Causality and Kramers–Kronig Relations in Gravitational Lensing

wave

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- TS, ApJ 896, (2020) (arXiv:2003.11748 [gr-qc])
- S.Tanaka and TS, PRD 108, 044015 (2023) (arXiv:2303.05650[gr-qc])
- S.Tanaka, G.Prabhu, S.Kapadia, TS, e-Print: 2504.21320 [gr-qc] (accepted for PRD)
- TS and S.Kapadia, PRD 112, 063529 (2025) (arXiv: 2506.02430 [gr-qc])

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2026 March 4 KEK

Gravitational lensing (GL) of light

powerful for probing

- dark matter
- cosmological parameters
- growth of large-scale structure
- gravitational physics
- ・・・

https://commons.wikimedia.org/wiki/Category:Gravitational_lensing?uselang=ja

GWs are also lensed

Difference between GL of light and GWs

e.g. Takahashi&Nakamura 2003

- **Coherent GW waveform**
- **Wave effects** $\lambda_{\mathrm{light}} \sim 5 \times 10^{-7}$ m, $\lambda_{\mathrm{GW}} \sim$ **3000 km**

Additional probe of cosmological parameters

Precise measurements of dark matter

New probe of sub-galactic-scale DM distribution

⋮

A new frontier in cosmology!

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No detection yet

Search for gravitational-lensing signatures in the full third observing run of the LIGO–Virgo network

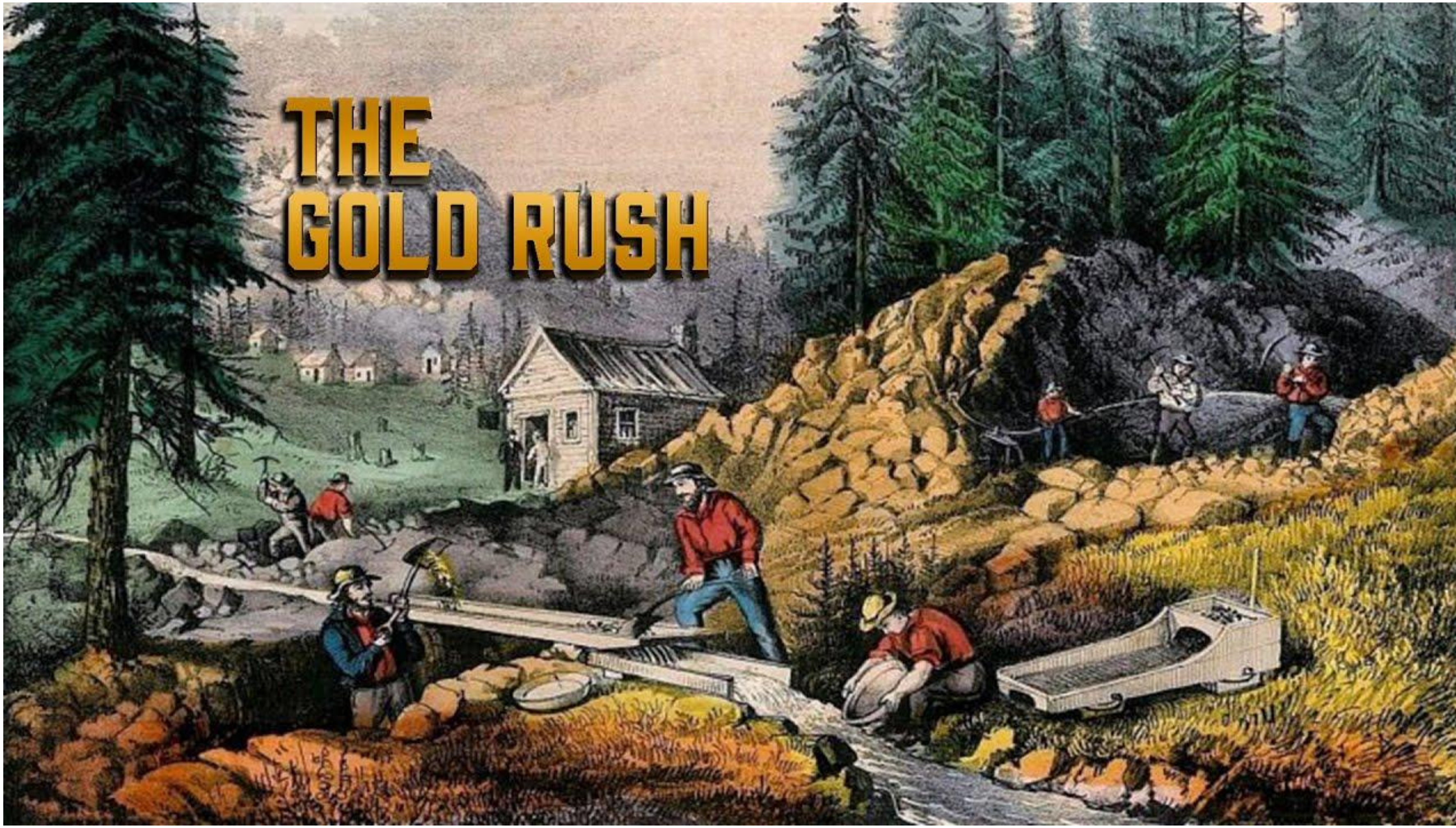
THE LIGO SCIENTIFIC COLLABORATION, THE VIRGO COLLABORATION AND THE KAGRA COLLABORATION

ABSTRACT

Gravitational lensing by massive objects along the line of sight to the source causes distortions of gravitational wave-signals; such distortions may reveal information about fundamental physics, cosmology and astrophysics. In this work, we have extended the search for lensing signatures to all binary black hole events from the third observing run of the LIGO–Virgo network. We search for repeated signals from strong lensing by 1) performing targeted searches for subthreshold signals, 2) calculating the degree of overlap amongst the intrinsic parameters and sky location of pairs of signals, 3) comparing the similarities of the spectrograms amongst pairs of signals, and 4) performing dual-signal Bayesian analysis that takes into account selection effects and astrophysical knowledge. We also search for distortions to the gravitational waveform caused by 1) frequency-independent phase shifts in strongly lensed images, and 2) frequency-dependent modulation of the amplitude and phase due to point masses. None of these searches yields significant evidence for lensing. Finally, we use the non-detection of gravitational-wave lensing to constrain the lensing rate based on the latest merger-rate estimates and the fraction of dark matter composed of compact objects.

arXiv:2304.08393

In the future, lensing of the GWs will become daily events.



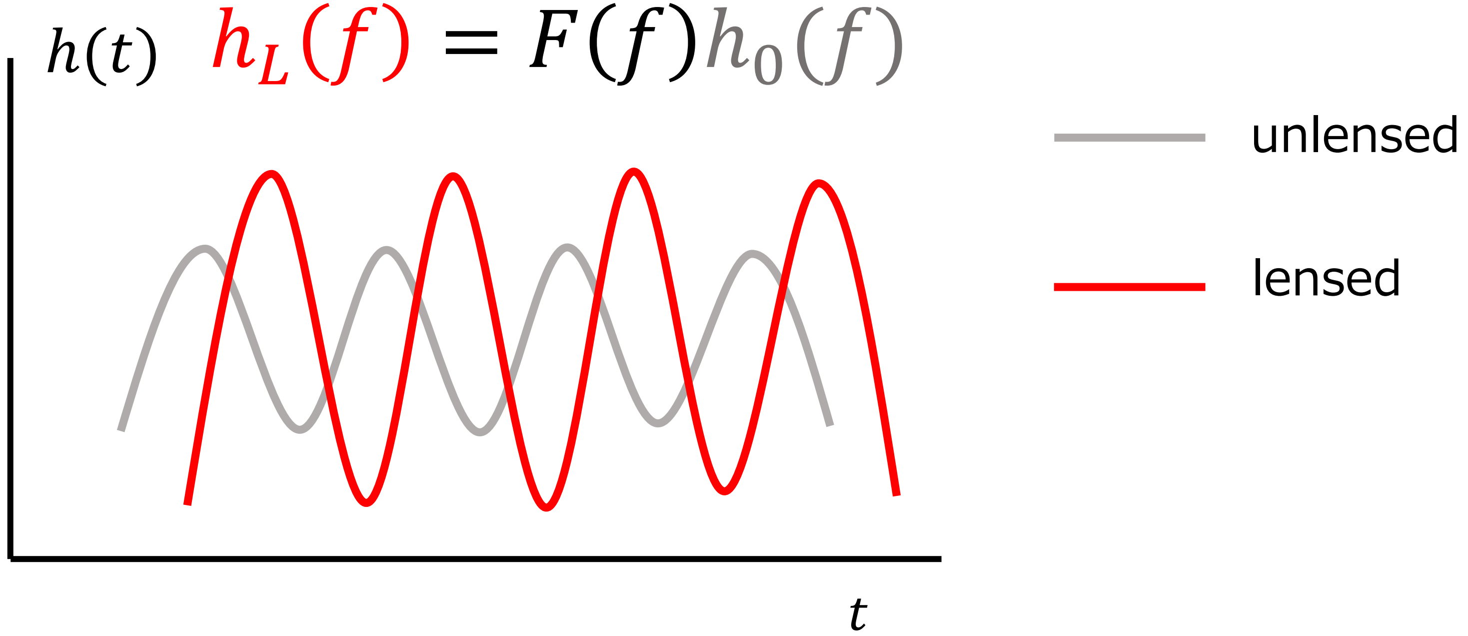
The golden era of the GL of GWs coming soon

Theoretical understandings in wave optics

Lensing of GWs

Lensing deforms the waveform in f-dependent manner

$$h_L(f) = F(f)h_0(f)$$

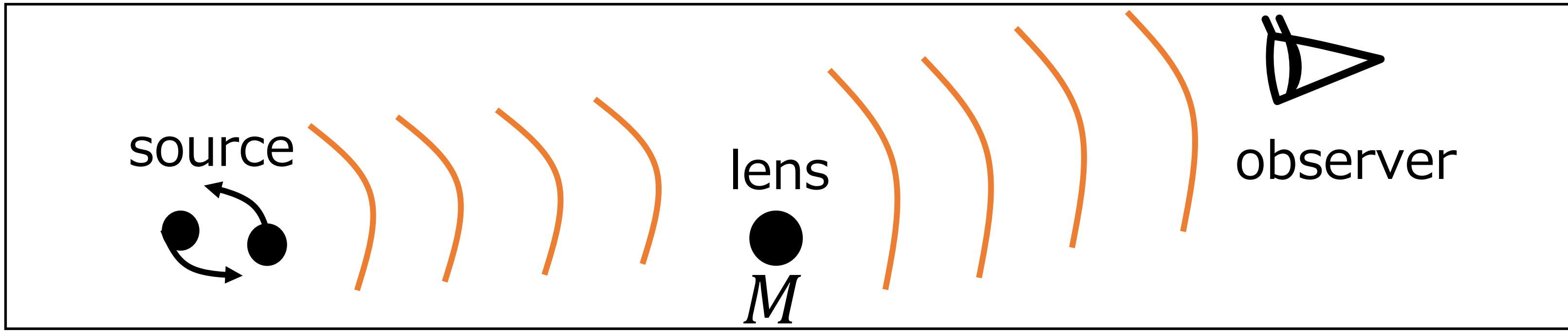


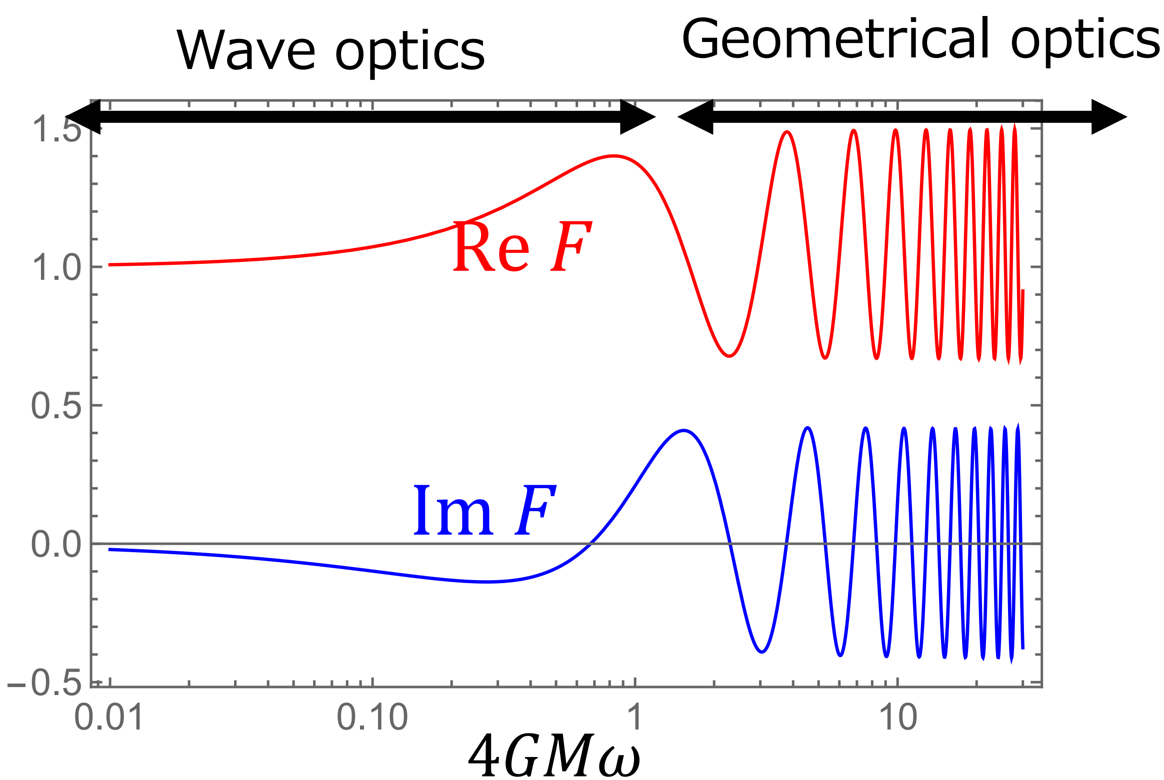
Amplification factor (complex number)

$$F(f) = A(f)e^{i\phi(f)}$$

Measurement of lens = measurement of $F(f)$

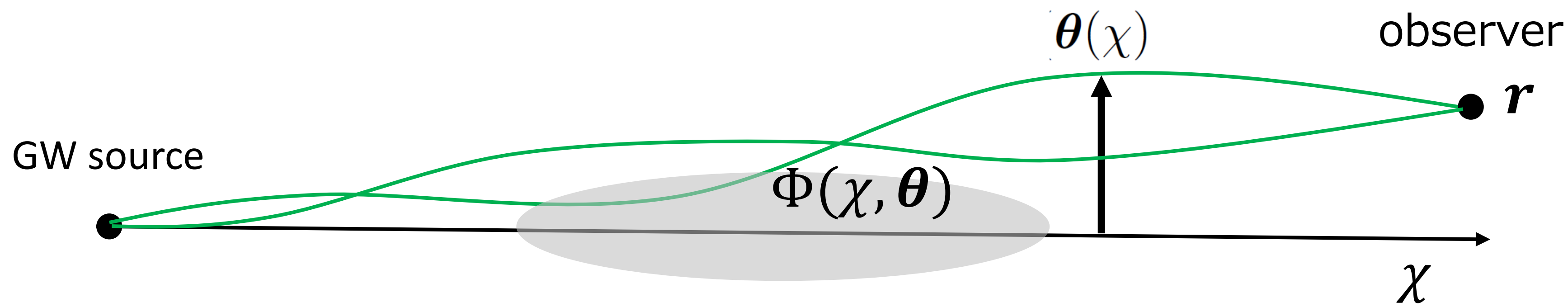
Example: point-mass lens





Path integral formula

Nakamura&Deguchi 1999, Leung et al. 2023



$$F(\boldsymbol{r},f)=\int D\theta\,\exp\left(i\omega\int_0^{\chi_s}L(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\chi)d\chi\right)$$

$$\int_0^{\chi_s}L\,d\chi=t_d[\boldsymbol{\theta}]$$

t_d: light travel time

$$L\big(\boldsymbol{\theta}(\chi),\dot{\boldsymbol{\theta}}(\chi),\chi\big)=\frac{1}{2}\chi^2\dot{\boldsymbol{\theta}}^2-2\Phi(\chi,\boldsymbol{\theta})$$

Equivalent to 2d quantum mechanics with time-dependent mass and potential

Geometrical optics

High frequency limit $\omega \to \infty$ (the classical limit $\hbar \to 0$)

$$F(\boldsymbol{r}, f) = \int D\theta \exp\left(i\omega \int_0^{\chi_s} L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \chi) d\chi\right)$$

Only paths satisfying $\frac{\delta t_d}{\delta \boldsymbol{\theta}} = 0$ contribute. (Fermat's principle)

$$F(\omega) = \sqrt{\mu_1} + \sum_{j=2}^{n} \sqrt{\mu_j} \exp\left[i(\omega T_{1j} - \pi n_j)\right]$$

μ_1:magnification of the first arrival image

$T_{1j} = T_1 - T_j$

Superluminal GWs?

THE ASTROPHYSICAL JOURNAL, 835:103 (9pp), 2017 January 20
doi:10.3847/1538-4357/835/1/103

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ARRIVAL TIME DIFFERENCES BETWEEN GRAVITATIONAL WAVES AND ELECTROMAGNETIC SIGNALS DUE TO GRAVITATIONAL LENSING

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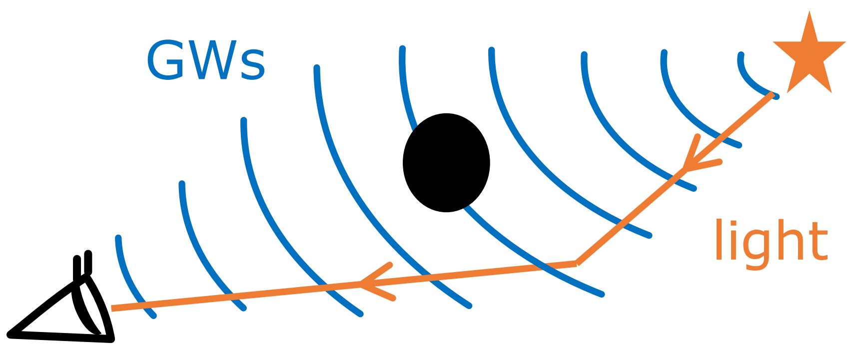
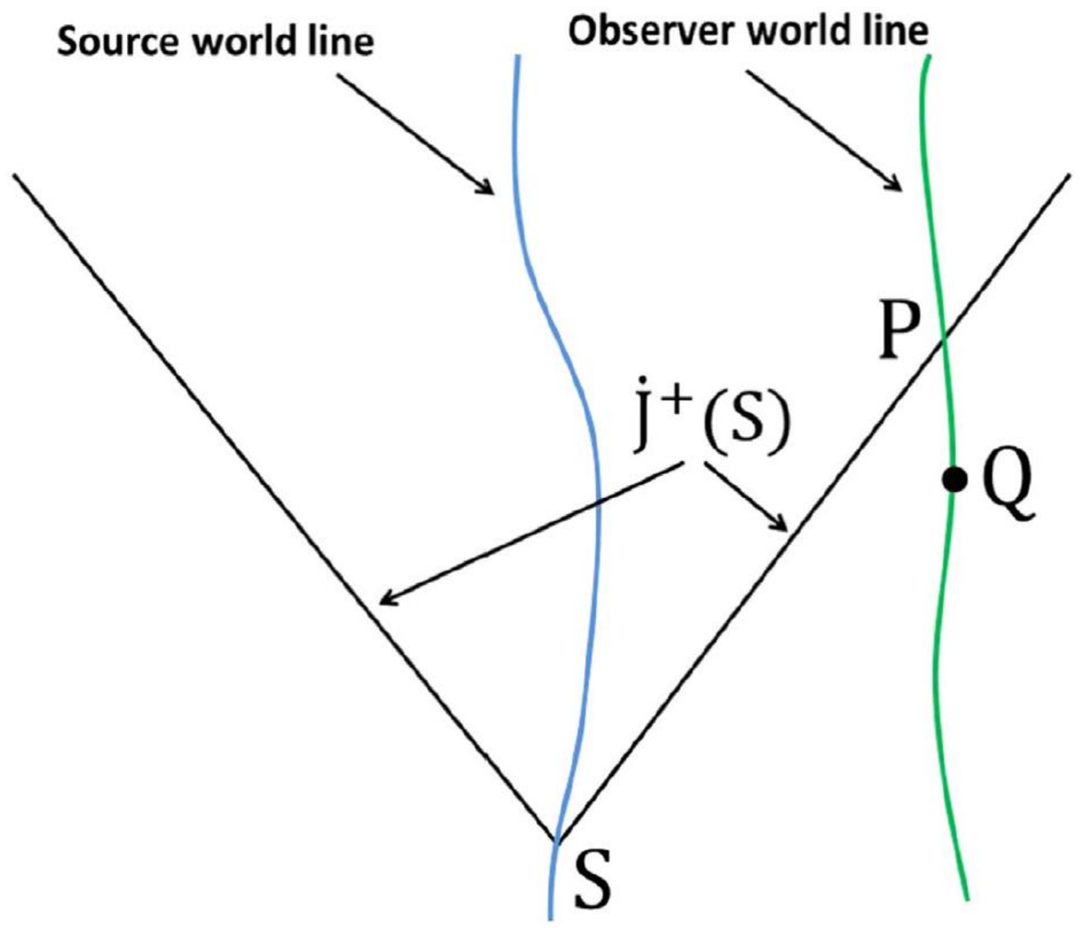
ABSTRACT

In this study we demonstrate that general relativity predicts arrival time differences between gravitational wave (GW) and electromagnetic (EM) signals caused by the wave effects in gravitational lensing. The GW signals can arrive *earlier* than the EM signals in some cases if the GW/EM signals have passed through a lens, even if both signals were emitted simultaneously by a source. GW wavelengths are much larger than EM wavelengths; therefore, the propagation of the GWs does not follow the laws of geometrical optics, including the Shapiro time delay, if the lens mass is less than approximately $10^5\, M_\odot (f/\mathrm{Hz})^{-1}$, where f is the GW frequency. The arrival time difference can reach $\sim 0.1\,\mathrm{s}\,(f/\mathrm{Hz})^{-1}$ if the signals have passed by a lens of mass $\sim 8000\, M_\odot (f/\mathrm{Hz})^{-1}$ with the impact parameter smaller than the Einstein radius; therefore, it is more prominent for lower GW frequencies. For example, when a distant supermassive black hole binary (SMBHB) in a galactic center is lensed by an intervening



Phase velocity, group velocity are superluminal



Physical signals propagating from S to Q break causality

The Astrophysical Journal, 896:46 (4pp), 2020 June 10
https://doi.org/10.3847/1538-4357/ab8d3f

On Arrival Time Difference Between Lensed Gravitational Waves and Light

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Abstract

It is known that geometrical optics no longer applies to gravitational lensing if the wavelength of a propagating wave becomes comparable to or larger than the Schwarzschild radius of a lensing object. We investigate the propagation of gravitational waves in wave optics, particularly focusing on the difference between their arrival time and the arrival time of light. We argue that, contrary to the observation in the previous work, gravitational waves never arrive at an observer earlier than light when both gravitational waves and light are emitted from a same source simultaneously.

Unified Astronomy Thesaurus concepts: Gravitational lensing (670)

Lensed GWs never arrive earlier than light

Simple explanation

$$F(\omega) = \sum_{\text{path}:\boldsymbol{\theta}} e^{i\omega t_d[\boldsymbol{\theta}]}$$

$$h_L(t) = \int f(t-t')h_0(t')dt'$$

$$f(t) = \sum_{\boldsymbol{\theta}} \int \frac{d\omega}{2\pi} e^{i\omega t_d[\boldsymbol{\theta}] - i\omega t} = \sum_{\boldsymbol{\theta}} \delta(t - t_d[\boldsymbol{\theta}])$$

$f(t) = 0$ for $t < \min_{\boldsymbol{\theta}} t_d[\boldsymbol{\theta}] = T_{\min}$

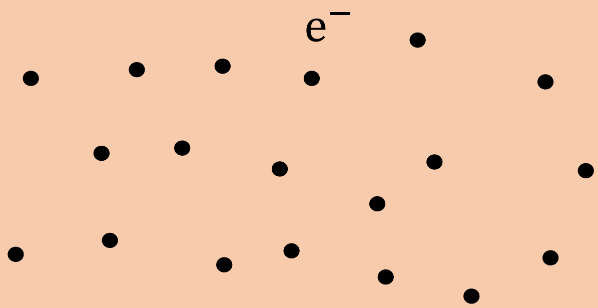
$T_{\min}$: travel time of geodesic (=light)

Lensed GWs never arrive earlier than light

Lensed signal comes only after its geometrical optics counterpart (remains inside the lightcone)

No violation of causality in GL

Kramers-Kronig (KK) relation



$$\boldsymbol{D}(\omega) = \epsilon(\omega)\boldsymbol{E}(\omega)$$

$n\ (= \mathrm{Re}\epsilon)$: refractive index

$\kappa(= \mathrm{Im}\epsilon)$: extinction coefficient

Causality ⇒ $\epsilon(t) = 0$ for $t < 0$

KK relation

$$n(\omega) - 1 = \frac{1}{\pi} \fint d\omega' \frac{\kappa(\omega')}{\omega'(\omega' - \omega)}$$

↖principal value integral

KK relation is satisfied in any materials

K-K relation in GL

Causality ⇒ KK relation

$$\mathrm{Re}\,F(\omega) \equiv 1 + K(\omega),$$
$$\mathrm{Im}\,F(\omega) \equiv S(\omega).$$

GL version of KK relation

Tanaka and TS
PRD 108, 044015 (2023)

$$\frac{K(\omega)}{\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\frac{S(\omega')}{\omega'},$$
$$\frac{S(\omega)}{\omega} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\frac{K(\omega')}{\omega'}.$$

KK relation is satisfied in any lens

Derivation of the K-K relation

- Causality

$$\Longleftrightarrow \quad F(\omega) = \int_{T_{min}}^{\infty} dt \; f(t) e^{i\omega t}$$

good convergence in I^+

$$\Longleftrightarrow \quad F(\omega) e^{-i\omega T_{min}} = \int_{0}^{\infty} dt \; f(t + T_{min}) e^{i\omega t}$$

redefine → $F(\omega)$

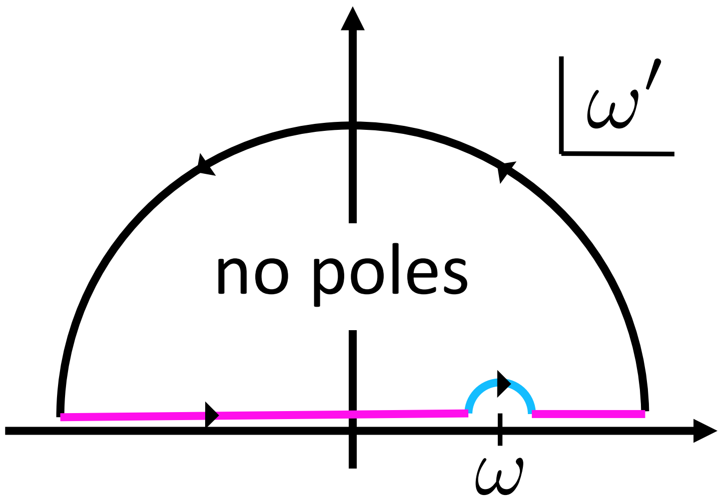
F is regular in upper half of complex ω-plane (I^+)

- $F(0) = 1, \quad \lim_{\omega \to \infty} |F(\omega)| < C$

$G(\omega) := \frac{F(\omega)-1}{\omega}$ has no poles on the real axis and in I^+ and $G \to 0$ $(\omega \to \infty)$.

Derivation of the K-K relation

- Cauchy's integral theorem



$$\oint_C \frac{d\omega'}{\omega' - \omega} \frac{F(\omega') - 1}{\omega'} = 0$$

$$\Longleftrightarrow \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \frac{F(\omega') - 1}{\omega'} - \pi i \frac{F(\omega) - 1}{\omega} = 0$$

$$\Longleftrightarrow \quad \mathrm{Re}\, F(\omega) = 1 + \frac{\omega}{\pi} -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \frac{\mathrm{Im}\, F(\omega')}{\omega'}$$

Sum rules (implications of the K-K relation)

Condensed matter

$$\int_0^\infty d\omega\, \kappa(\omega) = \frac{\pi}{2}\omega_p^2$$

ω_p:plasma mass

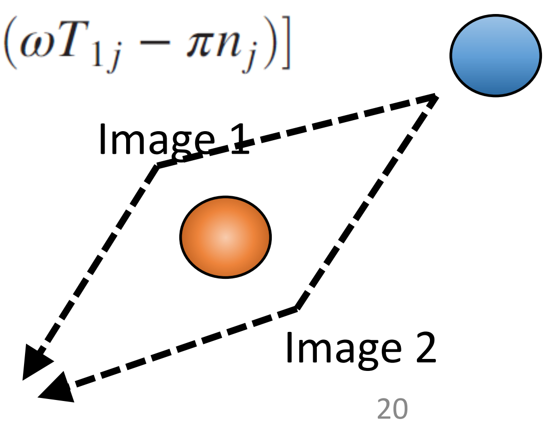
GL of GWs

$$\frac{2}{\pi}\int_0^\infty d\omega \frac{S(\omega)}{\omega} = 1 - \sqrt{\mu_1}$$

I don't know an intuitive explanation for this...

Geometrical optics $F(\omega) = \sqrt{\mu_1} + \sum_{j=2}^{n} \sqrt{\mu_j} \exp[i(\omega T_{1j} - \pi n_j)]$

μ_1:magnification of the first arrival image



Reformulation of the KK relation

TS and Kapadia
PRD **112**, 063529(2025)

$$F(\omega) = |F(\omega)| e^{i\theta(\omega)}$$

The magnitude and the phase have clear physical meaning

Does KK relation for $|F|$ and θ exist?

※ YES for condensed matter

e.g. Stern 1963

$$\ln F(\omega) = \ln|F(\omega)| + i\theta(\omega)$$

Physics experiment class (undergraduate, Science Tokyo)

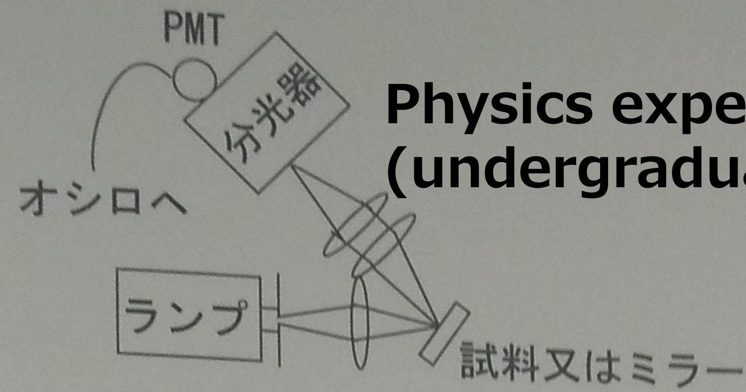


図 18: 反射測定の配置図

多くの物質は吸収スペクトルが測定できないため，一般には垂直反射の反射スペクトル $R(\omega)(=|r(\omega)|^2)$ の測定から誘電率を求めることが多い。図のような測定系を組み立て，試料からの反射強度 $I_R(\omega)$ とアルミミラーからの反射強度 $I_{R0}(\omega)$ を測定する。出来るだけ垂直反射の条件に近づけるため，入射角は小さいほうが良い。反射率 $R(\omega)$ は，$R(\omega)=I_R(\omega)/I_{R0}(\omega)$，から求める。ここではアルミミラーの反射率を 1 であると近似する。垂直反射の場合の振幅反射率 $r(\omega)$ はフレネルの反射公式 (物理学実験第一"光の波動的性質"参照) より，

$$r(\omega)=\frac{n+i\kappa-1}{n+i\kappa+1}=\sqrt{R(\omega)}\exp[i\theta(\omega)]. \quad (91)$$

または，

$$\ln r(\omega)=\ln\sqrt{R(\omega)}+i\theta(\omega). \quad (92)$$

ここで，θ は反射の際の光の位相変化を示す。クラマース・クローニッヒの関係[14] より，

$$\theta(\omega)=\frac{2\omega}{\pi}P\int_0^\infty \frac{\ln\sqrt{R(\nu)}}{\nu^2-\omega^2}d\nu. \quad (96)$$

したがって，反射スペクトルが十分広い光エネルギー領域で測定できれば，(96) 式より $\theta(\omega)$ を求めることができる。(91) 式より，屈折率の実部と虚部は，

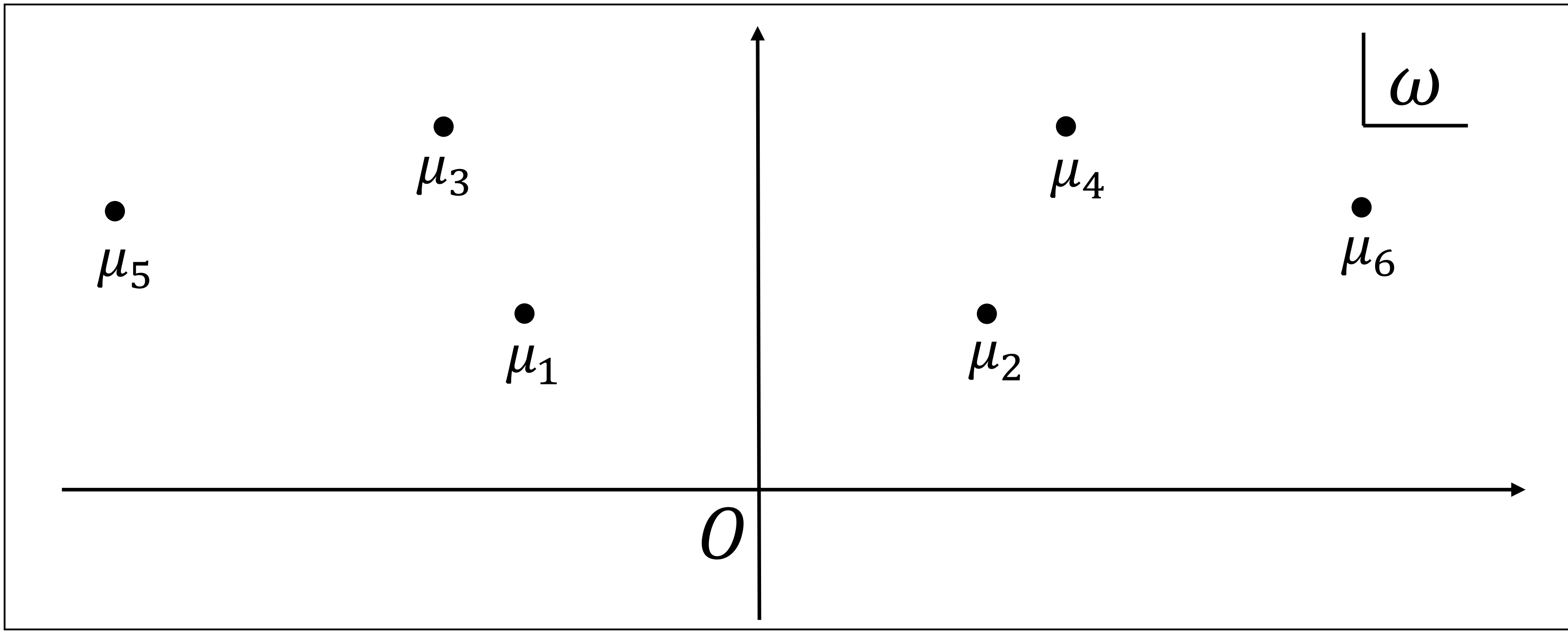
KK relation for $|F|$ and θ in condensed matter physics is well known.

$A(\omega)$: analytic in upper-half plane

$A_{new}(\omega) = B(\omega)A(\omega)$ is analytic

$$B(\omega) = \prod_n \left(\frac{\omega - \mu_n}{\mu_n^* - \omega} \right)$$

Blaschke product



$|B(\omega)| = 1$ for $\omega \in \mathbb{R}$

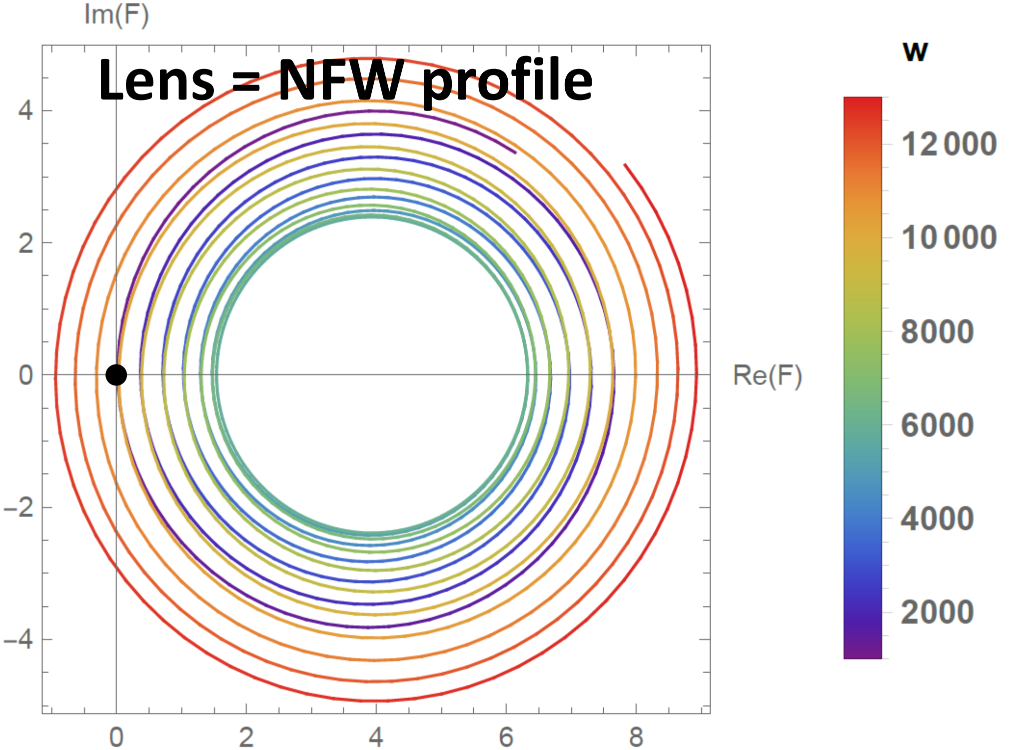
$|A_{new}(\omega)| = |A(\omega)|$ for $\omega \in \mathbb{R}$

$|A(\omega)|$ does not uniquely determine the phase of $A(\omega)$.

Possible existence of the B-product in GL

$$\arg(\sqrt{\mu_1}+\sum_{j=2}^{N}\sqrt{\mu_j}e^{i\omega\Delta t_j-\pi i n_j \mathrm{sgn}(\omega)})$$

$$=-\frac{\omega}{\pi}\int_0^\infty \ln\left(\sum_{j=1}^{N}\mu_j+2\sum_{j>k}^{N}\sqrt{\mu_j\mu_k}\cos(\omega'\Delta t_{jk}-\pi n_{jk})\right)\frac{d\omega'}{\omega'^2-\omega^2}-i\ln B(\omega)$$



The Blaschke product exists in this case

KK Relations for the magnitude and the phase

$$\theta(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\ln|F(\omega')|}{\omega'^2-\omega^2}d\omega' - i\sum_n \ln\left(\frac{\omega-\mu_n}{\mu_n^*-\omega}\right)$$

$$\ln|F(\omega)| = \frac{2}{\pi}\int_0^\infty \left(\frac{\omega'}{\omega'^2-\omega^2} - \frac{1}{\omega'}\right)\theta(\omega')d\omega'$$

The phase is not uniquely determined from the magnitude.

The magnitude is uniquely determined from the phase.

I don’t know an intuitive explanation for this...

Low-frequency behavior

Choi et al. 2021, Tambalo et al. 2023

$$F(\omega) = \begin{cases} 1 + 2^{-\frac{k}{2}} e^{-i\frac{k\pi}{4}} \Gamma\left(1 - \frac{k}{2}\right) w^{\frac{k}{2}} + \cdots & (\text{Generalized SIS}) \\ 1 + \frac{w}{2}\left(\pi + 2i \ln w\right) + \cdots & (\text{Point} - \text{mass lens}) \end{cases}$$

$$\rho(r) \propto \frac{1}{r^{k+1}}$$

Now assume $|F(\omega)| = 1 + A\omega^{\alpha} + \cdots$

Point mass ($\alpha = 1$), SIS ($\alpha = \frac{1}{2}$), gSIS ($\alpha = \frac{k}{2}$)

KK relation

$$\theta(\omega) = \begin{cases} -\tan\left(\frac{\pi\alpha}{2}\right) \ln|F(\omega)| + \cdots & 0 \le \alpha \le 1 \\ \ln|F(\omega)| \ln \omega + \cdots & \alpha = 1 \end{cases}$$

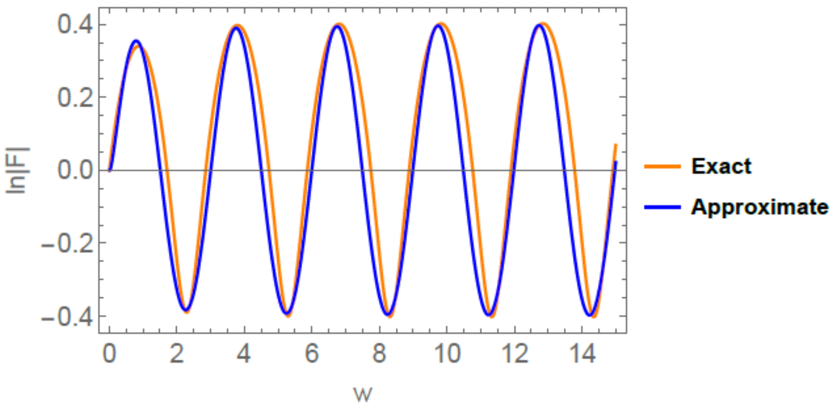
Causality strongly restricts the low-frequency behavior of the lens signal.

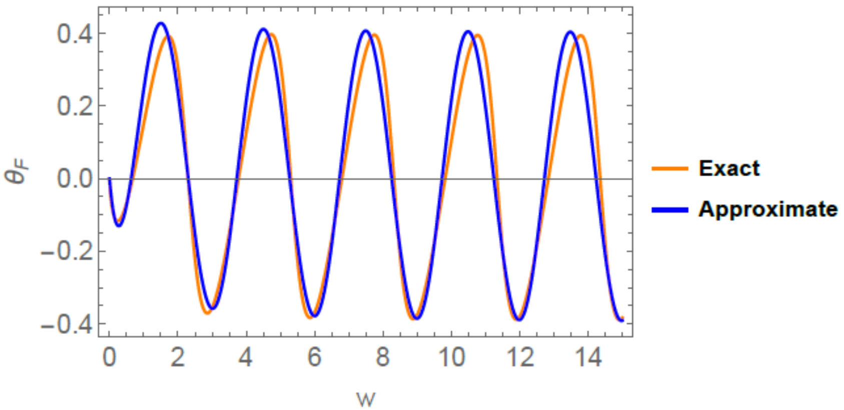
Phenomenology

It may be useful if there is an example which gives analytic form of the phase

$$\ln|F(\omega)| = \frac{\alpha\omega}{\omega+\beta}\cos(\gamma\omega+\phi)$$

$$\theta(\omega) = -\frac{\alpha\,\omega}{\pi(\beta-\omega)(\beta+\omega)}\Big[-2\beta\cos(\beta\gamma-\phi)\,\mathrm{Ci}(\beta\gamma) + 2\,\mathrm{Ci}(\gamma\omega)\,(\beta\cos\phi\cos(\gamma\omega)+\omega\sin\phi\sin(\gamma\omega)) + \beta\sin(\beta\gamma-\phi)\,(\pi-2\,\mathrm{Si}(\beta\gamma)) + \cos\phi\sin(\gamma\omega)\,(-\pi\omega+2\beta\,\mathrm{Si}(\gamma\omega)) + \cos(\gamma\omega)\sin\phi\,(\pi\beta-2\omega\,\mathrm{Si}(\gamma\omega))\Big],$$





Application to lensing measurement

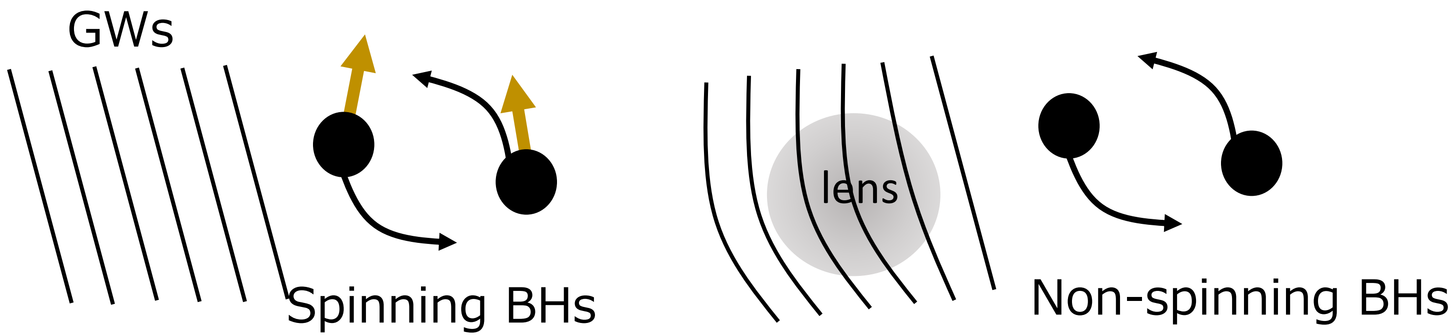
Tanaka et al. 2025

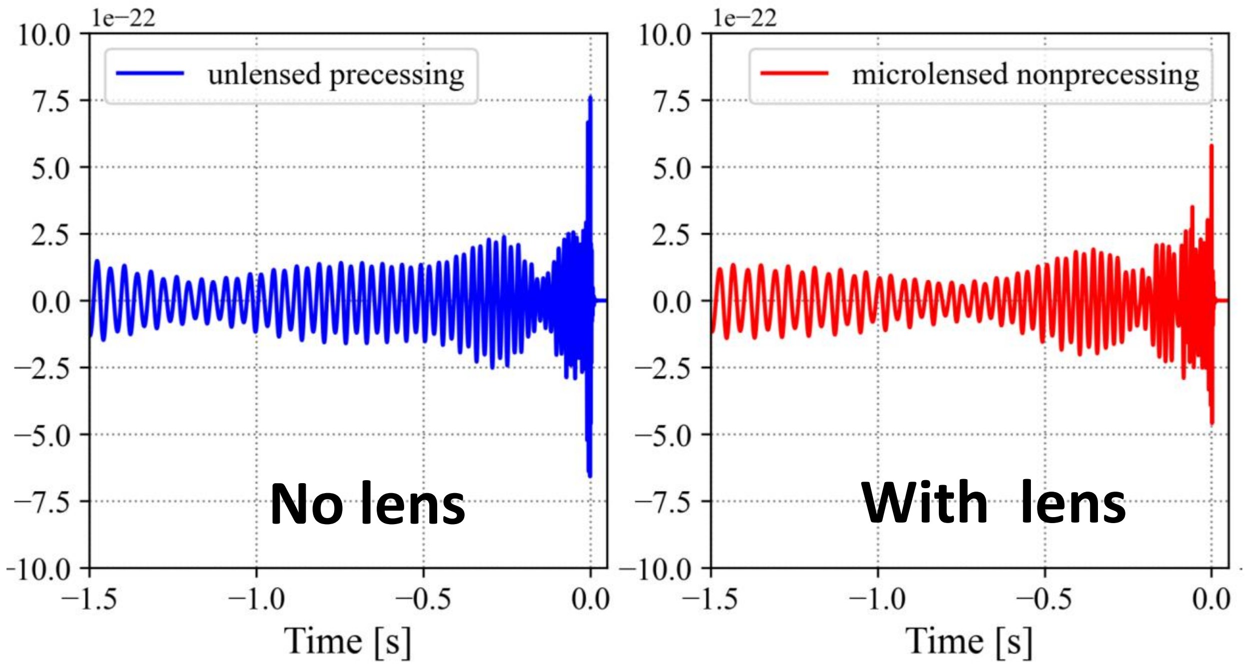
$$\boldsymbol{h_L(f)} = F(f)h_0(f)$$

How can we know it is lensed or unlensed?

(This question persists even if the GW data is free from errors)

Lensing of GWs





Kim&Liu 2023

Some GW sources mimic lens signal

Application to lensing measurement

Tanaka, Prabhu, Kapadia, TS (2025)

Lensing hypothesis

$h_{obs}(f)$ is a waveform lensed from template $h_0(f;\boldsymbol{\theta_S})$

$$F_{obs}(f) := \frac{h_{obs}(f)}{h_0(f;\boldsymbol{\theta_S})}$$

Any $F_{obs}(f)$ which does not satisfy the KK relation is ruled out as **false** GL signal.

***This is independent of the lensing model.**

Can the KK relation be used to reject the hypothesis even when the frequency band is finite?

Application to lensing measurement

Quantification of the violation of the KK

$$\Delta \coloneqq (LHS) - (RHS)$$

Assumptions on measurements:

1. $\omega_{min} \ll \omega \ll \omega_{max}$, 2. wave optics regime at ω_{min} , 3. no noise

If $F(f)$ is a correct lens signal,

$$\mathrm{Re}\Delta = O(1) \times K(\omega_{min})$$

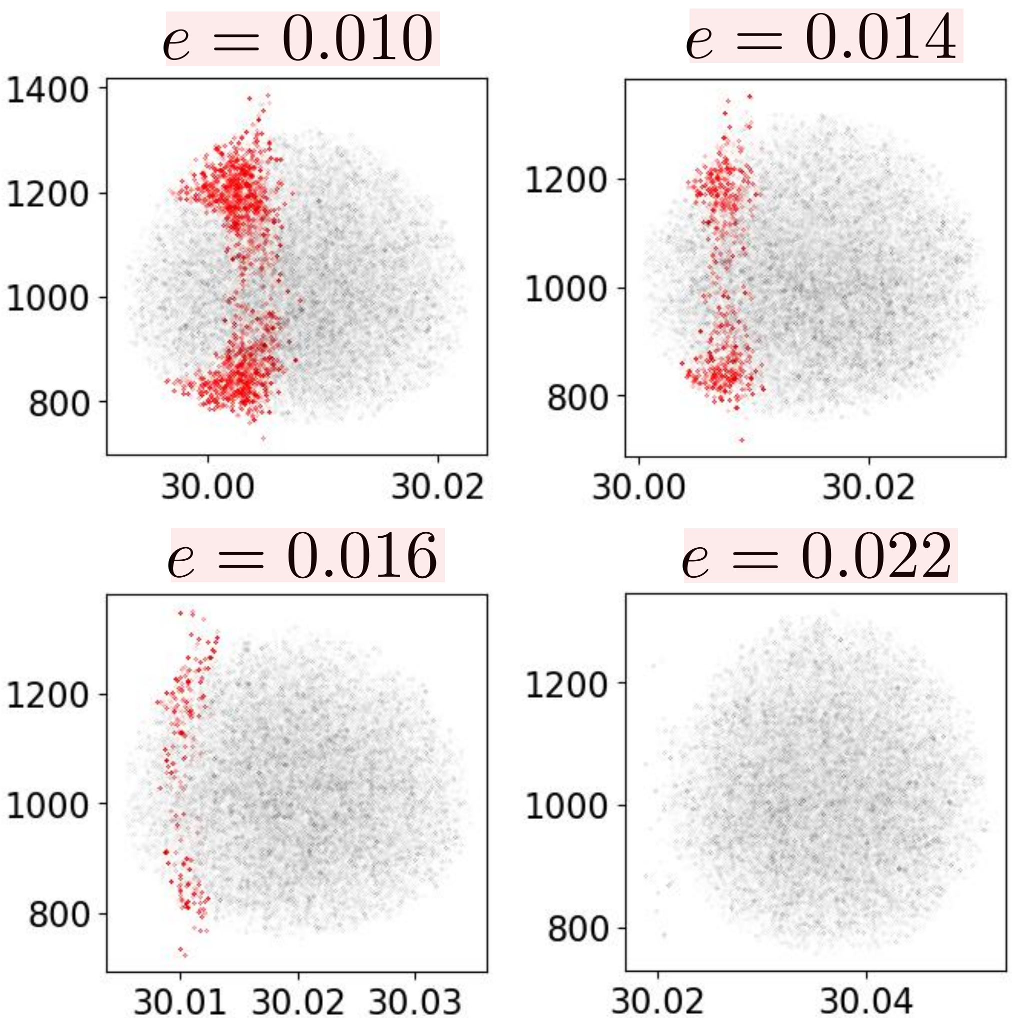
We reject unlensed template if $\mathrm{Re}\Delta > r_{\mathrm{th}} K(\omega_{min})$

Proof of concept

Tanaka, Prabhu, Kapadia, TS (2025)

GWs from an eccentric orbit

- $\phi_{obs} = \phi_{ecc}$ $m - d$ plot



ϕ_{ecc}'s parameters

$m = 30 M_\odot$

$d = 1000$ Mpc

$t_{coal} = 0$

$\varphi_{coal} = 0$

● ruled out

● not ruled out

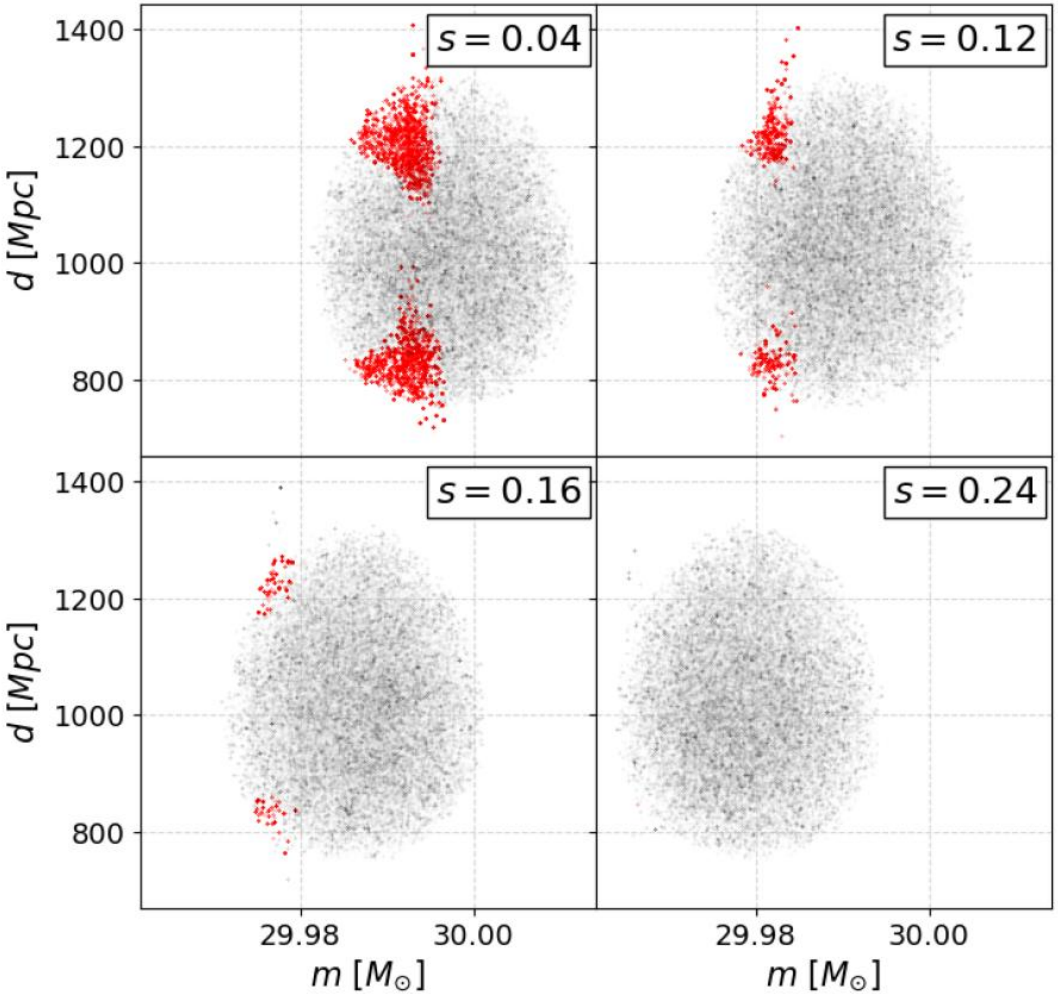
Large eccentric orbit can be distinguished from lensing

Proof of concept

Tanaka, Prabhu, Kapadia, TS (2025)

GWs from a spinning orbit

- $\phi_{obs} = \phi_{spin}$ $m - d$ plot



parameters

$m = 30 M_\odot$

$d = 1000\ \mathrm{Mpc}$

$t_{coal} = 0$

$\varphi_{coal} = 0$

● ruled out

● not ruled out

Orbit with large spin can be distinguished from lensing

Towards more realistic cases

There are cases where the lensing hypothesis is ruled out by using the KK relation even when the GW measurement is limited to finite frequency band. (No noise is assumed)

(Obvious) next step

Inclusion of measurement noise

Work in progress

Summary

Gravitational lensing is indispensable tool in cosmology.

GL signal obeys the Kramers-Kronig relation.

KK relations provide a model-independent diagnostic for gravitational-wave lensing