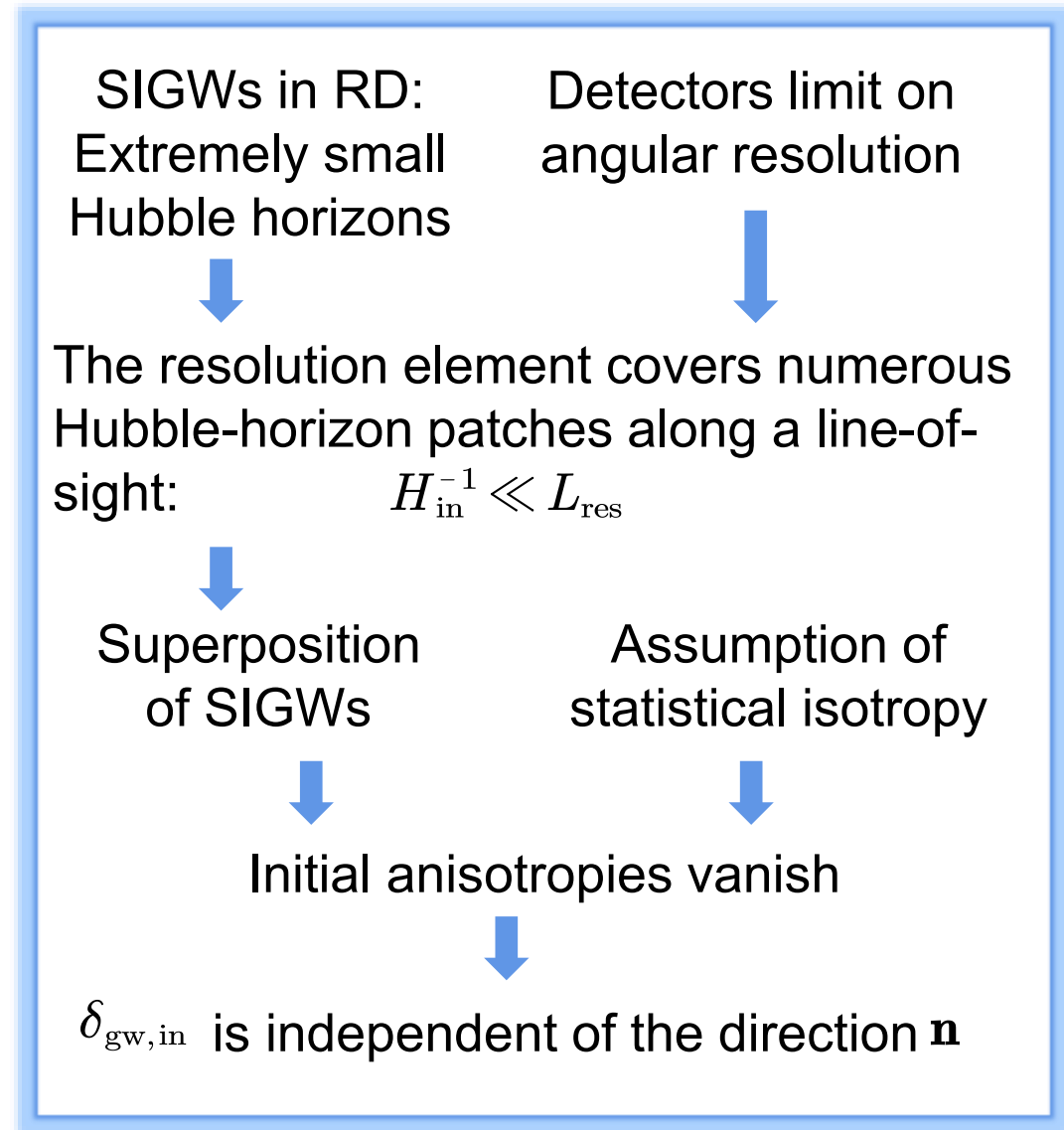
$$\delta_{\text{gw},0}(\mathbf{q}) = \delta_{\text{gw},\text{in}}(\mathbf{q}) + (6 - n_{\text{gw}}(\nu))\Phi_{\text{in}} \quad \mathbf{q} = q\mathbf{n}$$



$$\text{in} \Rightarrow (\eta_{\text{in}}, \mathbf{x}_{\text{in}})$$

$$0 \Rightarrow (\eta_0, \mathbf{x}_0)$$

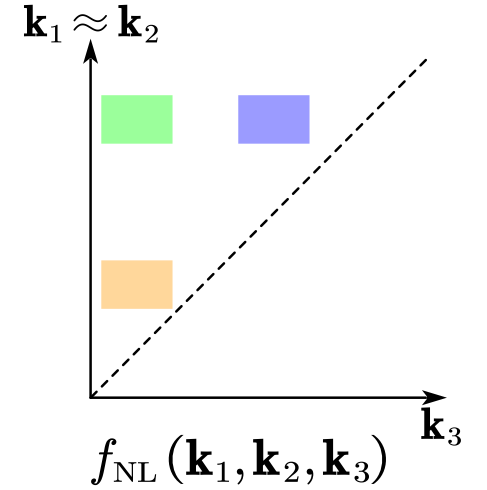
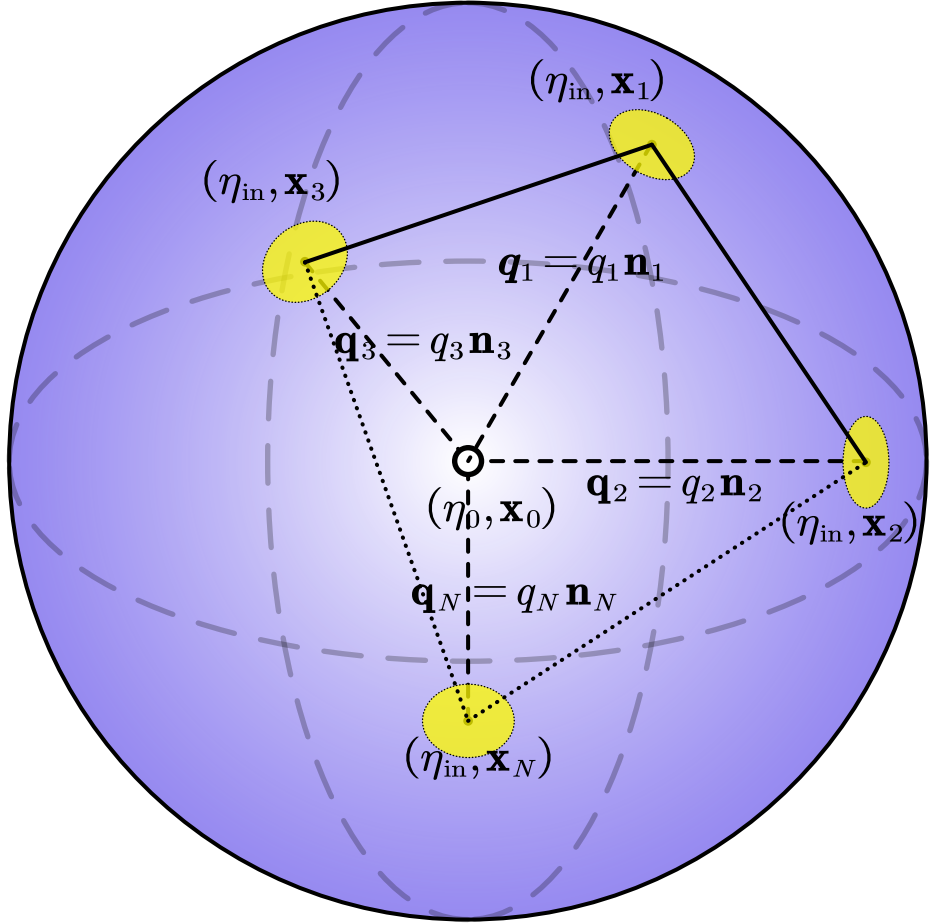
$$\delta_{\text{gw},0}(q, \mathbf{n}) = \delta_{\text{gw},\text{in}}(q) + (6 - n_{\text{gw}}(\nu))\Phi_{\text{in}} \quad \mathbf{n} = \frac{\mathbf{x}_0 - \mathbf{x}_{\text{in}}}{\eta_0 - \eta_{\text{in}}}$$



Multi-Point Correlator

$$\delta_{\text{gw},0}(q, \mathbf{n}) = \delta_{\text{gw},\text{in}}(q) + (6 - n_{\text{gw}}(\nu)) \Phi_{\text{in}}$$

$$\mathbf{n} = \frac{\mathbf{x}_0 - \mathbf{x}_{\text{in}}}{\eta_0 - \eta_{\text{in}}} \quad \begin{array}{l} \text{in} \Rightarrow (\eta_{\text{in}}, \mathbf{x}_{\text{in}}) \\ 0 \Rightarrow (\eta_0, \mathbf{x}_0) \end{array}$$



$$\left\langle \prod_{i=1}^N \omega_{\text{gw},\text{in},i}(q) \right\rangle$$

$$\delta_{\text{gw},\text{in}}(q) = \frac{\omega_{\text{gw},\text{in}}(q)}{\bar{\omega}_{\text{gw},\text{in}}(q)} - 1$$

$$\begin{aligned} \left\langle \prod_{i=1}^N \delta_{\text{gw},0}(\mathbf{q}) \right\rangle &= \left\langle \prod_{i=1}^N \delta_{\text{gw},\text{in},i}(q) \right\rangle + (6 - n_{\text{gw}}(\nu)) \left\langle \Phi_{\text{L},\text{in},N} \prod_{i=1}^{N-1} \delta_{\text{gw},\text{in},i}(q) \right\rangle \\ &+ \dots + (6 - n_{\text{gw}}(\nu))^{N-1} \left\langle \delta_{\text{gw},\text{in},1}(q) \prod_{i=2}^N \Phi_{\text{L},\text{in},i} \right\rangle \\ &+ (6 - n_{\text{gw}}(\nu))^N \left\langle \prod_{i=1}^N \Phi_{\text{L},\text{in},i} \right\rangle \end{aligned}$$

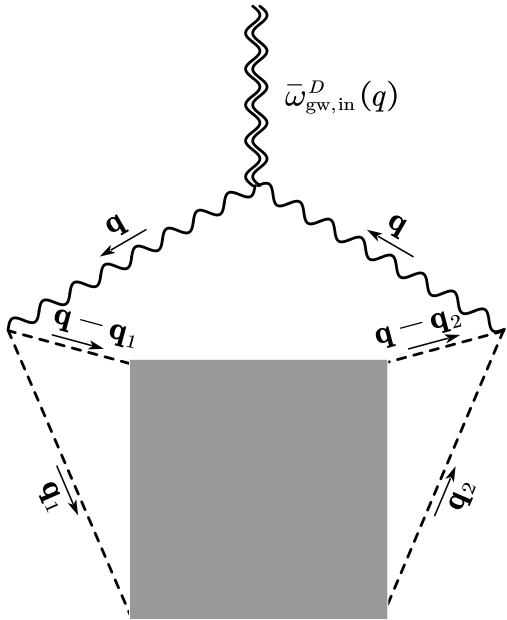
Diagrammatic Representation

$$\omega_{\text{gw,in}}(q) = \bar{\omega}_{\text{gw,in}}(q) + \omega_{\text{gw,in}}^{(1)}(q) + \omega_{\text{gw,in}}^{(2)}(q) + \omega_{\text{gw,in}}^{(3)}(q) + \dots$$

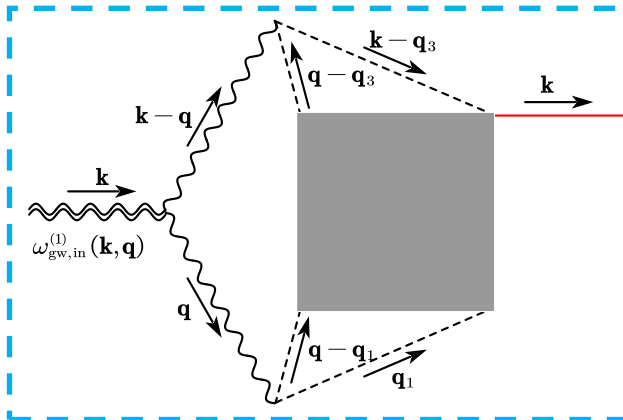
$\xrightarrow{\mathbf{k}} \frac{2\pi^2}{k^3} \Delta_{\text{L}}^2(k) \xrightarrow{\mathbf{q}} \frac{2\pi^2}{q^3} \Delta_{\text{S}}^2(q)$

$$\zeta_{\text{g}} = \zeta_{\text{gS}} + \zeta_{\text{gL}} \quad \langle \zeta_{\text{gX}}(\mathbf{k}) \zeta_{\text{gX}'}(\mathbf{k}') \rangle = \delta_{\text{XX}'} \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \Delta_{\text{gX}}^2(k), \quad \text{X} = \text{S, L}$$

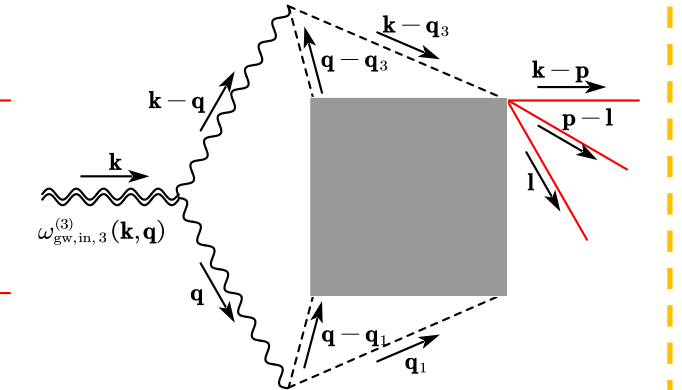
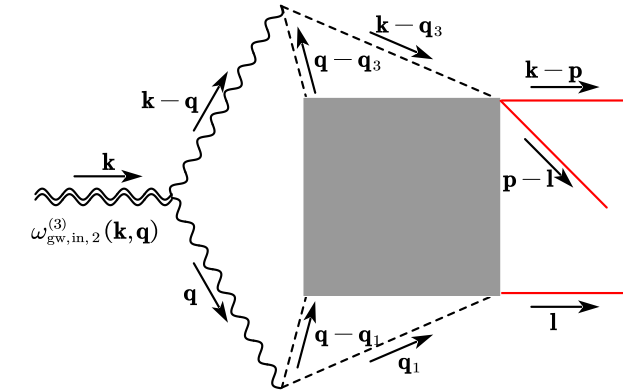
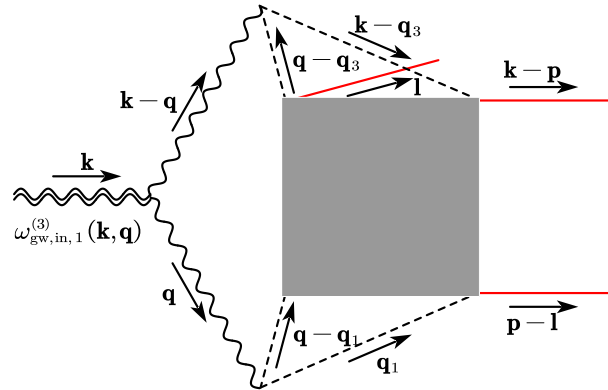
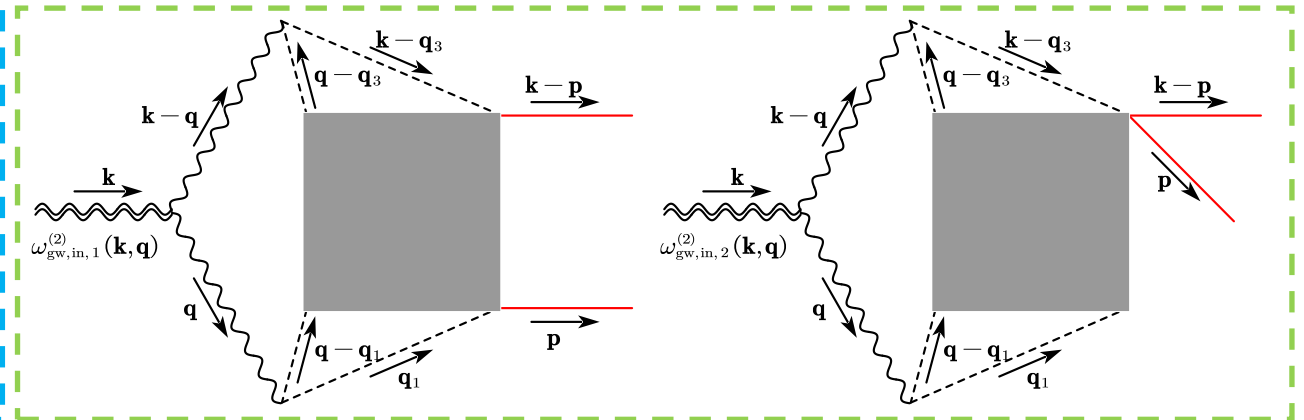
$$\bar{\omega}_{\text{gw,in}}(q)$$



$$\omega_{\text{gw,in}}^{(1)}(q) = \omega_{\text{ng}}^{(1)}(q) \zeta_{\text{gL}}$$



$$\omega_{\text{gw,in}}^{(2)}(q) = \omega_{\text{ng}}^{(2)}(q) \zeta_{\text{gL}}^2$$

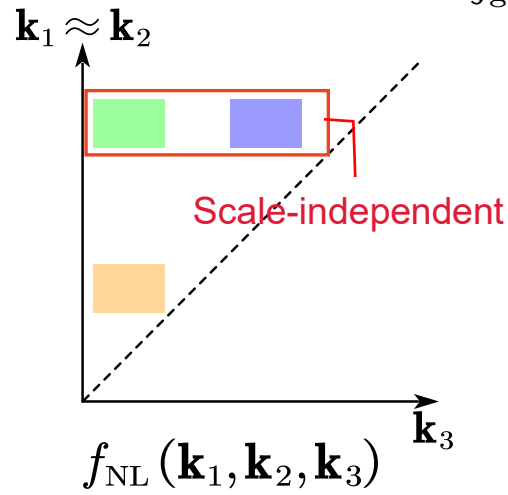


$$\omega_{\text{gw,in}}^{(3)}(q) = \omega_{\text{ng}}^{(3)}(q) \zeta_{\text{gL}}^3$$

Diagrammatic Approach

$$\zeta = \zeta_g + F_{\text{NL}} (\zeta_g^2 - \langle \zeta_g^2 \rangle) + G_{\text{NL}} \zeta_g^3 + H_{\text{NL}} (\zeta_g^4 - 3 \langle \zeta_g^2 \rangle^2) + \dots$$

$$\zeta_g = \zeta_{\text{gS}} + \zeta_{\text{gL}}$$

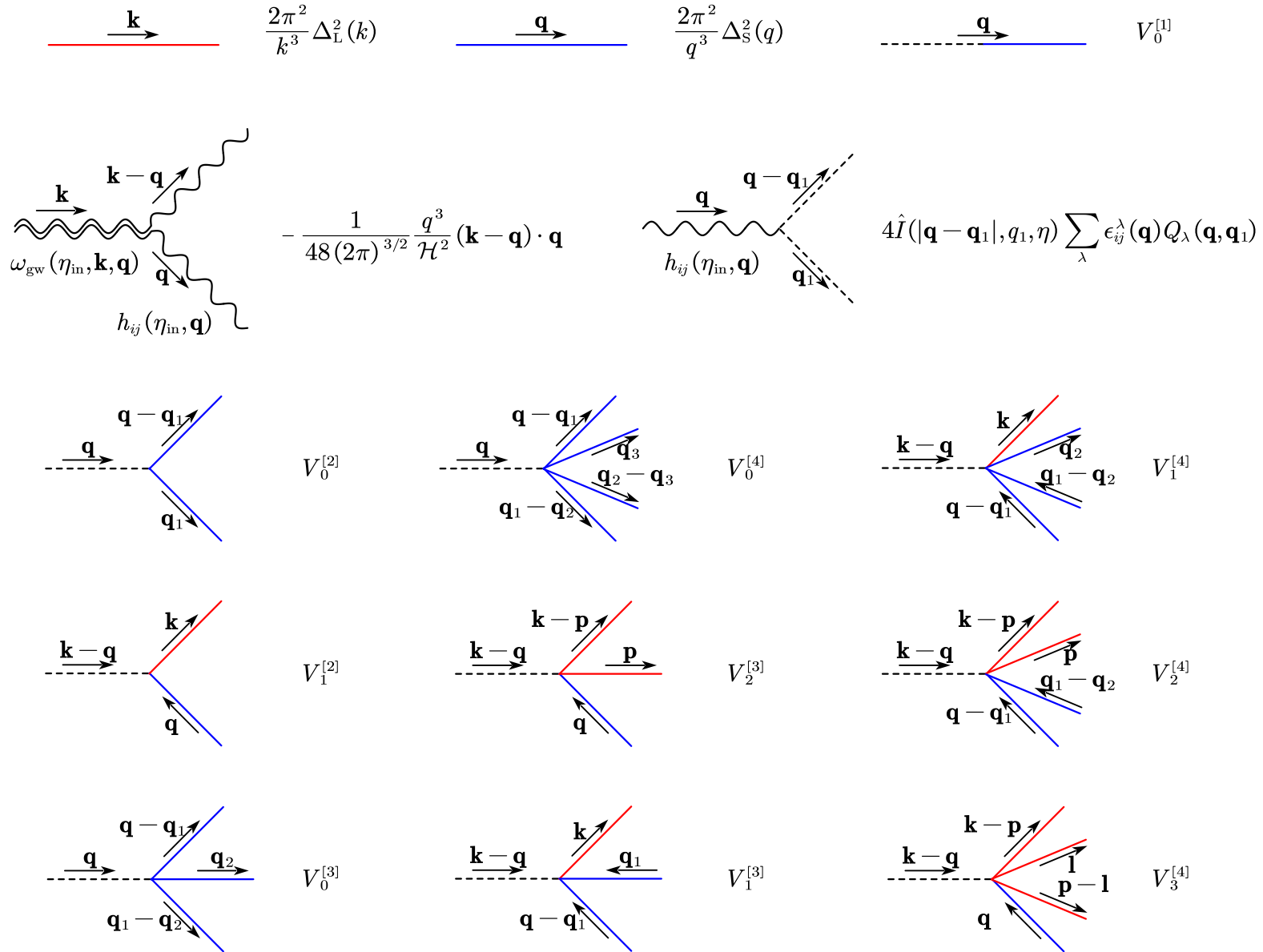


$$V_0^{[1]} = 1$$

$$V_0^{[2]} = V_1^{[2]} = F_{\text{NL}}$$

$$V_0^{[3]} = V_1^{[3]} = V_2^{[3]} = G_{\text{NL}}$$

$$V_0^{[4]} = V_1^{[4]} = V_2^{[4]} = V_3^{[4]} = H_{\text{NL}}$$

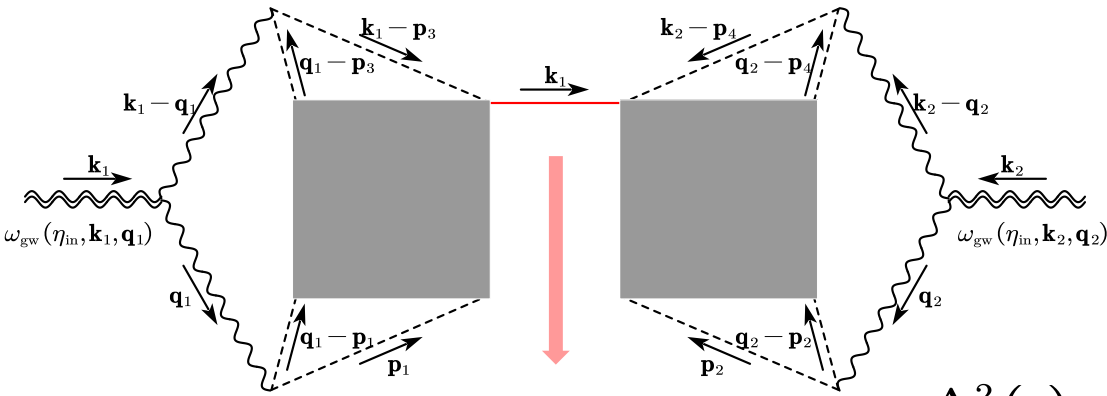


Two-point Angular Correlator

$$\zeta = \zeta_g + F_{\text{NL}} (\zeta_g^2 - \langle \zeta_g^2 \rangle) + G_{\text{NL}} \zeta_g^3 + H_{\text{NL}} (\zeta_g^4 - 3 \langle \zeta_g^2 \rangle^2) + \dots$$

$$\zeta_g = \zeta_{\text{gS}} + \zeta_{\text{gL}}$$

$$\langle \zeta_{\text{gX}}(\mathbf{k}) \zeta_{\text{gX}'}(\mathbf{k}') \rangle = \delta_{\text{XX}'} \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \Delta_{\text{gX}}^2(k), \quad \text{X} = \text{S, L}$$



Non-Gaussian bridge

$$\langle \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_1, q_1) \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_2, q_2) \rangle$$

$$\sim \langle \langle \zeta_{\text{gS}} \dots \zeta_{\text{gS}} \zeta_{\text{gL}} \rangle \langle \zeta_{\text{gS}} \dots \zeta_{\text{gS}} \zeta_{\text{gL}} \rangle \rangle$$

$$\sim \langle \zeta_{\text{gS}} \dots \zeta_{\text{gS}} \rangle \langle \zeta_{\text{gL}} \zeta_{\text{gL}} \rangle \langle \zeta_{\text{gS}} \dots \zeta_{\text{gS}} \rangle$$

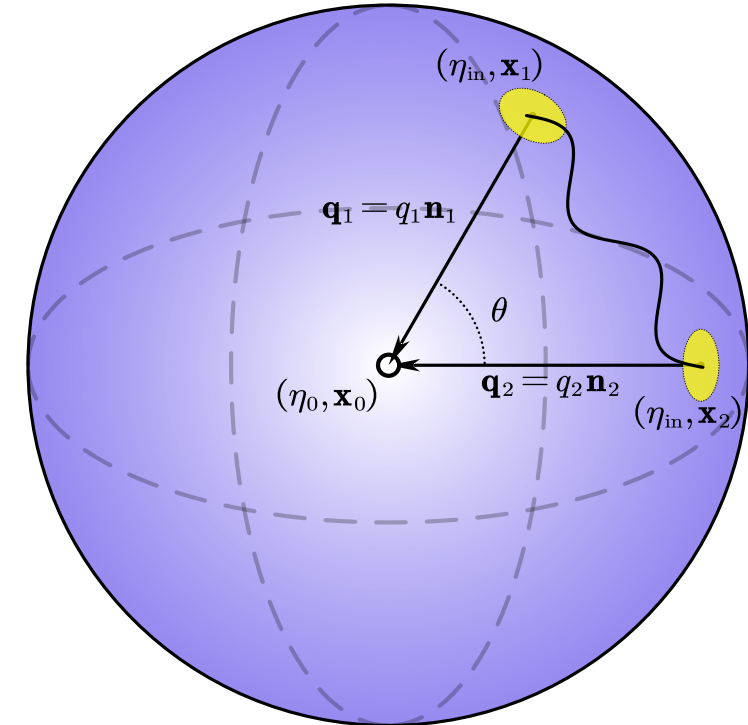
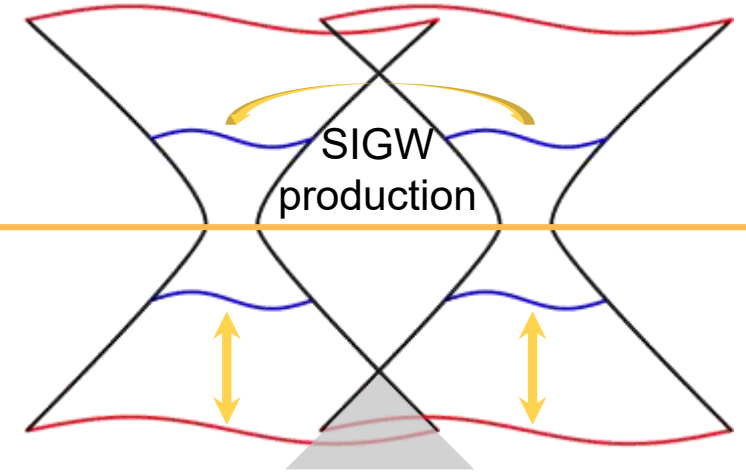
$$\Delta_{\text{S}}^2(q) = \frac{A_{\text{S}}}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\ln^2(q/q_*)}{2\sigma^2}\right)$$

$$\Delta_{\text{L}}^2(q) = A_{\text{L}} \simeq 2.1 \times 10^{-9}$$

ζ_{gS} at \mathbf{x}_1 and \mathbf{x}_2 are not correlated.

Raidation
Domination

Inflation



1st-order PNG-induced Initial Inhomogeneity

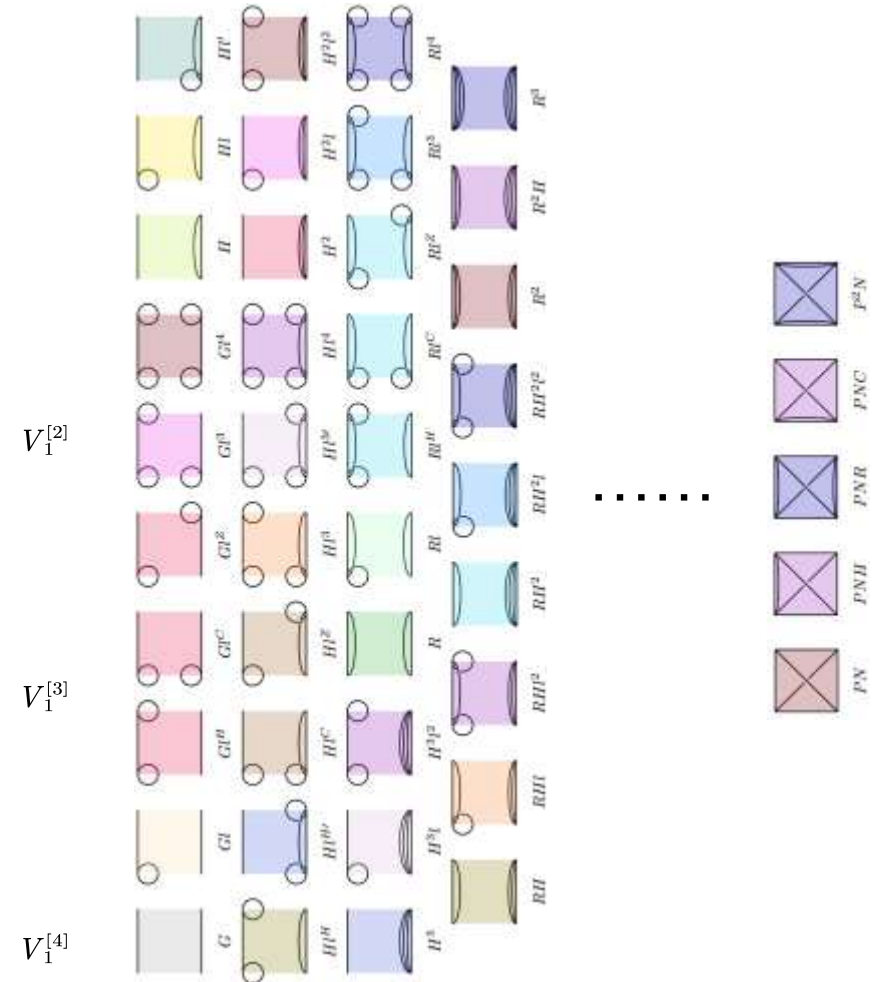
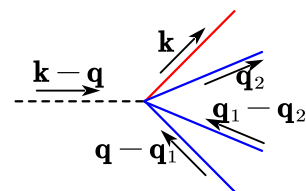
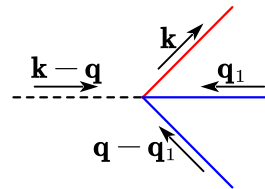
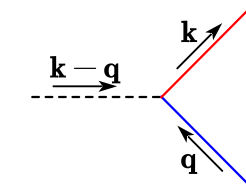
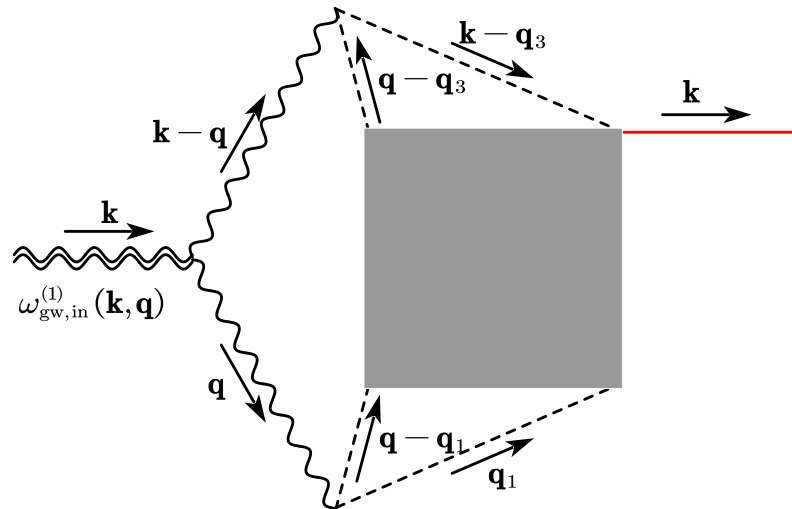
$$\omega_{\text{gw,in}}^{(1)}(q) = \omega_{\text{ng}}^{(1)}(q) \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k} \cdot \mathbf{x}} \zeta_{\text{gL}}(\mathbf{k})$$

$$\omega_{\text{ng}}^{(1)}(q) = \sum_{c=0}^4 \sum_{b=0}^{[4-c]} \sum_{a=0}^{[4-b-c]} \bar{\omega}_{\text{gw,in}}^{(a,b,c)}(q) \sum_{i=1}^{o-1} N_i (i+1) \frac{V_1^{[i+1]}}{V_0^{[i]}}$$

$$\bar{\omega}_{\text{gw,in}}^{(a,b,c)} \propto F_{\text{NL}}^a G_{\text{NL}}^b H_{\text{NL}}^c A^{(a+2b+3c)/2}$$

N_i : number of $V_0^{[i]}$ -vertices. $N_1 = 4 - a - b - c$

$$N_2 = a \quad N_3 = b \quad N_4 = c$$



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Reduced Angular Power Spectrum

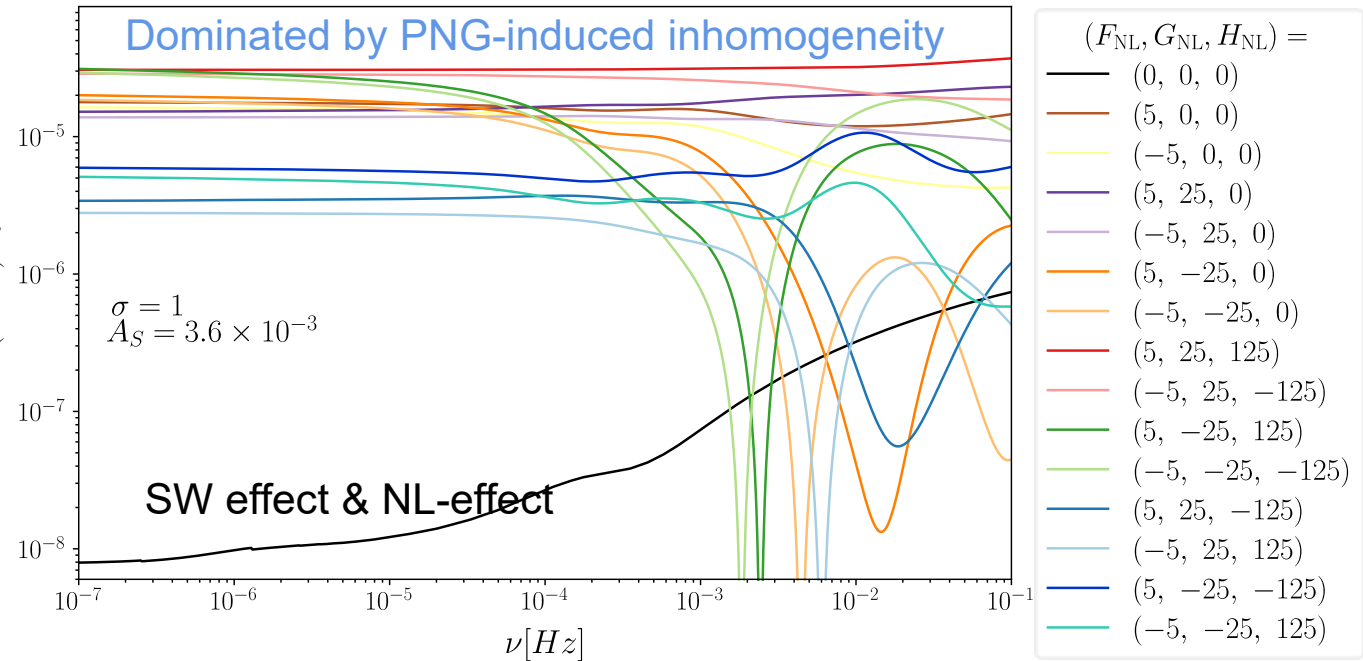
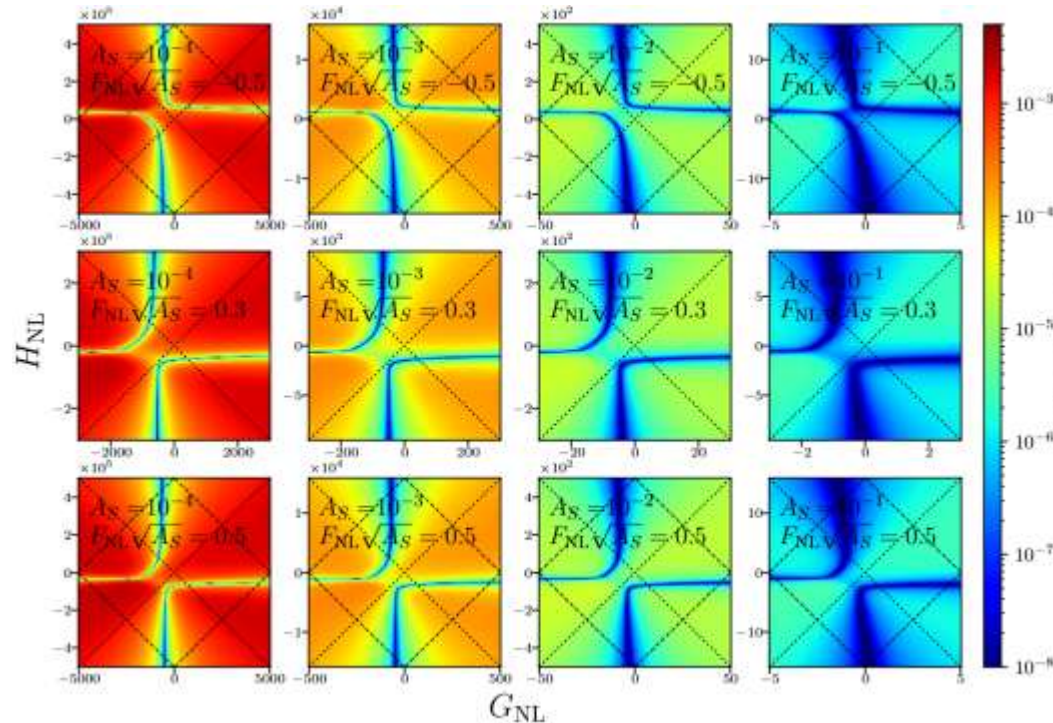
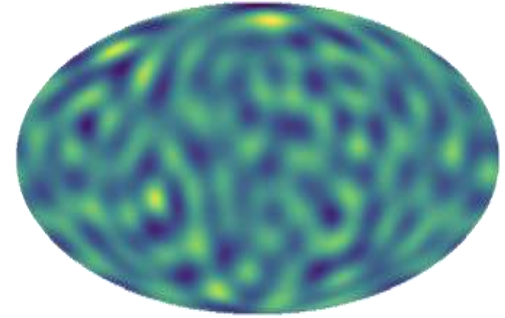
PNG-induced Inhomogeneity SW effect & NL-effect

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$$\delta_{\text{gw},0}^{(1)}(q) = \left[\frac{\omega_{\text{ng}}^{(1)}(q)}{\bar{\omega}_{\text{gw},\text{in}}(q)} + \frac{3}{5} (6 - n_{\text{gw}}(\nu)) \right] \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \zeta_{\text{gL}}(\mathbf{k})$$

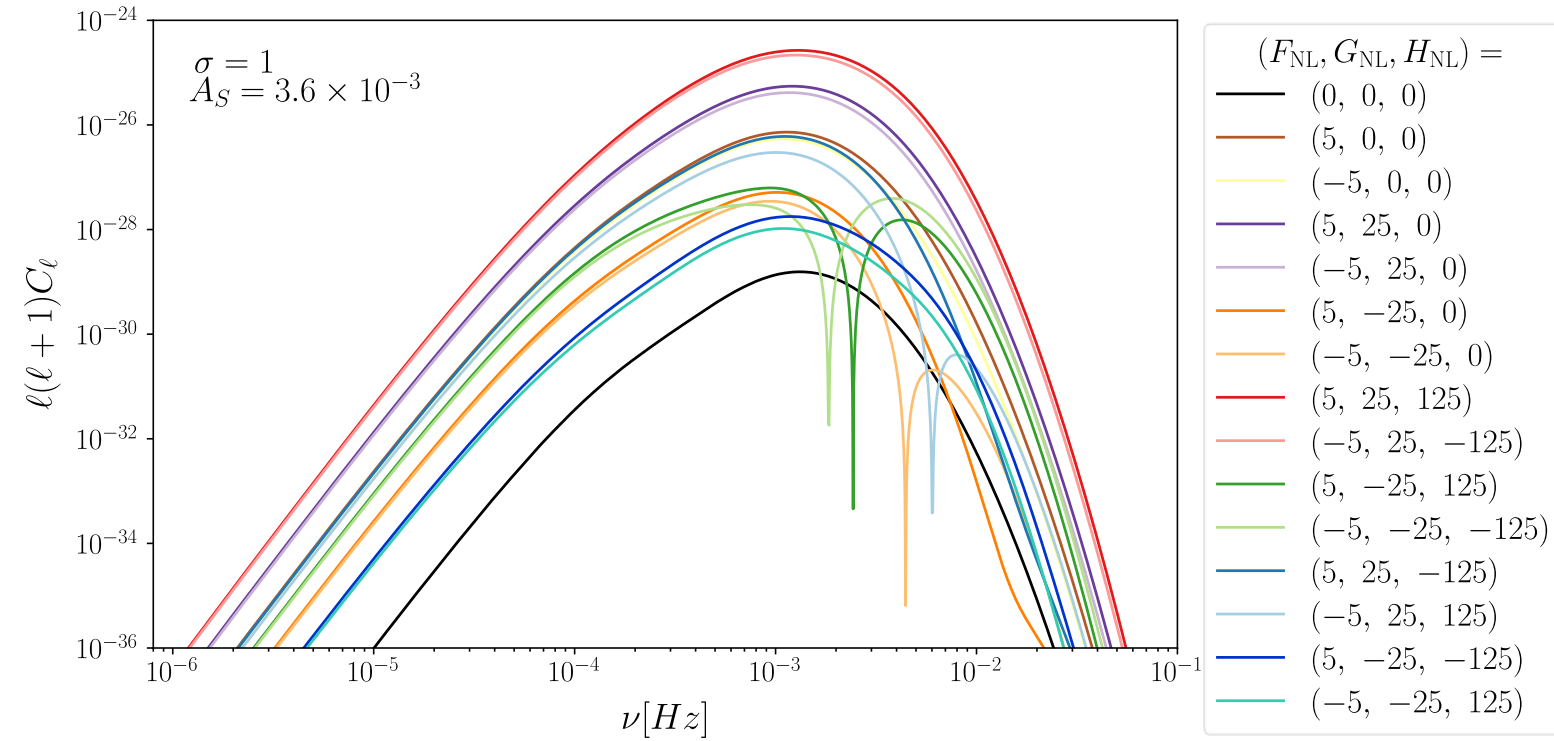
$$\tilde{C}_\ell(\nu) = \frac{2\pi A_L}{\ell(\ell+1)} \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw},\text{in}}(2\pi\nu)} + \frac{3}{5} (6 - n_{\text{gw}}(\nu)) \right]^2$$



Angular Power Spectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

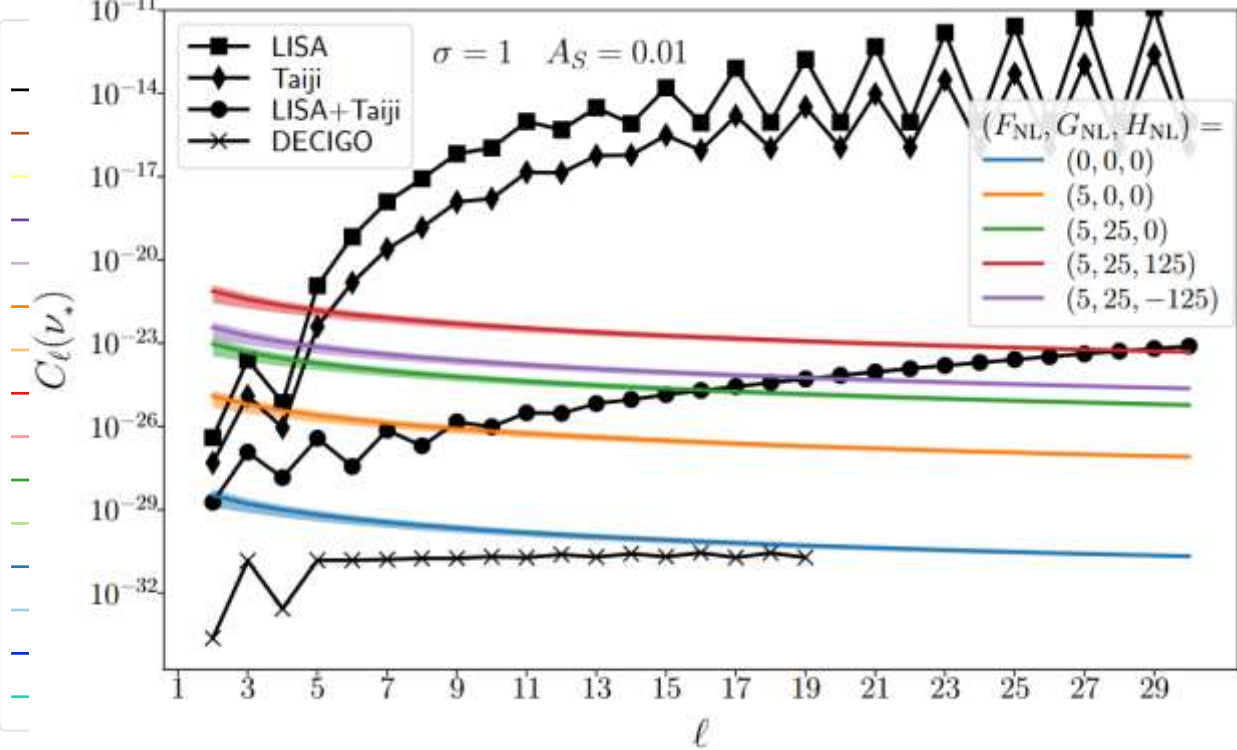
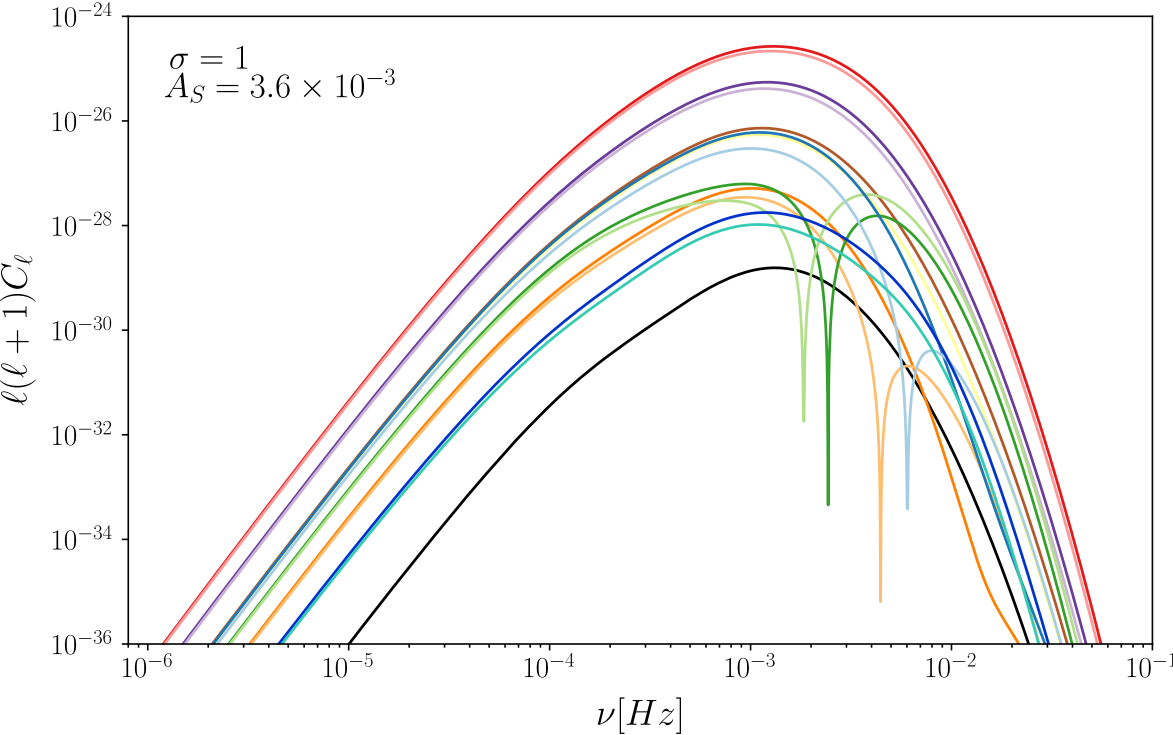
$$C_\ell(\nu) = \left[\frac{\bar{\Omega}_{\text{gw},0}(\nu)}{4\pi} \right]^2 \tilde{C}_\ell(\nu)$$



Angular Power Spectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

$$C_\ell(\nu) = \left[\frac{\bar{\Omega}_{\text{gw},0}(\nu)}{4\pi} \right]^2 \tilde{C}_\ell(\nu)$$



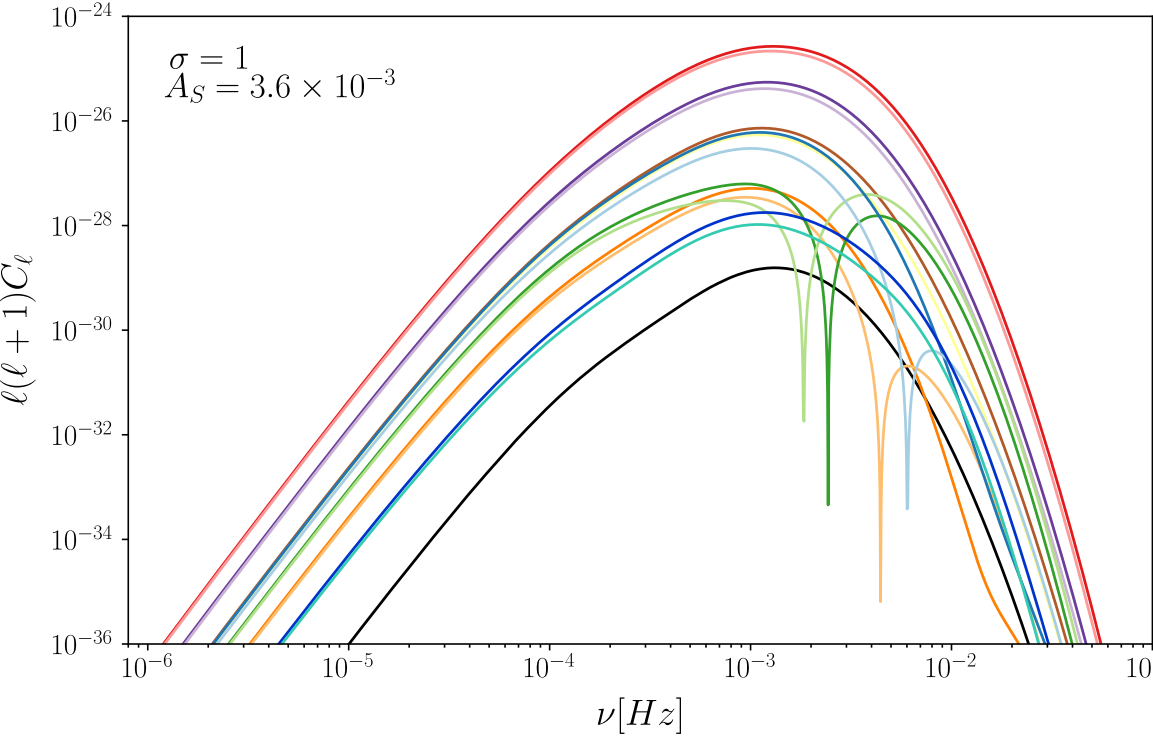
SIGWs: $C_\ell(\nu) \propto [\ell(\ell + 1)]^{-1}$

Astrophysical GWs: $C_\ell(\nu) \propto (\ell + 1/2)^{-1}$

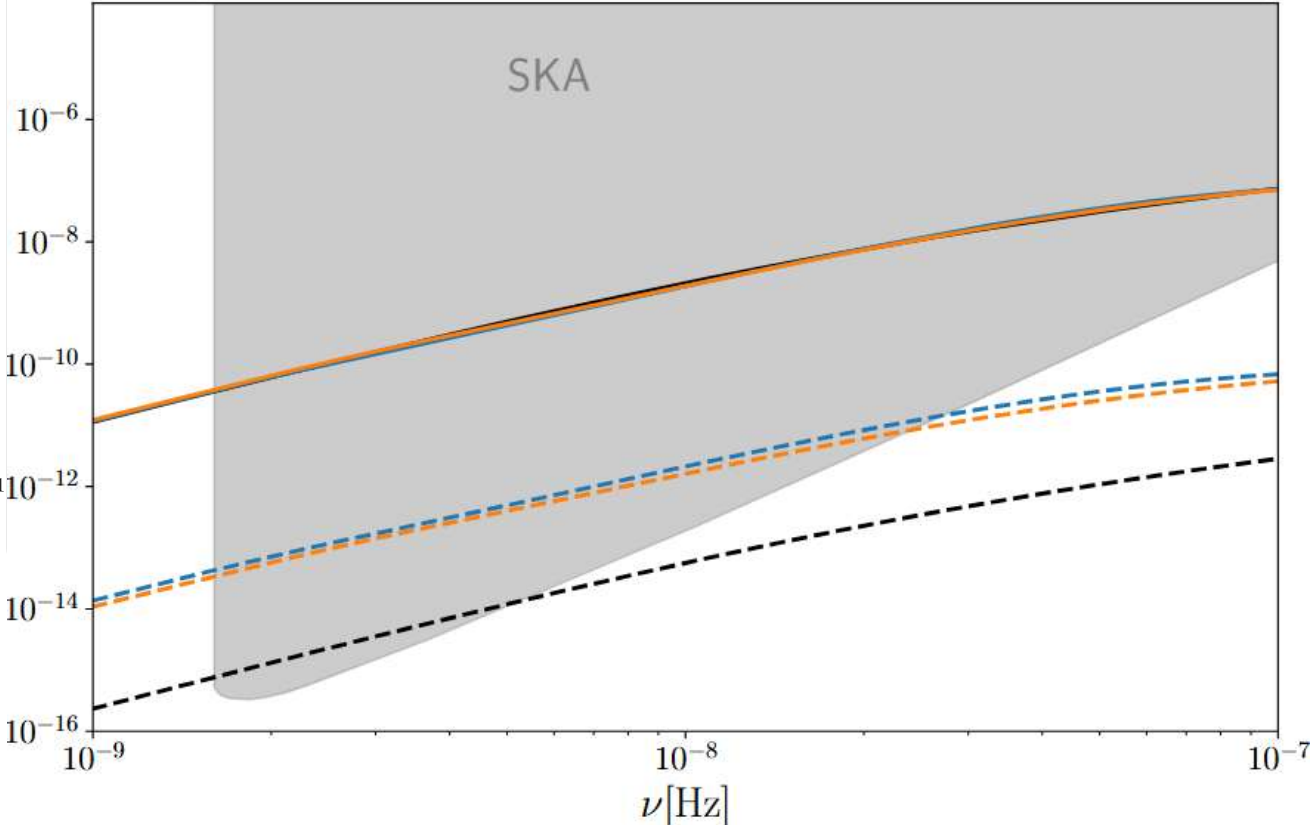
Angular Power Spectrum

JL, S. Wang, Z.-C. Zhao, and K. Kohri, arXiv:2309.07792

$$C_\ell(\nu) = \left[\frac{\bar{\Omega}_{\text{gw},0}(\nu)}{4\pi} \right]^2 \tilde{C}_\ell(\nu)$$



- $h^2 \Omega_{\text{gw},0}(\nu)$, $A_S = 0.417$, $\sigma = 1.35$, $F_{\text{NL}} = 0$, $G_{\text{NL}} = 0$, $H_{\text{NL}} = 0$, $\nu_* = 478.6\text{mHz}$
- - - $h^2 \sqrt{\frac{\ell(\ell+1)}{2\pi}} C_\ell(\nu)$, $A_S = 0.417$, $\sigma = 1.35$, $F_{\text{NL}} = 0$, $G_{\text{NL}} = 0$, $H_{\text{NL}} = 0$, $\nu_* = 478.6\text{mHz}$
- $h^2 \Omega_{\text{gw},0}(\nu)$, $A_S = 0.009$, $\sigma = 1$, $F_{\text{NL}} = -11$, $G_{\text{NL}} = -120$, $H_{\text{NL}} = 0$, $\nu_* = 135.0\text{mHz}$
- - - $h^2 \sqrt{\frac{\ell(\ell+1)}{2\pi}} C_\ell(\nu)$, $A_S = 0.009$, $\sigma = 1$, $F_{\text{NL}} = -11$, $G_{\text{NL}} = -120$, $H_{\text{NL}} = 0$, $\nu_* = 135.0\text{mHz}$
- $h^2 \Omega_{\text{gw},0}(\nu)$, $A_S = 0.015$, $\sigma = 1$, $F_{\text{NL}} = -11$, $G_{\text{NL}} = 30$, $H_{\text{NL}} = 100$, $\nu_* = 180.0\text{mHz}$
- - - $h^2 \sqrt{\frac{\ell(\ell+1)}{2\pi}} C_\ell(\nu)$, $A_S = 0.015$, $\sigma = 1$, $F_{\text{NL}} = -11$, $G_{\text{NL}} = 30$, $H_{\text{NL}} = 100$, $\nu_* = 180.0\text{mHz}$



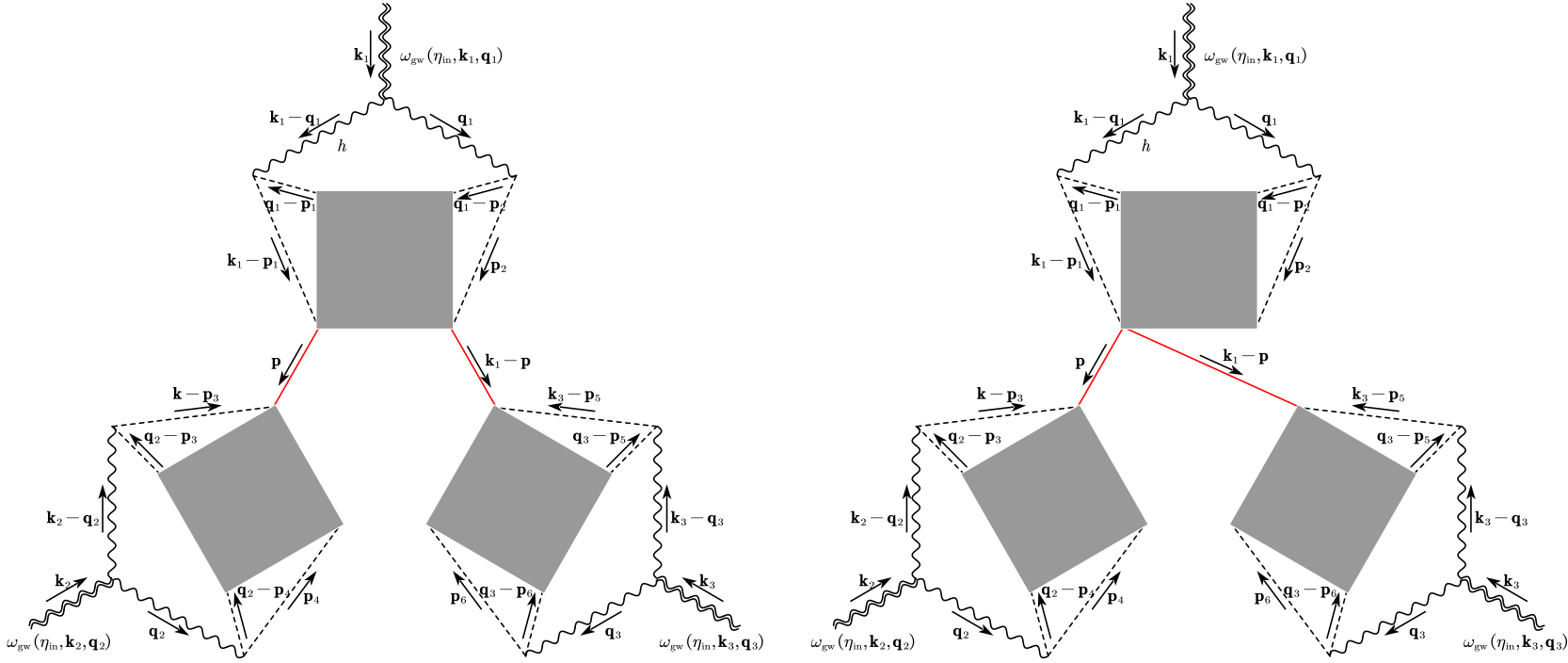
Parameter degeneracy breaking

SIGWs: $C_\ell(\nu) \propto [\ell(\ell + 1)]^{-1}$

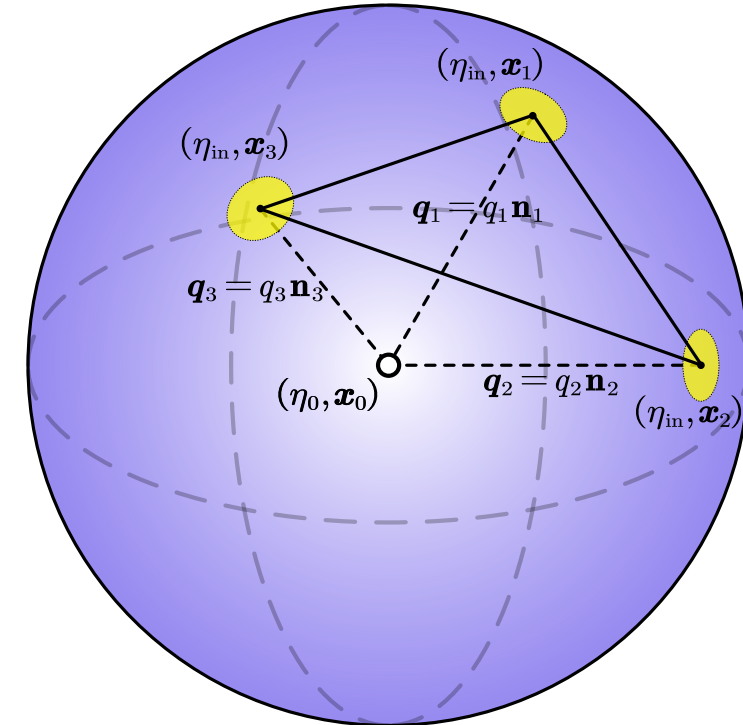
Astrophysical GWs: $C_\ell(\nu) \propto (\ell + 1/2)^{-1}$

Three-point Angular Correlator

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$$\left\langle \prod_{i=1}^3 \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_i, q) \right\rangle$$



$$\omega_{\text{gw}, \text{in}}(q) = \bar{\omega}_{\text{gw}, \text{in}}(q) + \omega_{\text{gw}, \text{in}}^{(1)}(q) + \omega_{\text{gw}, \text{in}}^{(2)}(q)$$

$$\omega_{\text{gw}, \text{in}}^{(2)}(q) = \omega_{\text{ng}}^{(2)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p}}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{p}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

Angular Bispectrum

$$B_{\ell_1 \ell_2 \ell_3}(\nu) = b(\nu) h_{\ell_1 \ell_2 \ell_3} \left[\frac{1}{\ell_1 \ell_2 (\ell_1 + 1) (\ell_2 + 1)} + (\ell_1 \leftrightarrow \ell_3) + (\ell_2 \leftrightarrow \ell_3) \right]$$

$$b(\nu) = 2(2\pi A_L)^2 \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^2 \left[\frac{\omega_{\text{ng}}^{(2)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}F_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]$$

PNG-induced Inhomogeneity SW effect & NL effect

$$\omega_{\text{ng}}^{(2)}(q) = \sum_{c=0}^4 \sum_{b=0}^{\lfloor 4-c \rfloor} \sum_{a=0}^{\lfloor 4-b-c \rfloor} \bar{\omega}_{\text{gw,in}}^{(a,b,c)}(q) (\mathcal{I}_1 + \mathcal{I}_2)$$

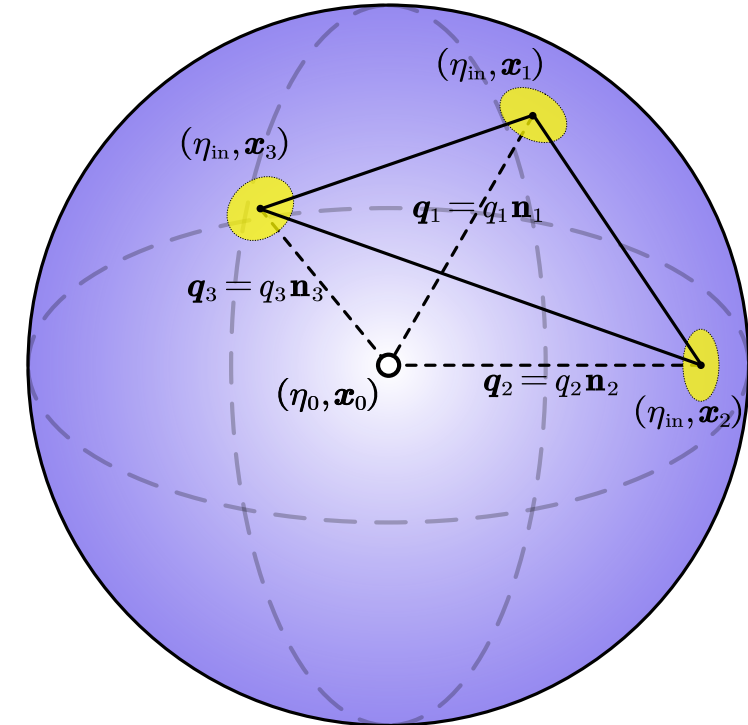
$$\mathcal{I}_1 = \sum_{l=1}^2 \sum_{n=1}^2 \sum_{i=1}^{o-1} \left(\frac{(i+l)!}{i!} \frac{V_1^{[i+1]}}{V_0^{[i]}} \right)^n \binom{N_i}{n} \delta_{ln,2}$$

$$\mathcal{I}_2 = \sum_{i=1}^3 \sum_{j=1}^{\min(i-1, o-1)} \sum_{i=1}^{o-1} N_i N_j (i+1)(j+1) \frac{V_1^{[i+1]} V_1^{[j+1]}}{V_0^{[i]} V_0^{[j]}}$$

$$\bar{\omega}_{\text{gw,in}}^{(a,b,c)} \propto F_{\text{NL}}^a G_{\text{NL}}^b H_{\text{NL}}^c A^{(a+2b+3c)/2}$$

$$N_1 = 4 - a - b - c$$

$$N_2 = a \quad N_3 = b \quad N_4 = c$$



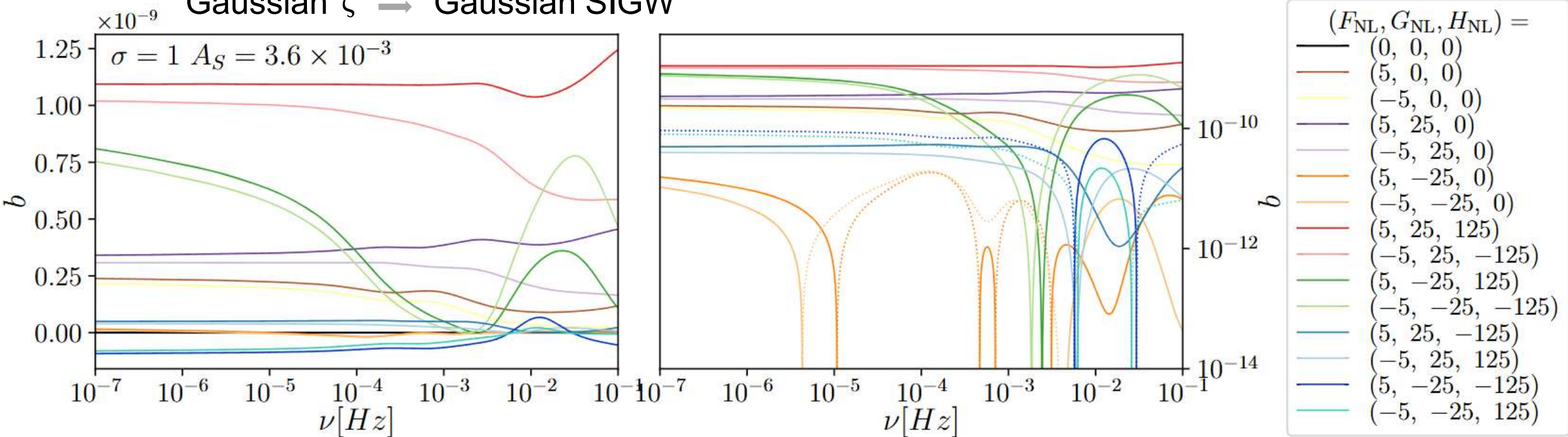
Angular Bispectrum

$$B_{\ell_1 \ell_2 \ell_3}(\nu) = b(\nu) h_{\ell_1 \ell_2 \ell_3} \left[\frac{1}{\ell_1 \ell_2 (\ell_1 + 1) (\ell_2 + 1)} + (\ell_1 \leftrightarrow \ell_3) + (\ell_2 \leftrightarrow \ell_3) \right]$$

$$b(\nu) = 2(2\pi A_L)^2 \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^2 \left[\frac{\omega_{\text{ng}}^{(2)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}F_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]$$

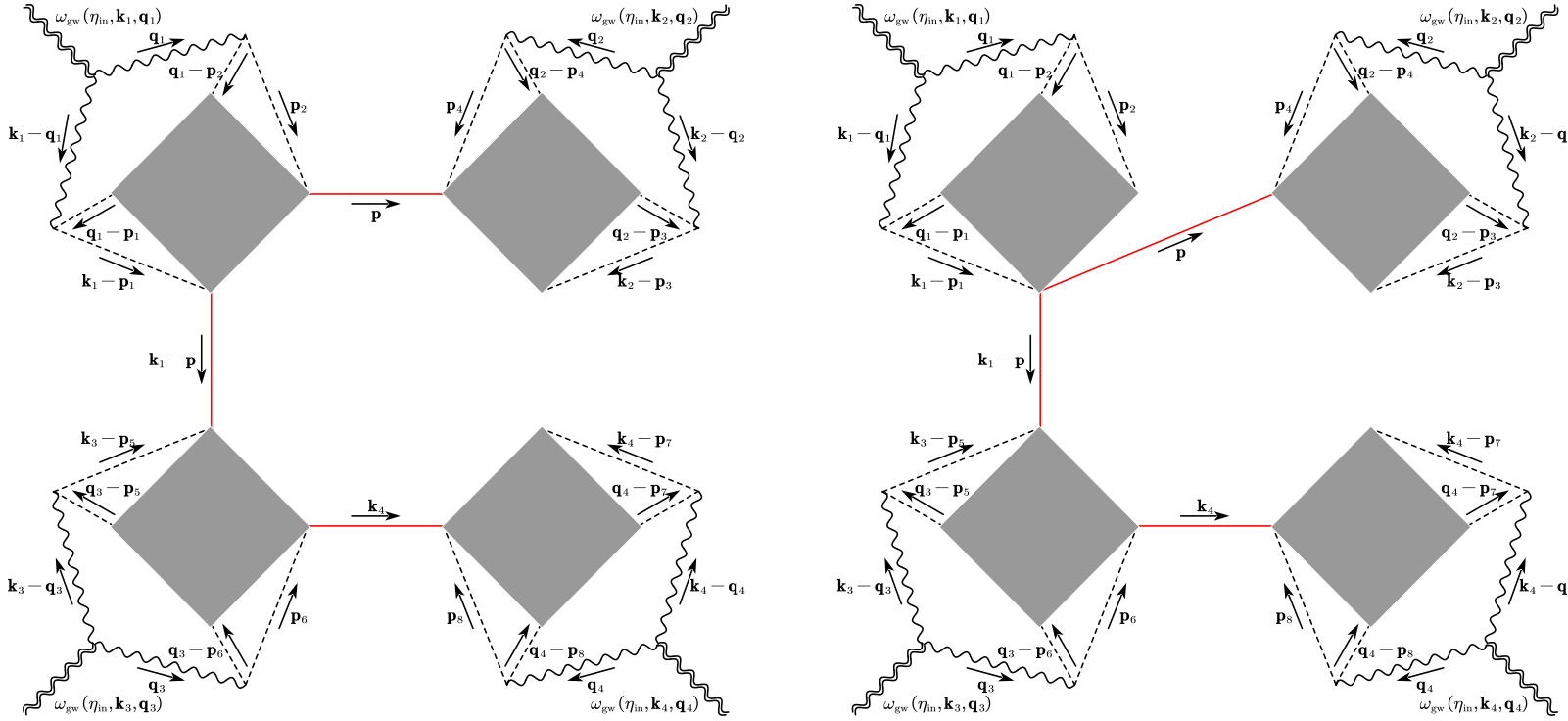
PNG-induced Inhomogeneity SW effect & NL effect

Gaussian $\zeta \rightarrow$ Gaussian SIGW

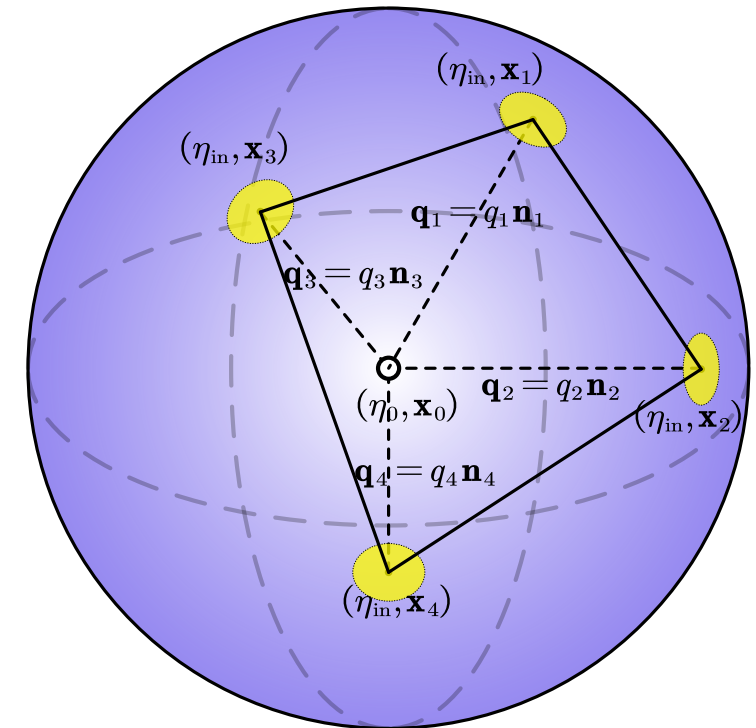


Four-point Angular Correlator

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056



$$\left\langle \prod_{i=1}^4 \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_i, q) \right\rangle$$

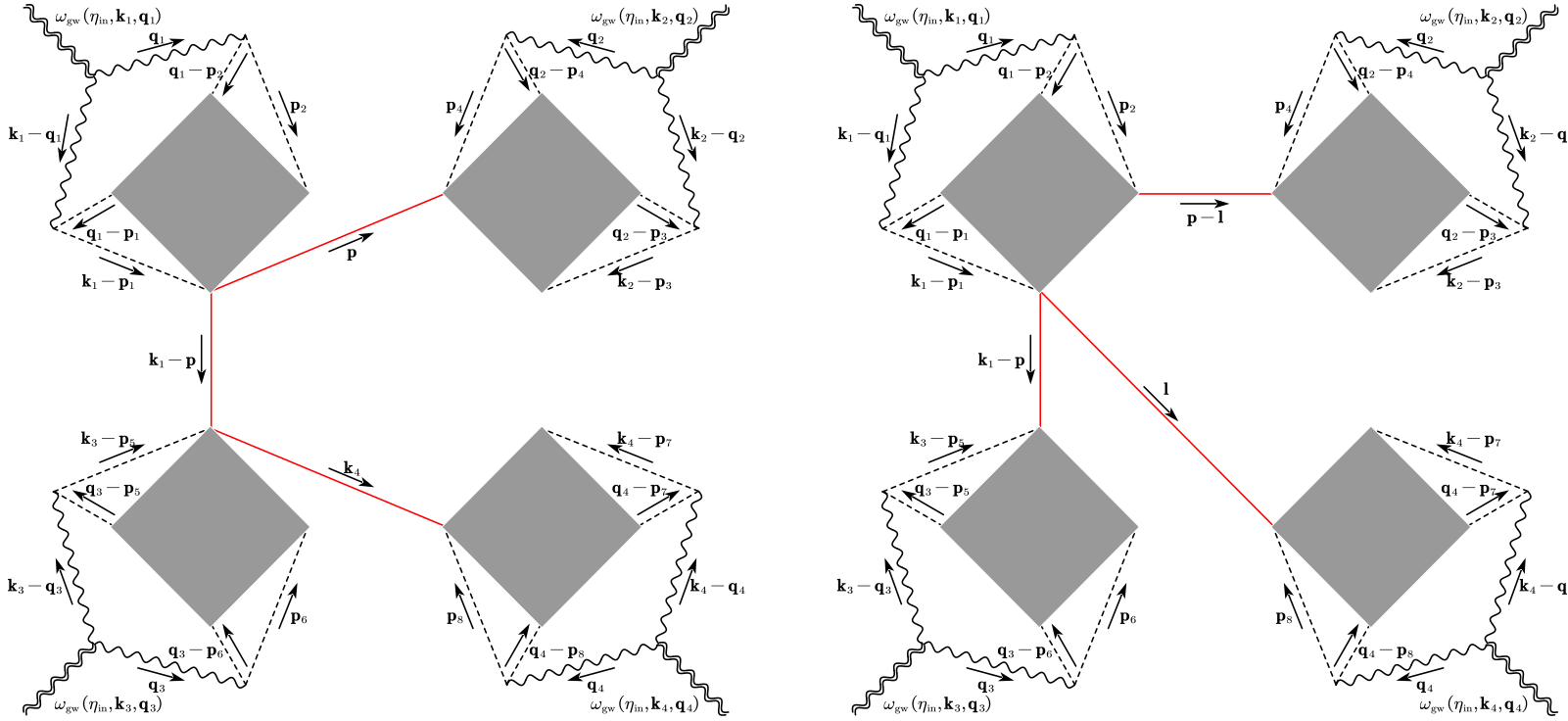


$$\omega_{\text{gw}, \text{in}}(q) = \bar{\omega}_{\text{gw}, \text{in}}(q) + \omega_{\text{gw}, \text{in}}^{(1)}(q) + \omega_{\text{gw}, \text{in}}^{(2)}(q)$$

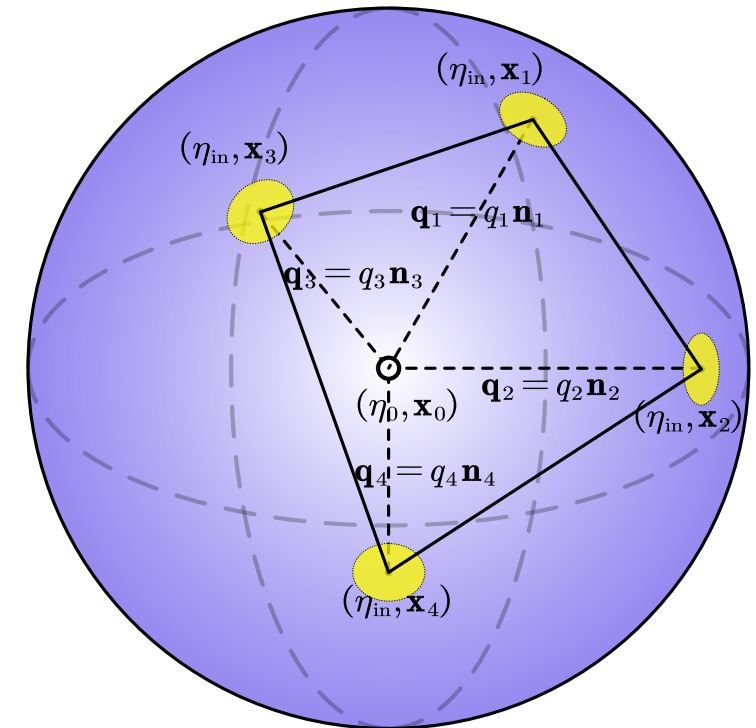
$$\omega_{\text{gw}, \text{in}}^{(2)}(q) = \omega_{\text{ng}}^{(2)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p}}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{p}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

Four-point Angular Correlator

JL, S. Wang, Z.-C. Zhao, and K. Kohri, JCAP 10 (2023), 056



$$\left\langle \prod_{i=1}^4 \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_i, q) \right\rangle$$



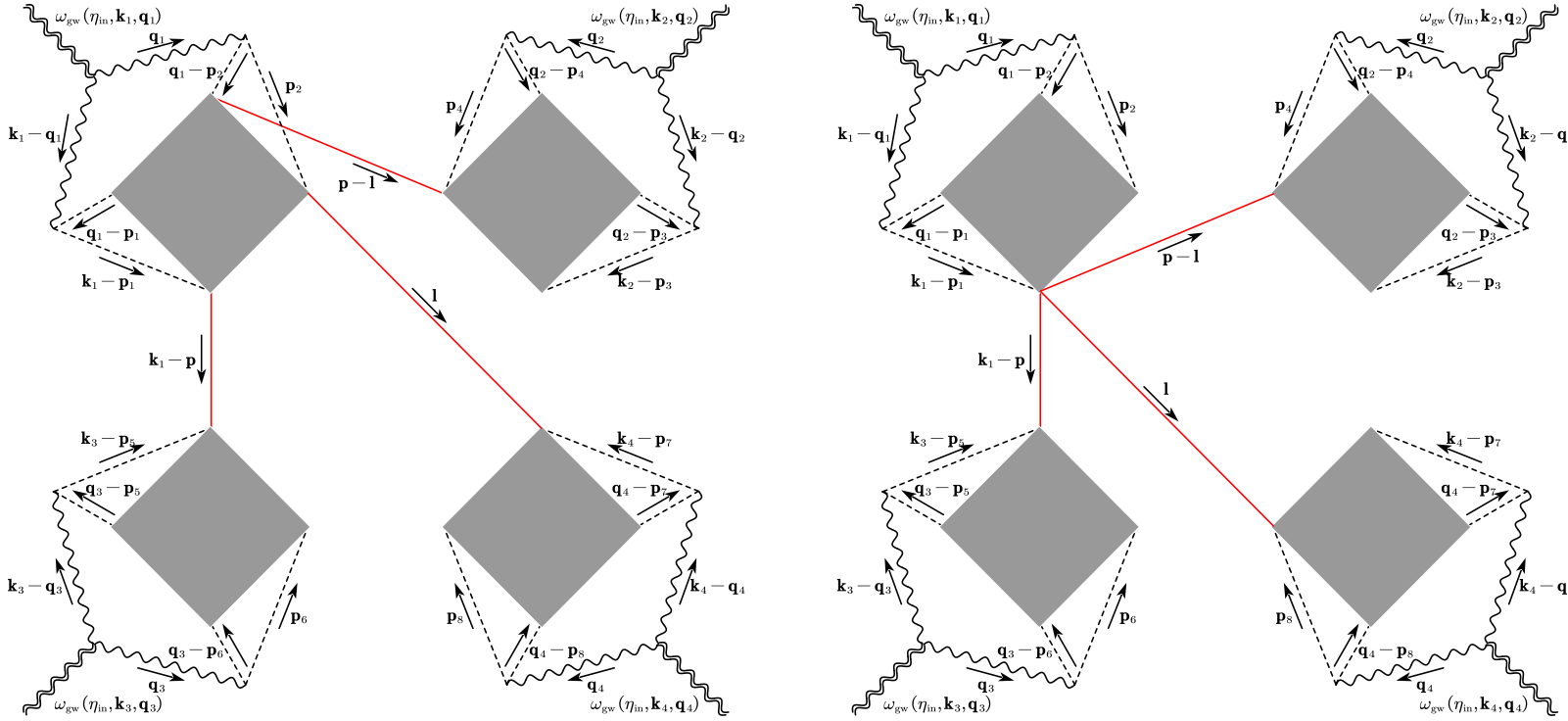
$$\omega_{\text{gw}, \text{in}}(q) = \bar{\omega}_{\text{gw}, \text{in}}(q) + \omega_{\text{gw}, \text{in}}^{(1)}(q) + \omega_{\text{gw}, \text{in}}^{(2)}(q) + \omega_{\text{gw}, \text{in}}^{(3)}(q)$$

$$\omega_{\text{gw}, \text{in}}^{(2)}(q) = \omega_{\text{ng}}^{(2)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p}}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{p}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

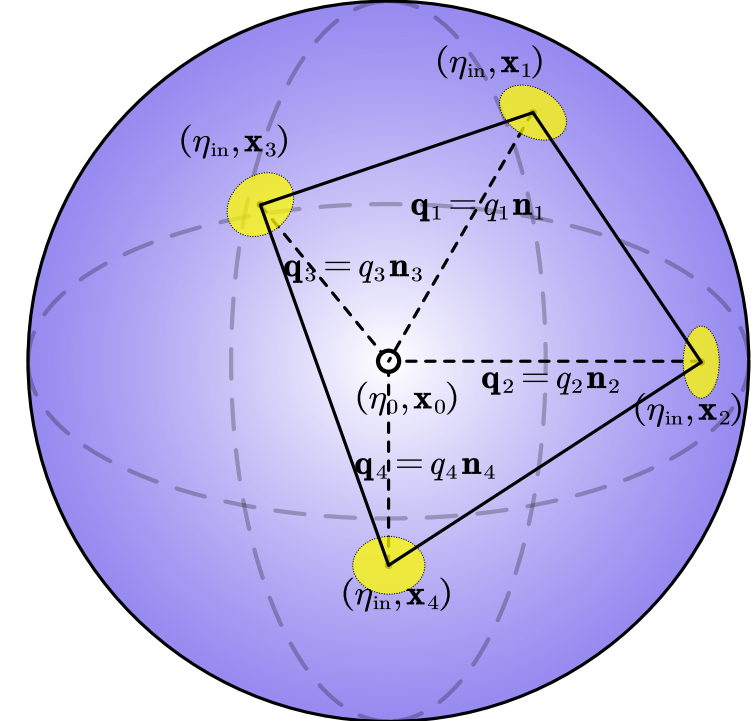
$$\omega_{\text{gw}, \text{in}}^{(3)}(q) = \omega_{\text{ng}}^{(3)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p} d^3 \mathbf{l}}{(2\pi)^{9/2}} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{l}) \zeta_{\text{gL}}(\mathbf{p} - \mathbf{l}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

Four-point Angular Correlator

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$$\left\langle \prod_{i=1}^4 \omega_{\text{gw}}(\eta_{\text{in}}, \mathbf{k}_i, q) \right\rangle$$



$$\omega_{\text{gw}, \text{in}}(q) = \bar{\omega}_{\text{gw}, \text{in}}(q) + \omega_{\text{gw}, \text{in}}^{(1)}(q) + \omega_{\text{gw}, \text{in}}^{(2)}(q) + \omega_{\text{gw}, \text{in}}^{(3)}(q)$$

$$\omega_{\text{gw}, \text{in}}^{(2)}(q) = \omega_{\text{ng}}^{(2)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p}}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{p}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

$$\omega_{\text{gw}, \text{in}}^{(3)}(q) = \omega_{\text{ng}}^{(3)}(q) \int \frac{d^3 \mathbf{k} d^3 \mathbf{p} d^3 \mathbf{l}}{(2\pi)^{9/2}} e^{i \mathbf{k} \cdot \mathbf{x}_{\text{in}}} \zeta_{\text{gL}}(\mathbf{l}) \zeta_{\text{gL}}(\mathbf{p} - \mathbf{l}) \zeta_{\text{gL}}(\mathbf{k} - \mathbf{p})$$

Angular Trispectrum

$$t_{\ell_3 \ell_4}^{\ell_1 \ell_2}(L, \nu) = \left[\frac{t_1(\nu)}{L(L+1)} + \frac{t_2(\nu)}{\ell_1(\ell_1+1)} + \frac{t_2(\nu)}{\ell_3(\ell_3+1)} \right] \frac{h_{\ell_1 \ell_2 L}}{\ell_2(\ell_2+1)} \frac{h_{\ell_3 \ell_4 L}}{\ell_4(\ell_4+1)}$$

$$t_1(\nu) = 4(2\pi A_L)^3 \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^2 \left[\frac{\omega_{\text{ng}}^{(2)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}F_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]^2$$

$$t_2(\nu) = \frac{(2\pi A_L)^3}{2\pi} \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^3 \left[\frac{\omega_{\text{ng}}^{(3)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}G_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]$$

PNG-induced Inhomogeneity

SW effect & NL effect

$$t_1(\nu) = \frac{b^2(\nu)}{\ell(\ell+1)\tilde{C}_\ell(\nu)}$$

Different multipole combinations \rightarrow $t_1(\nu)$ and $t_1(\nu)$ are distinguishable

Angular Trispectrum

$$t_2(\nu) = \frac{(2\pi A_L)^3}{2\pi} \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5} (6 - n_{\text{gw}}(\nu)) \right]^3 \left[\frac{\omega_{\text{ng}}^{(3)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5} G_{\text{NL}} (6 - n_{\text{gw}}(\nu)) \right]$$

$$\omega_{\text{ng}}^{(3)}(q) = \sum_{c=0}^4 \sum_{b=0}^{[4-c]} \sum_{a=0}^{[4-b-c]} \bar{\omega}_{\text{gw,in}}^{(a,b,c)}(q) (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4)$$

$$\bar{\omega}_{\text{gw,in}}^{(a,b,c)} \propto F_{\text{NL}}^a G_{\text{NL}}^b H_{\text{NL}}^c A^{(a+2b+3c)/2}$$

$$\mathcal{T}_1 = \sum_{l=1}^3 \sum_{n=1}^3 \sum_{i=1}^{o-1} \left(\frac{(i+l)!}{i!} \frac{V_1^{[i+1]}}{V_0^{[i]}} \right)^n \binom{N_i}{n} \delta_{ln,3}$$

$$N_1 = 4 - a - b - c$$

$$N_2 = a \quad N_3 = b \quad N_4 = c$$

$$\mathcal{T}_2 = \sum_{i=1}^3 \sum_{j=1}^{\min(i-1, o-1)} \sum_{i=1}^{o-1} N_i (N_i - 1) (i+2) (i+1)^2 \frac{V_1^{[i+2]} V_1^{[i+1]}}{(V_0^{[i]})^2}$$

$$\mathcal{T}_3 = \sum_{l_1=1}^3 \sum_{n_1=1}^3 \sum_{l_2=1}^3 \sum_{n_2=1}^{o-l_1} \sum_{i=2}^{o-l_1} \sum_{j=1}^{\min(i-1, o-l_2)} \left(\frac{(i+l_1)!}{i!} \frac{V_1^{[i+l_1]}}{V_0^{[i]}} \right)^{n_1} \binom{N_i}{n_1} \left(\frac{(j+l_2)!}{j!} \frac{V_1^{[j+l_2]}}{V_0^{[j]}} \right)^{n_2} \binom{N_j}{n_2} \delta_{l_1 n_1 + l_2 n_2, 3}$$

$$\mathcal{T}_4 = \sum_{i=1}^3 \sum_{j=1}^{\min(i-1, o-1)} \sum_{k=1}^{\min(j-1, o-1)} N_i N_j N_k (i+1) (j+1) (k+1) \frac{V_1^{[i+1]} V_1^{[j+1]} V_1^{[k+1]}}{V_0^{[i]} V_0^{[j]} V_0^{[k]}}$$

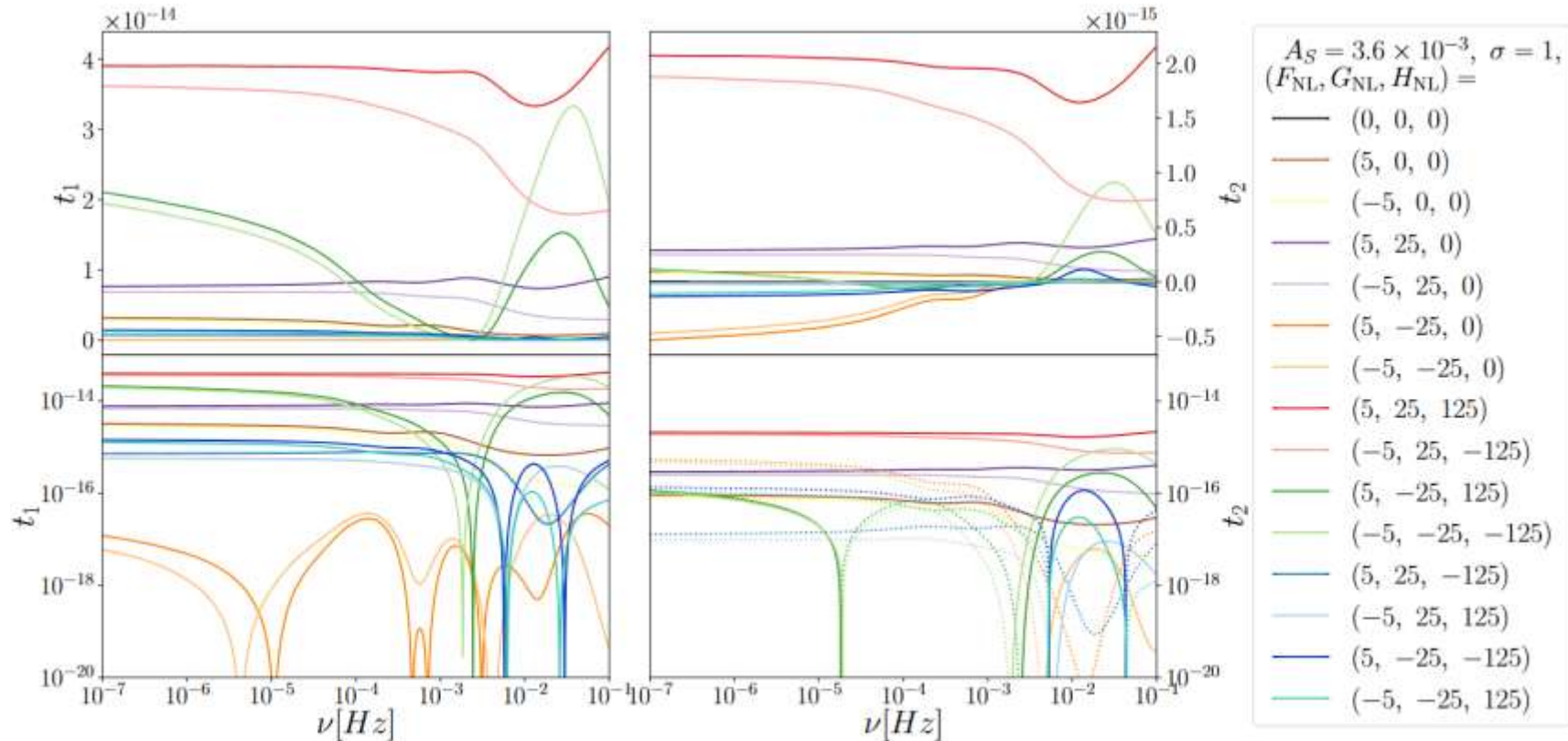
Angular Trispectrum

$$t_1(\nu) = 4(2\pi A_L)^3 \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^2 \left[\frac{\omega_{\text{ng}}^{(2)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}F_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]^2$$

$$t_2(\nu) = \frac{(2\pi A_L)^3}{2\pi} \left[\frac{\omega_{\text{ng}}^{(1)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}(6 - n_{\text{gw}}(\nu)) \right]^3 \left[\frac{\omega_{\text{ng}}^{(3)}(2\pi\nu)}{\bar{\omega}_{\text{gw,in}}(2\pi\nu)} + \frac{3}{5}G_{\text{NL}}(6 - n_{\text{gw}}(\nu)) \right]$$

PNG-induced Inhomogeneity

SW effect & NL effect



Gaussian ζ
 ↓
 Gaussian SIGW

Summary

Summary

What Have We Done?

- Developed a “renormalized” diagrammatic approach to compute the energy-density spectrum, angular power spectrum, bispectrum, and trispectrum of SIGWs for primordial non-Gaussianity up to arbitrary order, which enables the direct, automated generation of all necessary integrals for these spectra with only simple Python/Mathematica codes;
- Computed the numerical results for these observables for PNG up to h_{NL} order.

Scientific Implications & Future Prospects

- A supplementary probe for physics in the early universe that surpasses the reach of measurements from CMB and LSS;
- A promising tool for the search for PBHs;
- Extraction of SIGW signals from the stochastic background and inference of model parameters;

- ...
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Thanks!