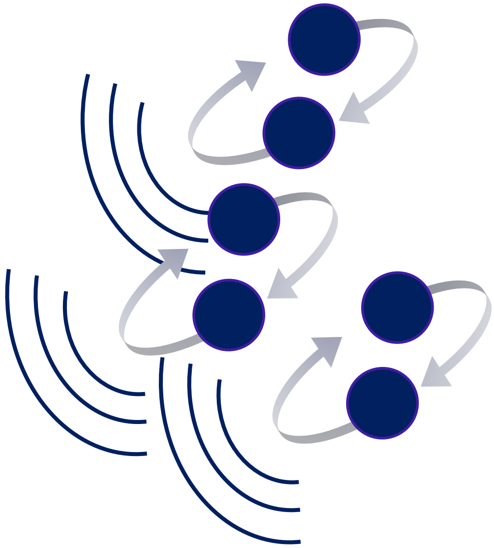
Resolving individual signals in the presence of stochastic background in future pulsar timing arrays



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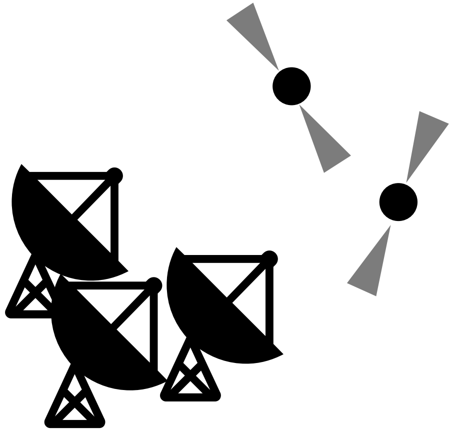
arxiv:2505.10284/MNRAS 543(2) 1010 2025 + some progress



What is pulsar timing array??

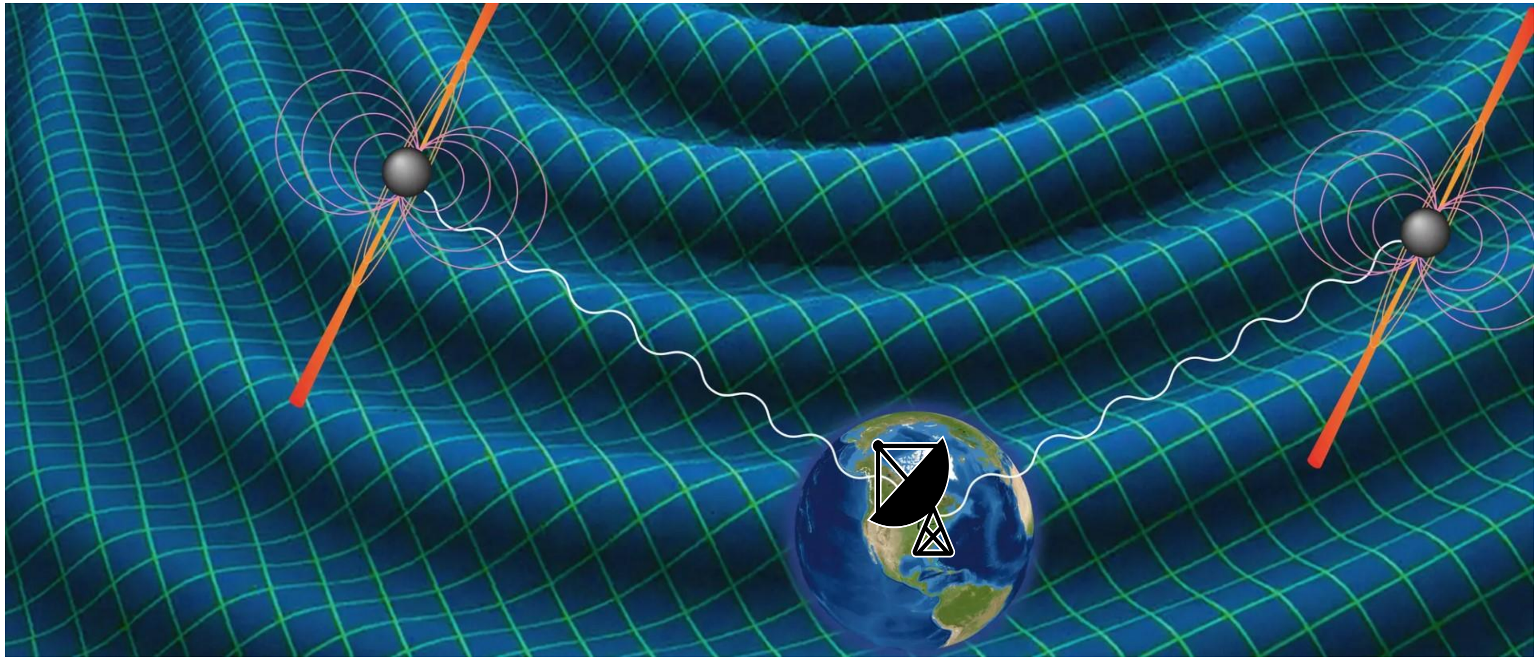


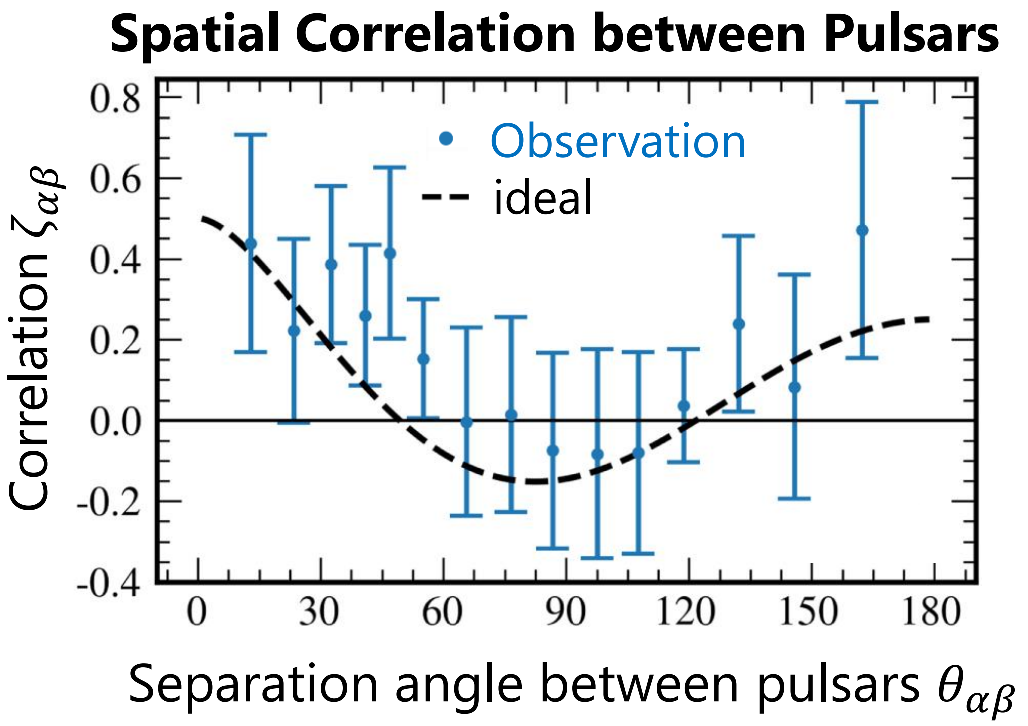
Image Credit: R. Hurt/CALTECH-JPL: NASA.

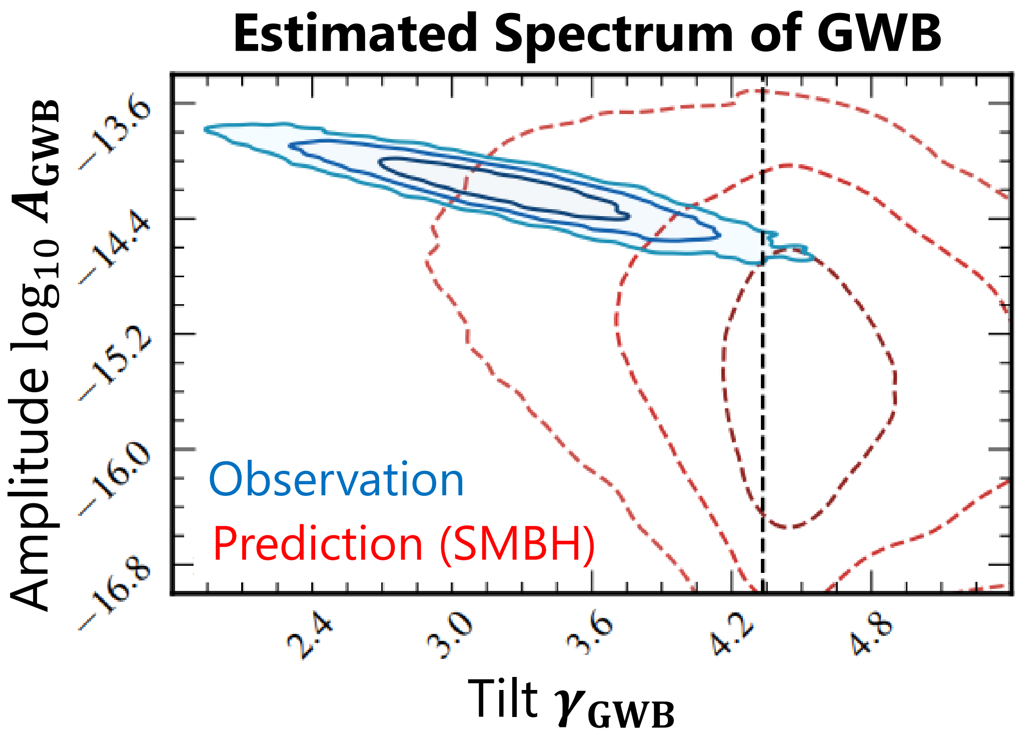
Pulsar Timing Array (PTA)

- probes gravitational waves (GWs) by monitering multiple pulsars over 10-20 yrs.
- has the sensitivity in the low GW frequency range ($\boldsymbol{f_{\mathrm{GW}} = 10^{-6}}$**-**$\boldsymbol{10^{-9}}$ **Hz**)

Recent PTA obs.

G. Agazie+(2023)





- **Evidence for GW Background (GWB)** $(\sim 3\sigma)$
- Primary Candidate: **Supermassive Black Holes (SMBH)**

  - Other GW signals : inflation, cosmic string, etc.

Spectrum of GWB

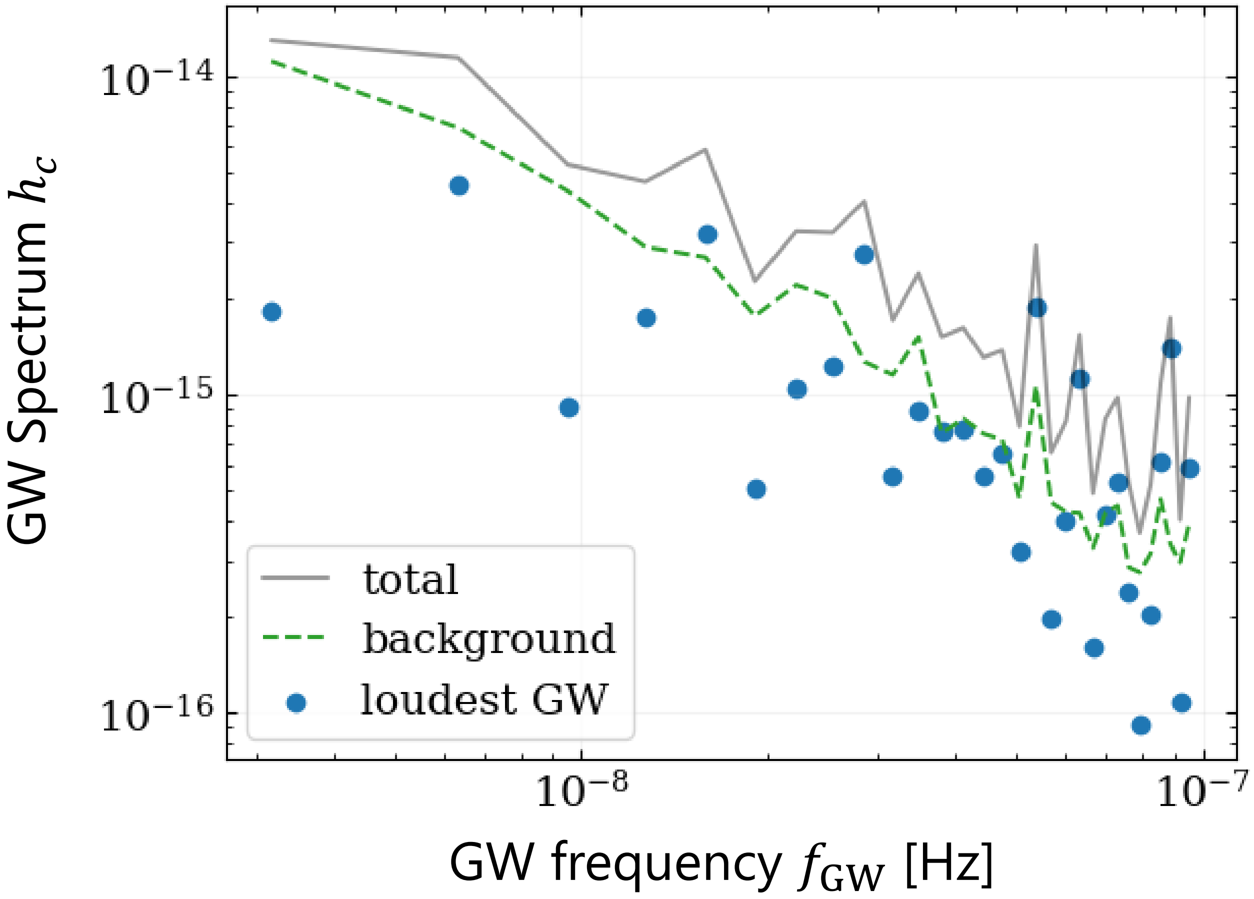
$$h_c = \underline{A_{\mathrm{GWB}}}\left(\frac{f}{f_{\mathrm{ref}}}\right)^{(3-\underline{\gamma_{\mathrm{GWB}}})/2}$$

CGW & GWB will coexist!!

G. Agazie+(2023)

Predicted GWB from SMBHBs

reproduced by holodeck



In future PTA measurement…

Continuous GWs (CGWs)

- **individual loud signals**

GW background (GWB)

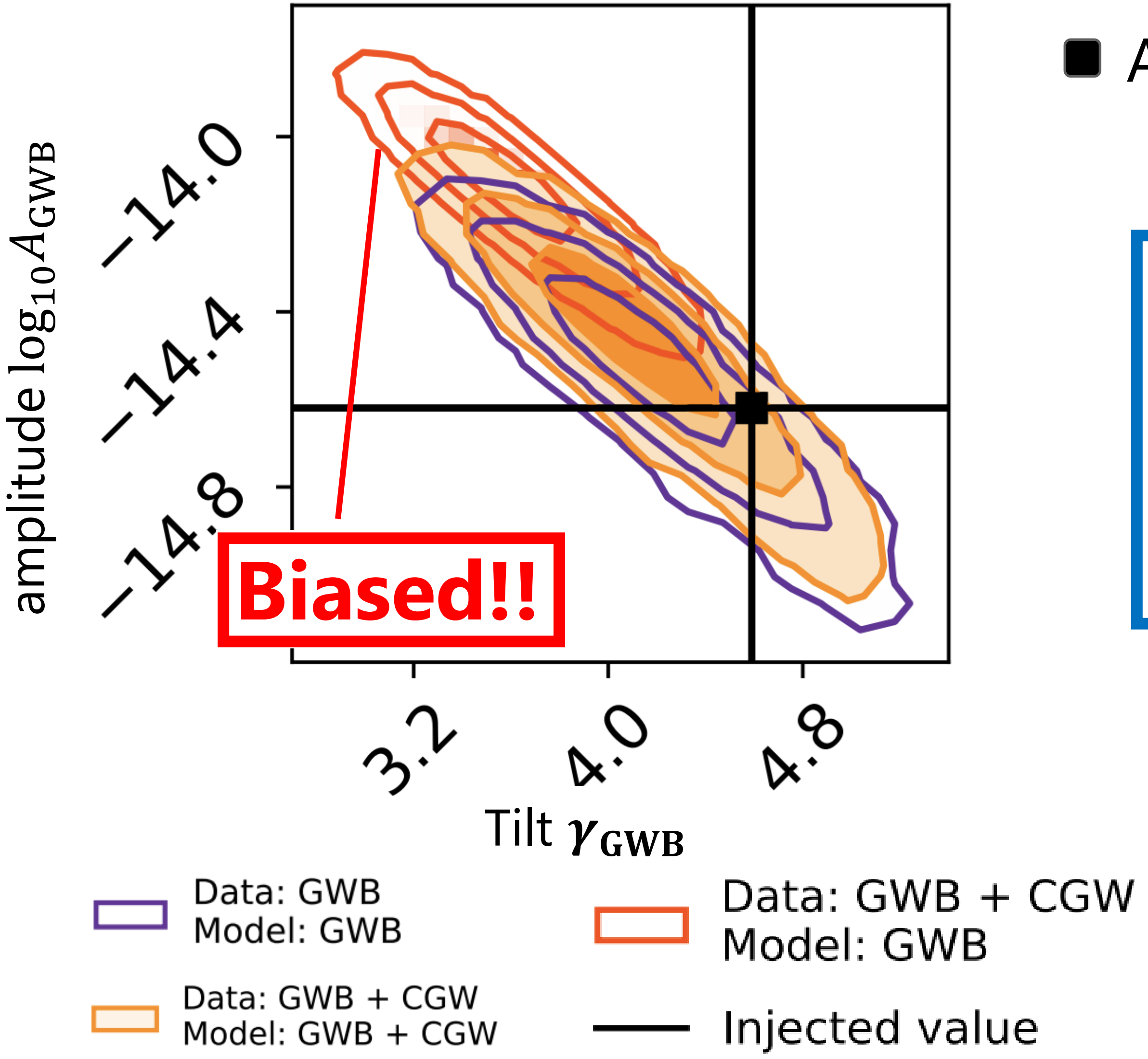
- assemble of unresolved signals

will be observed **simultaneously…**

Expected Problem in analysis

I. Ferranti+(2024)

Ex.) MCMC fitting of GWB



Biased!!

- Analysis of a mock dataset w/ 1CGW + GWB

→ estimation **bias** appear if assuming only GWB…

We need to establish the algorithm to identify multiple CGWs and GWB **simultaneously**, but **efficiently**…

💀 numerous num. of params. in CGWs…

▼

$\mathcal{F}_e$-statistic!!

Our study

S. Babak & A. Sesana (2012)
J. A. Ellis+ (2012) A. Petiteau+ (2013)

What is $\mathcal{F}_e$-statistic??

- An efficient method to detect CGWs with the reduced num. of params. $(8N_{\mathrm{GW}} + N_{\mathrm{p}} \rightarrow \mathbf{3N_{GW}})$

- In the strong noise limit (**white noise≫GWB)**,

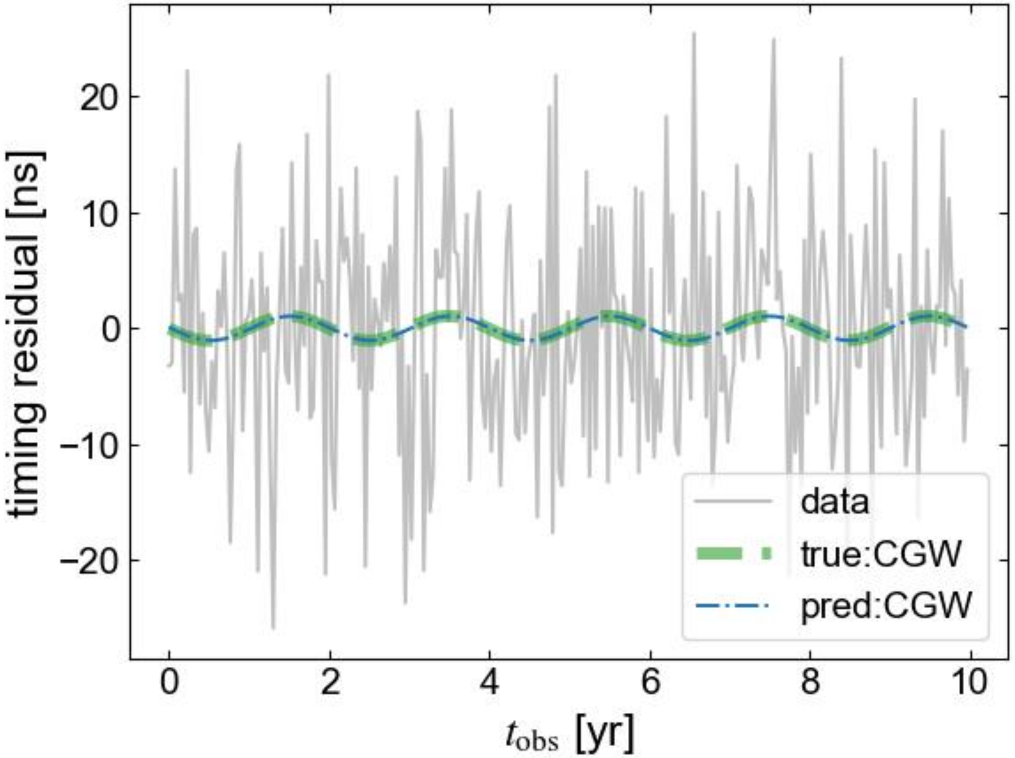
$\mathcal{F}_e$-stat. can work effectively in PTA datasets.

▼

Q. Is it effective in the presence of GWB?

→
- **introduce GWB-informed modelling**
- evaluate its performance through simple mock data simulations

Example of $\mathcal{F}_e$-statistic
(white noise≫GWB)



Timing Residual

Our Dataset (pulsar index: $\alpha = 0,1,2,\dots,N_\mathrm{p}$)

Timing residual: $$r_\alpha = \underline{s_\alpha} + \underline{n_{\mathrm{WN},\alpha}} + \underline{n_{\mathrm{GWB},\alpha}}$$

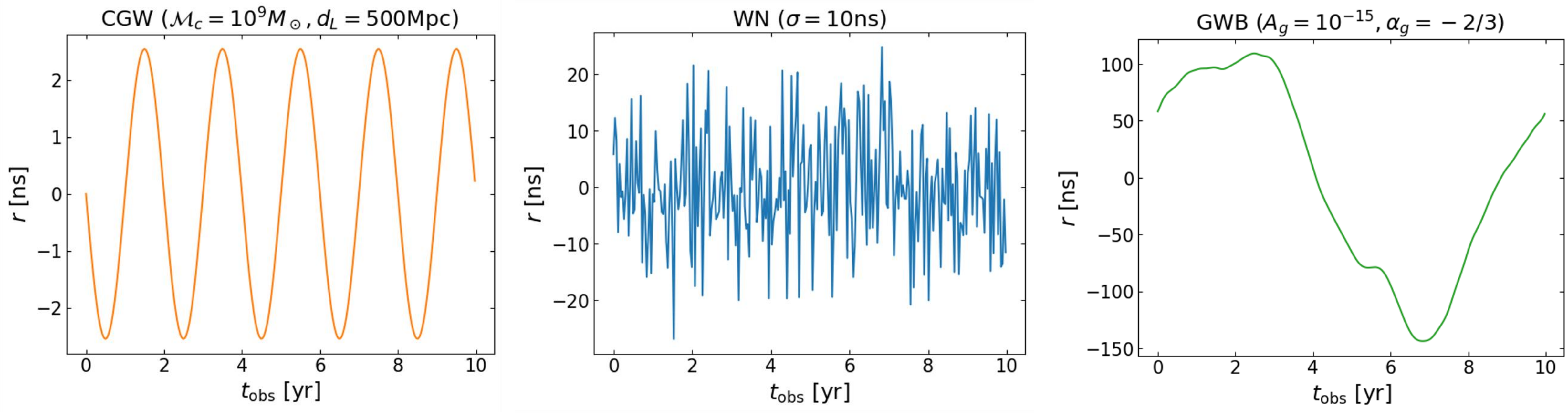
Continuous GW (CGW)

White noise (WN)

σ_α : noise level

GW Background (GWB)

Power-law spectrum : $h_c = A_g \left(\frac{f}{f_\mathrm{ref}}\right)^{\alpha_g}$



Signal (CGW): general

GW from single SMBHB (w/ circular orbit)

$$s_\alpha(t,\hat{\Omega}) = \sum_{A=+,\times} [\underline{s_A(t_{\mathrm{p},\alpha})} - \underline{s_A(t)}]\underline{F_\alpha^A(\hat{\Omega})}$$

Pulsar term Earth term Antenna pattern

- **Detector:** $F_\alpha^A(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}_\alpha^i \hat{p}_\alpha^j}{1+\hat{\Omega}\cdot\hat{p}_\alpha} e_{ij}^A(\hat{\Omega})$ → 2 params. (ϕ, θ)
- **Source:** $s_+(t) = 2\frac{\mathcal{M}_c^{5/3}}{d_\mathrm{L}\omega_\mathrm{obs}^{1/3}}\left[-\sin 2\Phi(t)\frac{1+\cos^2\iota}{2}\cos 2\psi - \cos 2\Phi(t)\cos\iota\sin 2\psi\right]$

  → 6 params. $(\mathcal{M}_c, d_L, \iota, \psi, \Phi_0, f_\mathrm{obs})$
- **Pulsar:** $t_{\mathrm{p},\alpha} = t - L_\mathrm{p}(1+\hat{\Omega}\cdot\hat{p}_\alpha)$ → N_P params. (L_p)

Signal (CGW): approx.

GW from single SMBHB (w/ circular orbit)

$$s_\alpha(t,\hat{\Omega}) = \sum_{A=+,\times} \left[\xcancel{s_A(t_{\mathrm{p},\alpha})} - \underline{s_A(t)}\right]\underline{F_\alpha^A(\hat{\Omega})}$$

↑ drop pulsar term!!

- **Detector:** $F_\alpha^A(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}_\alpha^i \hat{p}_\alpha^j}{1+\hat{\Omega}\cdot\hat{p}_\alpha}e_{ij}^A(\hat{\Omega})$ → 2 params. (ϕ, θ)
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→ 5 params. $(h_s \propto \mathcal{M}_c^{\frac{5}{3}}/d_L, \iota, \psi, \Phi_0, f_{\mathrm{obs}})$

▶ totally **7 parameters**： $(\phi, \theta, f_{\mathrm{obs}})$ & $(h_s, \iota, \psi, \Phi_0)$

intrinsic **extrinsic**

Maximum likelihood estimation

S. Babak & A. Sesana (2012) J. A. Ellis+. (2012)

- Setting: assume N_{GW} **CGW sources in a dataset**

- Define likelihood:

  - Datasets: $\mathbf{x} = \underline{\mathbf{s}} + \underline{\mathbf{n}}$

    Signal component (CGW)

    Noise component (WN+GWB)

  - **Likelihood:** $\mathcal{L}(\mathbf{n}) = \frac{1}{\sqrt{\det 2\pi \underline{\Sigma_{\mathrm{n}}}}} \exp\left(-\frac{1}{2} \mathbf{n}^T \underline{\Sigma_{\mathrm{n}}}^{-1} \mathbf{n} \right)$

    Noise Matrix

    $\Sigma_{\mathrm{n}} = \langle \mathbf{n}\mathbf{n}^T \rangle$

- Maximize likelihood ratio to determine parameters:

  $\mathcal{H}_0$: $\mathbf{x} = \mathbf{n}$

  $\mathcal{H}_1$: $\mathbf{x} = \mathbf{s} + \mathbf{n}$

$$\ln \Lambda = \ln \frac{\mathcal{L}(\mathbf{x}|\mathcal{H}_1)}{\mathcal{L}(\mathbf{x}|\mathcal{H}_0)}$$ → $7N_{\mathrm{GW}}$-parameter estimation…

$\mathcal{F}_e$-statistic

S. Babak & A. Sesana (2012) J. A. Ellis+. (2012)

- Rewrite the template for CGWs: $\mathbf{s} = \displaystyle\sum_{i=1}^{4N_{\mathrm{GW}}} a_i(\vec{h}_s, \vec{\iota}, \vec{\psi}, \vec{\mathbf{\Phi}}_0) A^i(t, \vec{\phi}, \vec{\theta}, \vec{f}_{\mathrm{obs}})$

  → Likelihood ratio: $\ln \Lambda = a_i X^i - \frac{1}{2} a_i M^{ij} a_j$

- **Analitically maximize** $\ln\Lambda$ over a_i $(i = 1, \ldots, 4N_{\mathrm{GW}})$:

$$0 = \left.\frac{\partial \ln \Lambda}{\partial a_k}\right|_{a=\hat{a}} = X^k - M^{ik} \underline{\hat{a}_i} \quad (\mathbf{4}\boldsymbol{N}_{\mathbf{GW}} \text{ equations})$$

Likelihood ratio: $\mathcal{F}_e \equiv \ln \Lambda(a = \hat{a}, \vec{\phi}, \vec{\theta}, \vec{f}_{\mathrm{obs}}) = \frac{1}{2} X^i M_{ij} X^j$

→ $\mathbf{3}\boldsymbol{N}_{\mathbf{GW}}$-parameter estimation!!

Noise modeling

Q. Does $\mathcal{F}_e$-stat. work well even in the presence of GWB??

⟶ A. Noise modeling is crucial!!

Noise Modeling

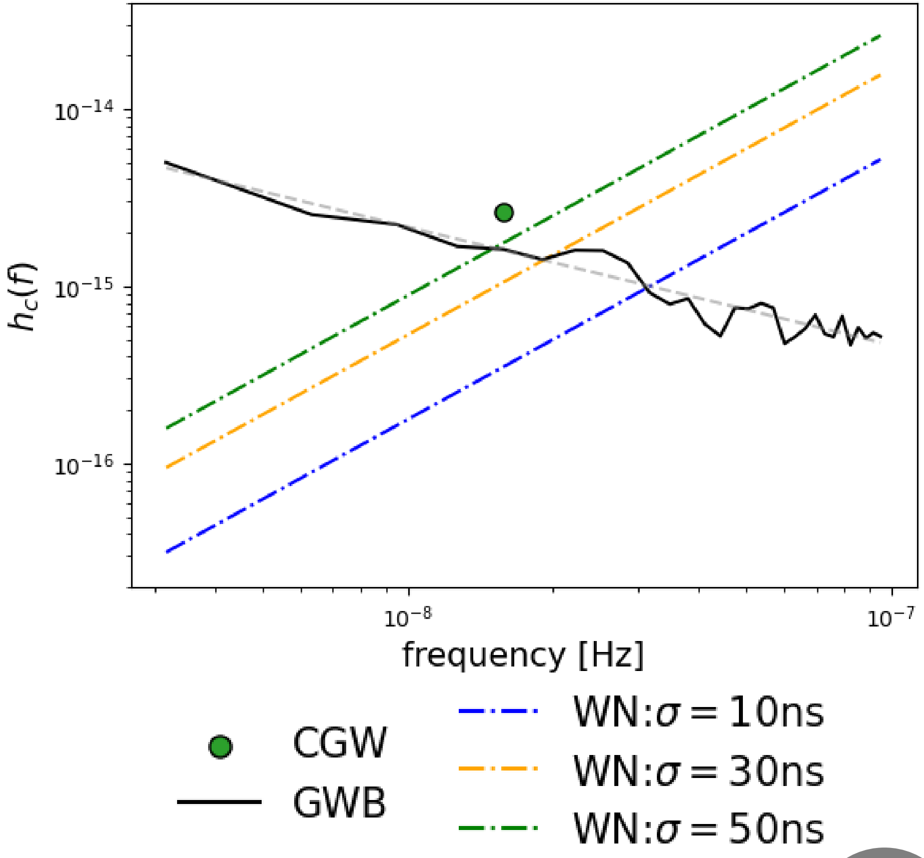
standard : $\Sigma_{\mathrm{n,pre}} = \Sigma_{\mathrm{n,WN}}$ S. Babak & A. Sesana (2012) J. A. Ellis+. (2012)

GWB-informed : $\Sigma_{\mathrm{n,new}} = \Sigma_{\mathrm{n,WN}} + \Sigma_{\mathrm{n,GWB}}$

Mock Data (1CGW+WN+GWB)

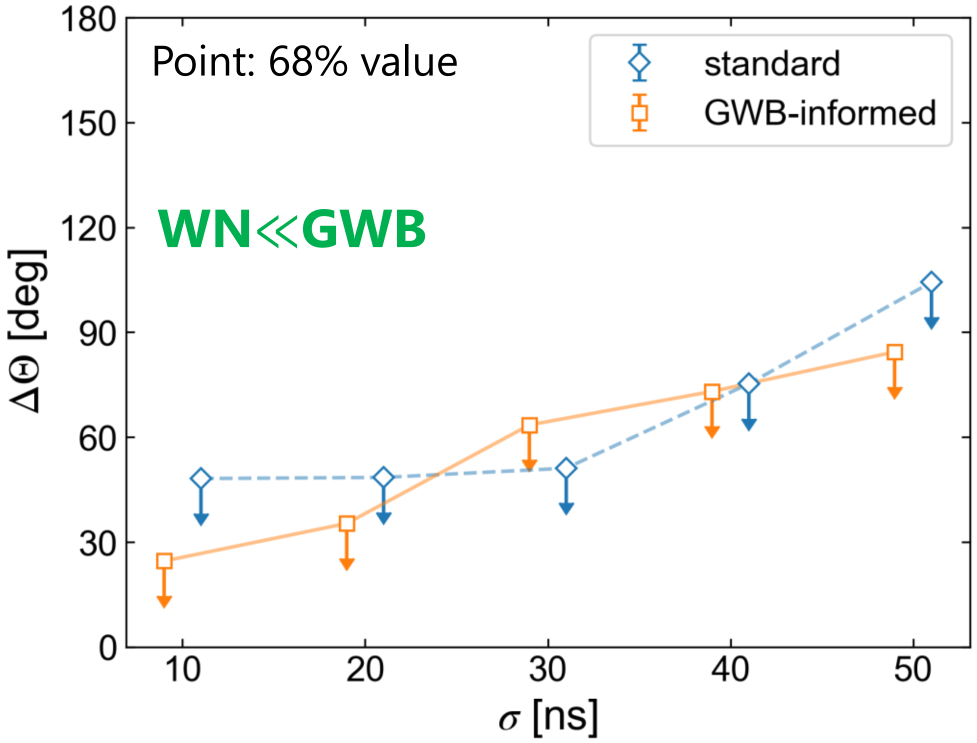
- PTA setup:
  T_{obs} = 10 yr, Δt = 2 week, $\boldsymbol{N_{\mathrm{p}} = 20}$
- Injected GW:
  CGW: $h_s(f_{\mathrm{obs}} = 16\,\mathrm{nHz}) = 1.1\ \times 10^{-15}$ ($\boldsymbol{N_{\mathrm{GW}} = 1}$)
  GWB: $A_g = 1.0 \times 10^{-15}, \alpha_g = -2/3$ (from SMBHs)

Spectra of injected sources

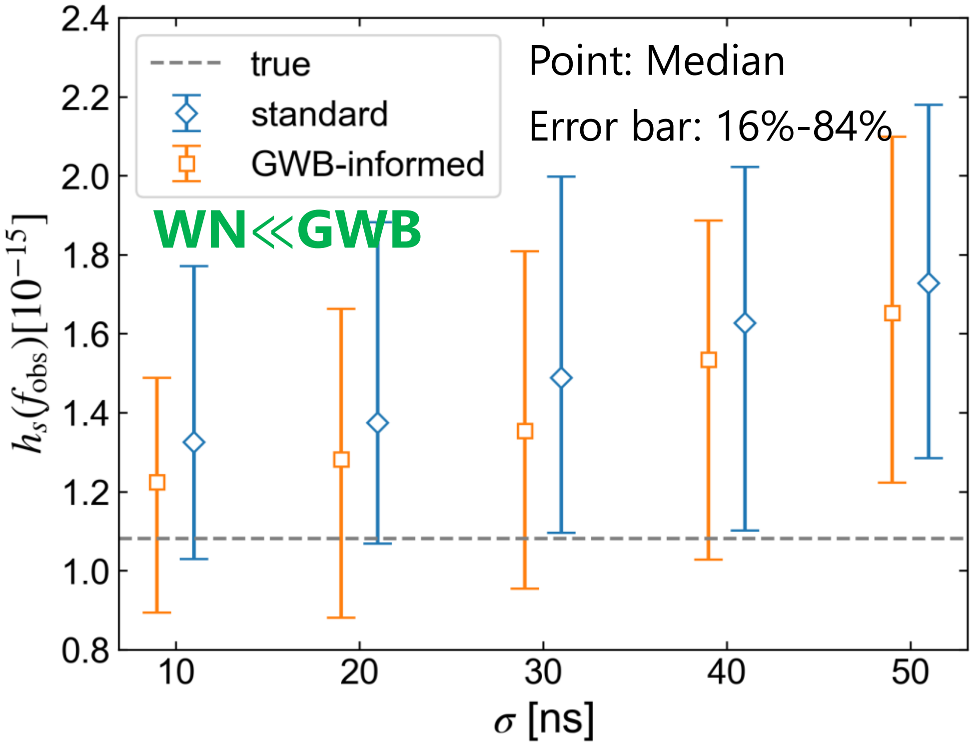


Standard vs GWB-informed

Error of sky position $\Delta\Theta$



CGW Amplitude $h_s(f_{\rm obs})$



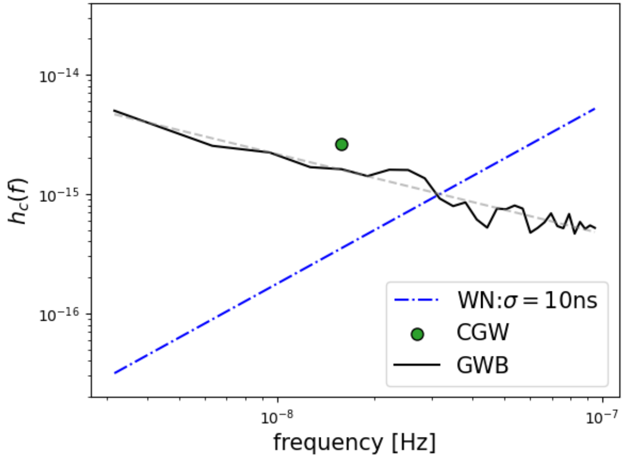
- Adopting **GWB-informed** modeling,

  $\mathcal{F}_e$-stat. can resolve a CGW even in the presence of GWB.

Simultaneous search

Realistic situation → **Simultaneous Search for CGW and GWB!!**

Spectra of injected sources



▶ Reduced log-likelihood

$$\boxed{\ln \mathcal{L} = -\frac{1}{2} \ln \det(2\pi\Sigma_{\mathrm{n}}) - \frac{1}{2}\mathbf{x}^T\Sigma_{\mathrm{n}}^{-1}\mathbf{x} + \mathcal{F}_e}$$

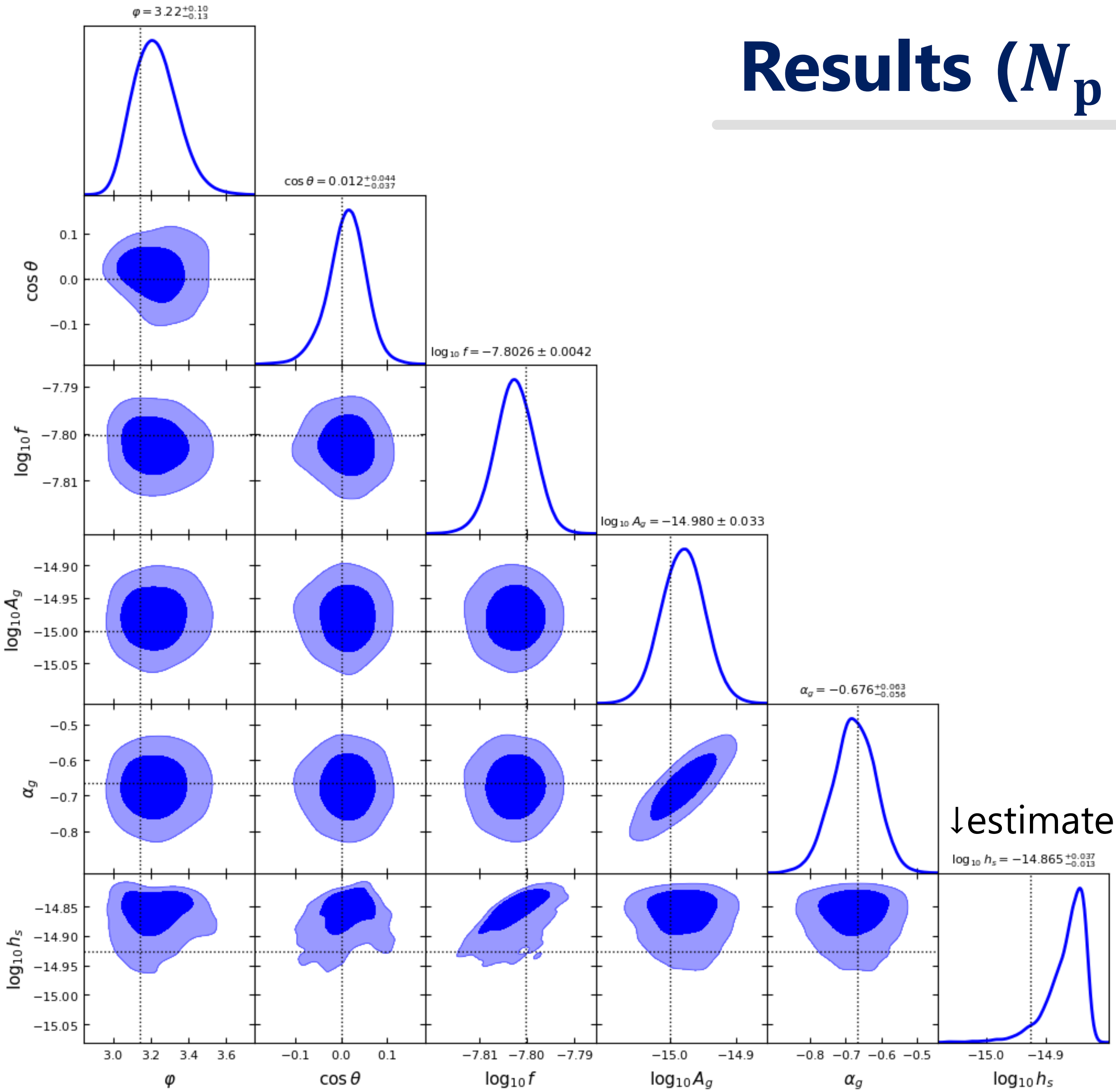
→ 3+2= **5** parameters $(\phi, \theta, f_{\mathrm{obs}}, A_g, \alpha_g)$

- Model: GWB-informed ($\Sigma_{\mathrm{n}} = \Sigma_{\mathrm{n,WN}} + \boldsymbol{\Sigma}_{\mathbf{n,GWB}}$)
- Algorithm: **MultiNest (Bayesian)**
- PTA setups: $\boldsymbol{N}_{\mathbf{p}} = \mathbf{20}$ or $\mathbf{40}$
- Datasets: 1CGW+WN+GWB (**WN≪GWB**)

Prior distributions

Parameter	Prior	Range
ϕ	Uniform	$[0, 2\pi]$
$\cos\theta$	Uniform	$[-1, 1]$
$\log_{10} f_{\mathrm{obs}}$	Uniform	$[-9, -7]$
$\log_{10} A_g$	Uniform	$[-18, 11]$
α_g	Uniform	$[-2.5, 1.0]$

Results ($N_\mathrm{p} = 20$)



↓estimated

Simultaneous search ($N_\mathrm{p} = 20$)

- successfully identify $(\phi, \theta, f_\mathrm{obs}, A_g, \alpha_g)$

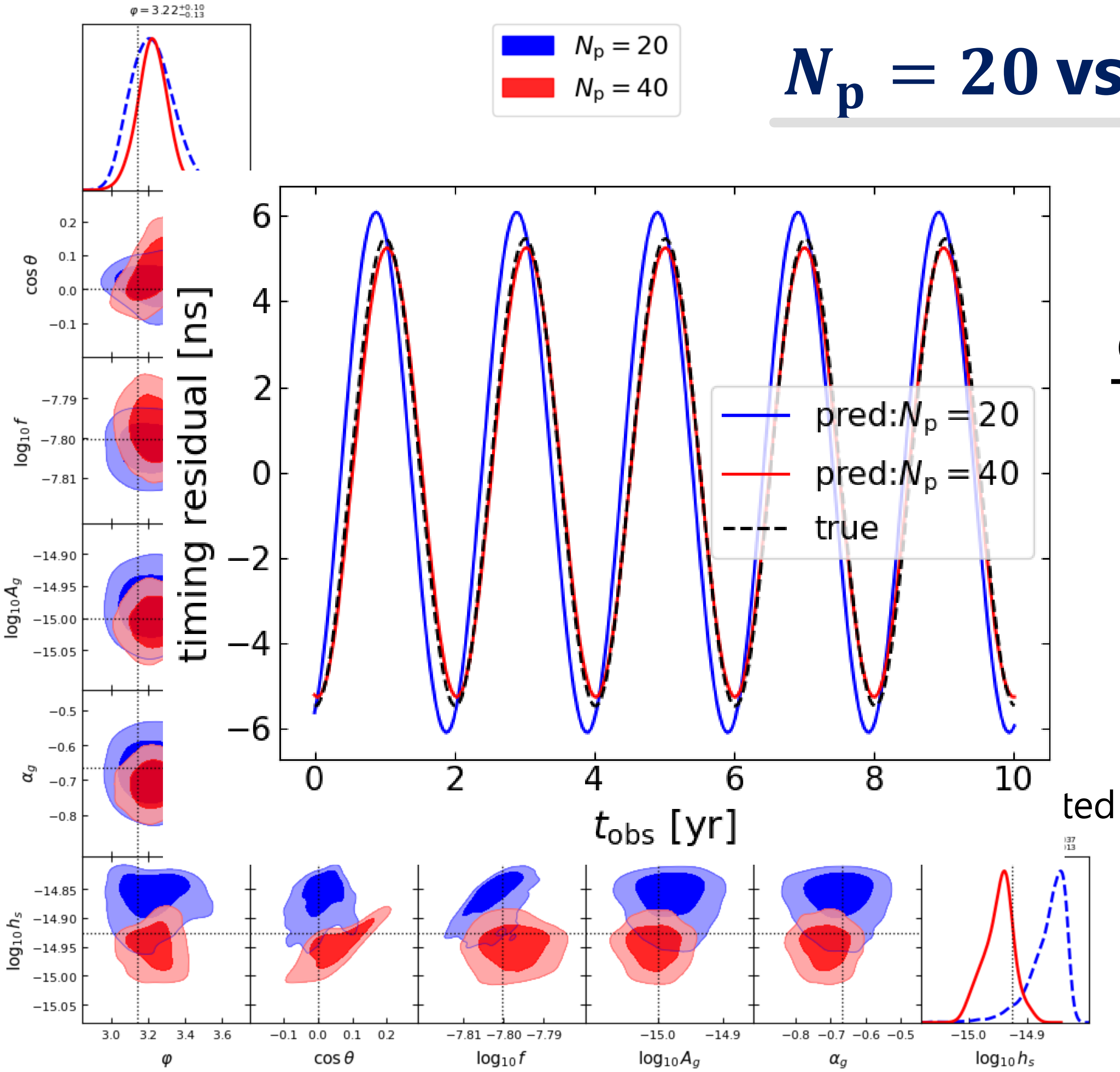
→ **Likelihood based on $\mathcal{F}_e$-stat.**

can separate CGW and GWB!!

- $\mathcal{F}_e$-stat. can recover $(h_s, \iota, \psi, \Phi_0)$

→ **overestimation** for CGW amplitude h_s…

$N_\mathrm{p} = 20$ vs 40

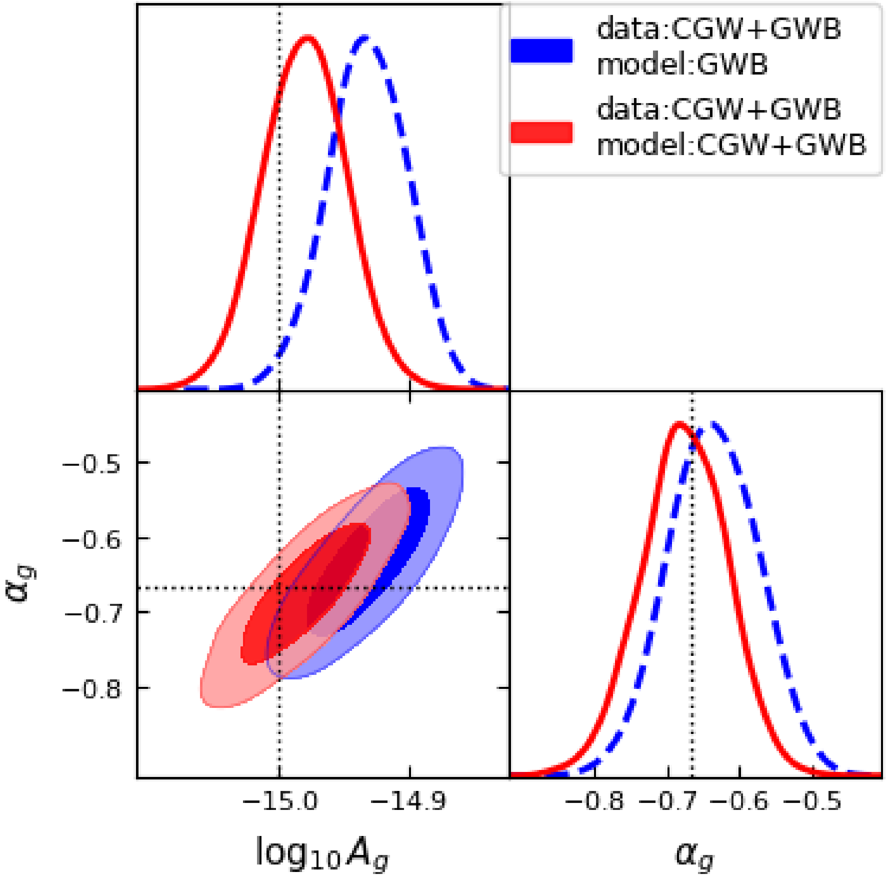


Comparison ($N_\mathrm{p} = 20$ & 40)

- CGW: Improvement of sky localization
- GWB: More cross correlations between pulsars

→ Mitigate the overestimation of $h_s(f_\mathrm{obs})$

How about GWB bias??



GWB estimation ($N_{\mathrm{p}} = 20$)

- Data: 1CGW + GWB (+WN)

- Model:

$$\underbrace{\underbrace{\ln \mathcal{L} = -\frac{1}{2}\ln\det(2\pi\Sigma_{\mathrm{n}}) - \frac{1}{2}\mathbf{x}^T\Sigma_{\mathrm{n}}^{-1}\mathbf{x}}_{\to \textbf{Model: GWB}} + \mathcal{F}_e}_{\to \textbf{Model: CGW+GWB}}$$

▶ Incorporating $\mathcal{F}_e$-stat. (only 3 params.),

we can recover GWB correctly!!

Summary

Summary

- **GW background** and **Continuous GW** will be observed **simultaneously** in upcoming PTAs.
- We revisit **$\mathcal{F}_e$-statistic** and this approach can
  - \- efficiently resolve CGWs with reduced number of CGW parameters $(8N_{\mathrm{GW}} + N_{\mathrm{p}} \rightarrow 3N_{\mathrm{GW}})$
  - \- also work well **in the presence of GWB** *confirmed!!*
  - \- **simultaneously capture 1 CGW and GWB** by the Bayesian analysis *confirmed!!*

Future works

- Apply to **multiple CGWs** and GWB
- reduce the computational cost of inverse noise matrix Σ_{n}^{-1}