

Unitarity test of lepton mixing via energy dependence of neutrino oscillation

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Motivation

- Neutrino mass remains an open question
- Studying the lepton mixing matrix holds the key to this problem
- Testing the unitarity of the PMNS matrix is one method to explore the general lepton mixing matrix

Introduction

Introduction

Neutrino mass remains a mystery

→ Origin of the lepton mixing matrix is also a mystery

Some models (e.g. eV-scale sterile neutrino oscillation) require mixing matrix larger than 3×3

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} & \dots \\ V_{\mu1} & V_{\mu2} & V_{\mu3} & \dots \\ V_{\tau1} & V_{\tau2} & V_{\tau3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \end{pmatrix}$$

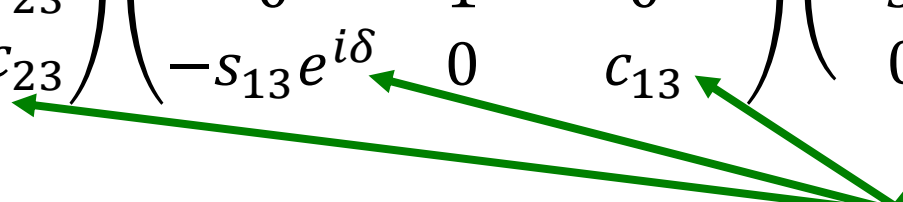
Exploring a general lepton mixing matrix is important

→ **Unitarity Test**

Introduction

Framework of standard 3 flavor ν oscillation

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$


If the PMNS matrix is 3×3 unitary, there are **only 4 parameters**
But this parametrization always **guarantees unitarity**

→ **Must remove this parametrization to test unitarity**

Neutrino oscillation and Unitarity

Neutrino oscillation

In vacuum

$$\begin{aligned} P\left(\nu_{\alpha\rightarrow\beta}(\bar{\nu}_{\alpha\rightarrow\beta})\right) \\ = \delta_{\alpha\beta} - 4 \sum_{j>k} \text{Re}[U_{\alpha j}U_{\beta j}^*U_{\alpha k}^*U_{\beta k}] \sin^2\left(\frac{\Delta m_{jk}^2}{4E}\right) \\ \mp 2 \sum_{j>k} \text{Im}[U_{\alpha j}U_{\beta j}^*U_{\alpha k}^*U_{\beta k}] \sin\left(\frac{\Delta m_{jk}^2}{2E}\right) \end{aligned}$$

Oscillation probability

= (**Coefficient**) × (**Energy-dependent function**)

Coefficients are just matrix elements



Observing energy-dependence can extract coefficients

Neutrino oscillation and Unitarity

In vacuum, 3-generation

Up to 2nd. order in $\Delta m_{21}^2/\Delta m_{31}^2$, U_{e3}

$$\begin{aligned}
 P(\nu_{\mu \rightarrow e}) &= 4|U_{\mu 3}U_{e 3}^*|^2 \sin^2(\Delta_{31}) + 4\text{Re}[U_{\mu 3}U_{e 3}^*U_{\mu 2}^*U_{e 2}] \Delta_{21} \sin(2\Delta_{31}) \\
 &\quad - 4\text{Re}[U_{\mu 2}U_{e 2}^*U_{\mu 1}^*U_{e 1}] \Delta_{21}^2 - 8\text{Im}[U_{\mu 3}U_{e 3}^*U_{\mu 2}^*U_{e 2}] \Delta_{21} \sin^2(\Delta_{31}) \\
 &= C_1 \cdot \sin^2(\Delta_{31}) + C_2 \cdot \Delta_{21} \sin(2\Delta_{31}) \\
 &\quad + C_3 \cdot \Delta_{21}^2 + C_4 \cdot \Delta_{21} \sin^2(\Delta_{31})
 \end{aligned}$$

$$\Delta_{jk} \equiv \frac{\Delta m_{jk}^2}{4E}$$

PMNS matrix is 3×3 unitary,

$$\begin{aligned}
 C_1 &= 4|U_{\mu 3}U_{e 3}^*|^2, \quad C_2 = 4\text{Re}[U_{\mu 3}U_{e 3}^*U_{\mu 2}^*U_{e 2}], \\
 C_3 &= -4\text{Re}[U_{\mu 2}U_{e 2}^*U_{\mu 1}^*U_{e 1}], \quad C_4 = -8\text{Im}[U_{\mu 3}U_{e 3}^*U_{\mu 2}^*U_{e 2}]
 \end{aligned}$$

$$C_1(C_3 - C_2) = C_2^2 + \frac{C_4^2}{4}$$

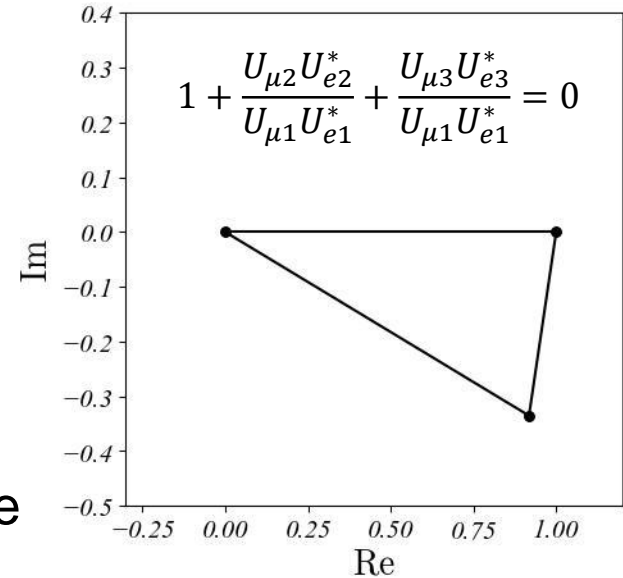
This relation is Independent of parametrization

Unitarity

The unitarity condition corresponds to drawing a triangle on the complex plane

$$U_{\mu 1} U_{e 1}^* + U_{\mu 2} U_{e 2}^* + U_{\mu 3} U_{e 3}^* = 0 \Rightarrow$$

If the unitarity condition is satisfied
 → The triangle closes as its vertices converge



Corresponding to unitarity condition of PMNS matrix

$$U_{\mu 1} U_{e 1}^* + U_{\mu 2} U_{e 2}^* + U_{\mu 3} U_{e 3}^* = 0 \Rightarrow$$

$$\xi \equiv C_1 (C_3 - C_2) - C_2^2 - \frac{C_4^2}{4}$$

If unitary → $\xi = 0$ ($\xi = 0$ is necessary condition for unitarity)

If $\xi \neq 0$ → non-unitary ($\xi \neq 0$ is sufficient condition for unitarity violation)

**ξ is a new criterion of Unitarity Test
 independent of parametrization**

Statistical analysis

Coefficient fit

Oscillation probability at a given energy bin E_j

$$P(\nu_{\mu \rightarrow e}, E_j) = C_1 \cdot B_1(E_j) + C_2 \cdot B_2(E_j) \\ + C_3 \cdot B_3(E_j) + C_4 \cdot B_4(E_j)$$

$$B_1 = \sin^2(\Delta_{31}), B_2 = \Delta_{21} \sin(\Delta_{31}), \\ B_3 = \Delta_{21}^2, B_4 = \Delta_{21} \sin^2(\Delta_{31})$$

$$\chi^2 \equiv \sum_j \left[\frac{P^{\text{obs}}(E_j) - \sum_{k=1}^4 C_k \cdot B_k}{\Delta P(E_j)} \right]^2$$

Minimum is obtained by

$$\frac{d}{dC_l} \chi^2 = 0$$

Coefficient fit

χ^2 can be defined as a form of matrix and vector

$$\chi^2 \equiv \sum_j \left[\frac{P^{\text{obs}}(E_j) - \sum_{k=1}^4 C_k \cdot B_k(E_j)}{\Delta P^{\text{obs}}(E_j)} \right]^2$$
$$= (\mathbf{P}^{\text{obs}} - \mathbf{B}\mathbf{C})^T \mathbf{W} (\mathbf{P}^{\text{obs}} - \mathbf{B}\mathbf{C})$$

$$\mathbf{P}^{\text{obs}} \equiv \begin{pmatrix} P^{\text{obs}}(E_1) \\ \vdots \\ P^{\text{obs}}(E_n) \end{pmatrix}, \mathbf{B} \equiv \begin{pmatrix} B_1(E_1) & \cdots & B_4(E_1) \\ \vdots & \ddots & \vdots \\ B_1(E_n) & \cdots & B_4(E_n) \end{pmatrix}, \mathbf{C} \equiv \begin{pmatrix} C_1 \\ \vdots \\ C_4 \end{pmatrix}$$

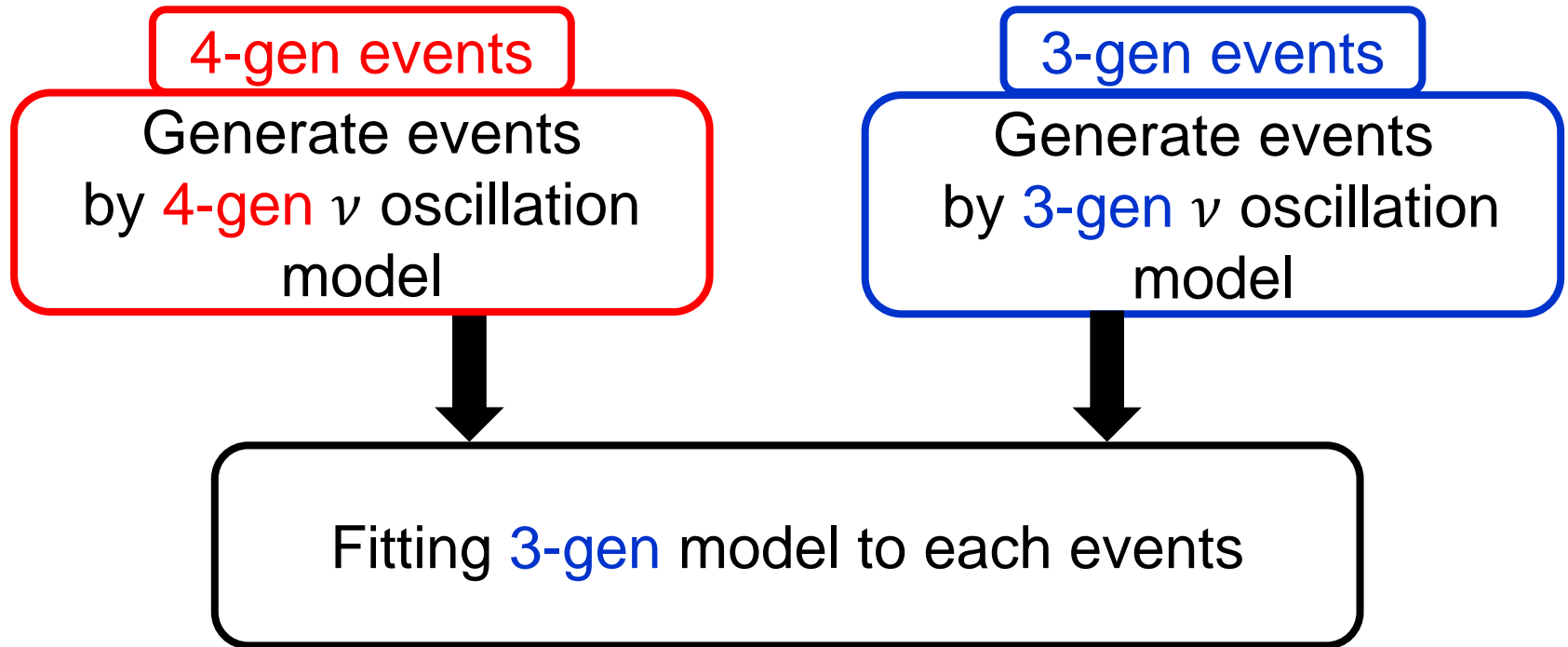
$$\mathbf{W} \equiv \text{diag} \left(\frac{1}{(\Delta P^{\text{obs}}(E_1))^2}, \dots, \frac{1}{(\Delta P^{\text{obs}}(E_n))^2} \right)$$

Best fit points of coefficient $C_1 \sim C_4$

$$\frac{d}{d\mathbf{C}} \chi^2 = 0 \implies \mathbf{C} = (\mathbf{B}^T \mathbf{W} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{W} \mathbf{P}^{\text{obs}}$$

Coefficient fit

Generate 10^6 virtual-experiments
and χ^2 fit with 3-gen model



※Probability distribution is assumed to be binomial distribution

Reference values

3-gen reference values (arithmetic average of bfp in PDG except for δ)

$\Delta m_{21}^2/10^{-5}$ eV	$\Delta m_{31}^2/10^{-3}$ eV	θ_{12}	θ_{13}	θ_{23}	δ
7.43	2.432	33.9°	8.49°	48.1°	270°

4-gen reference values (bfp in Parveen et al. 2025)

$\Delta m_{21}^2/10^{-5}$ eV			$\Delta m_{31}^2/10^{-3}$ eV			$\Delta m_{41}^2/10^{-3}$ eV		
7.5			2.55			1.0		
θ_{12}	θ_{13}	θ_{23}	θ_{14}	θ_{24}	θ_{34}	$\delta_{13}(\delta)$	δ_{24}	δ_{34}
34.3°	8.53°	49.3°	5.7°	5°	20°	-165.6°	0°	0°

In this study, we consider only Normal Mass Ordering and statistical errors.

Number of events at Hyper-Kamiokande

We assume T2HK and neutrino factory with ν_e at J-PARC

※ When observing ν from μ decay at HK, we need to identify the charges of muons generated by CC interactions

In principle, neutron tagging method can identify.

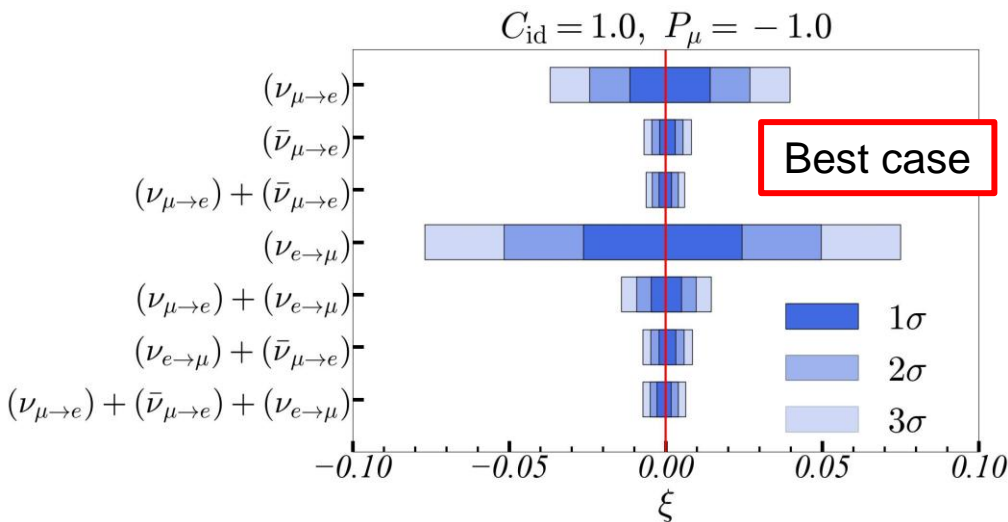
※ Muon beam polarization is also important, because neutrino flux depend on it.

In this talk, we show only 2 cases

- Charge identification efficiency and muon polarization are perfect (Best case)
- No charge identification and unpolarized muon (Worst case)

Unitarity Test

Fitting 3-gen model to 3-gen events



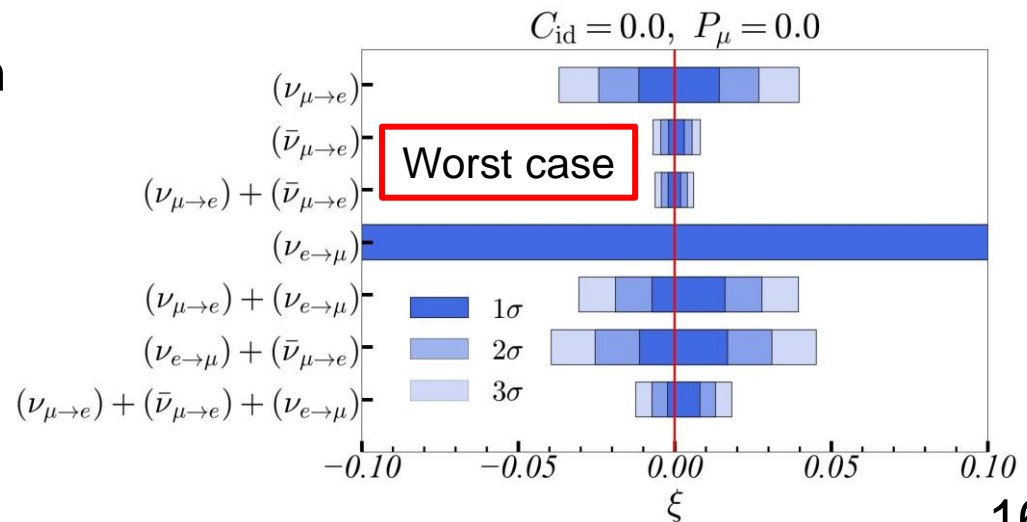
Test of consistency

between 3-gen model and 3-gen events in terms of ξ

- Combining multiple channels yields a much smaller statistical error

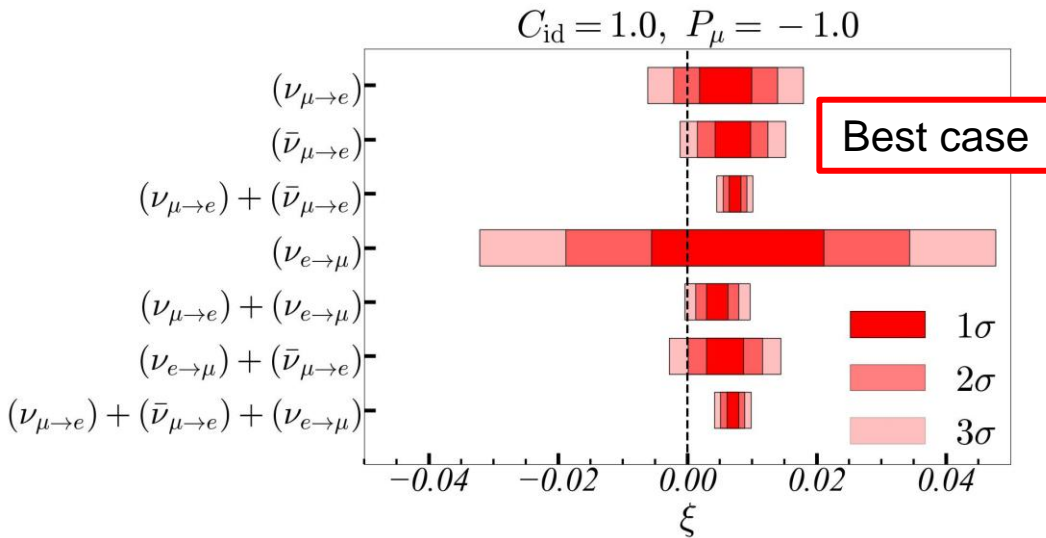
- Charge identification and muon polarization are important factor, particularly in the analysis using $(\nu_{e \rightarrow \mu})$

- All analyses are consistent with $\xi = 0$



Unitarity Test

Fitting 3-gen model to 4-gen events



Test of inconsistency

between 3-gen model and 4-gen events in terms of ξ

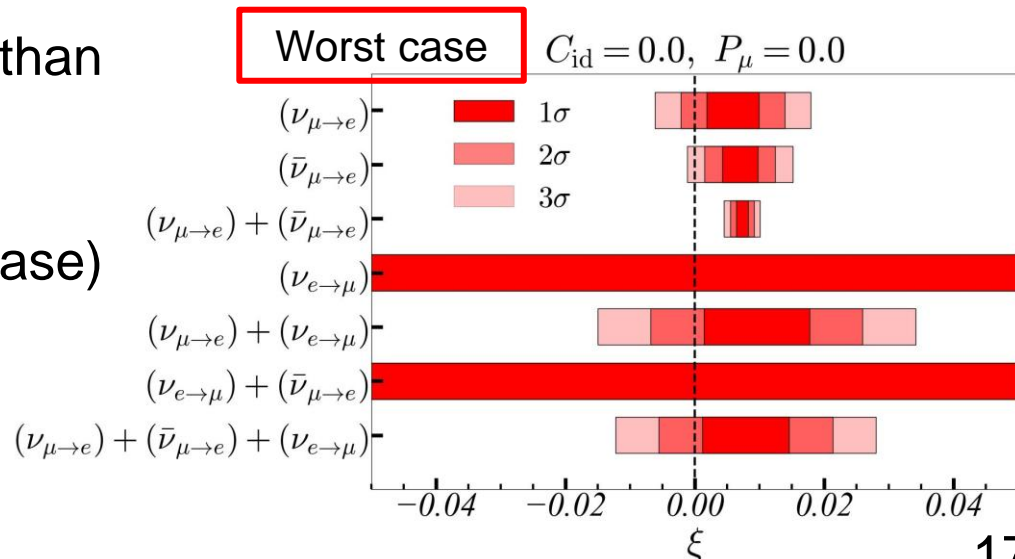
● Combining multiple channels yields a much smaller statistical error

● Inconsistent with $\xi = 0$ at more than 3σ ,

- $(\nu_{\mu \rightarrow e}) + (\bar{\nu}_{\mu \rightarrow e})$
- $(\nu_{\mu \rightarrow e}) + (\bar{\nu}_{\mu \rightarrow e}) + (\nu_{e \rightarrow \mu})$ (Best case)



Sufficient to test the violation of unitarity !!



Summary

- Studying a general lepton mixing is important to understand neutrino mass.
- One method to explore a general lepton mixing matrix is **testing unitarity** of it.
- We propose a method for testing unitarity that does **not depend on the parameterization** of the PMNS matrix, and **demonstrate its application using the new criterion ξ** .
- Fitting 3-gen model to 4-gen events results in a **3σ inconsistency with $\xi = 0$** , in the $(\nu_{\mu \rightarrow e}) + (\bar{\nu}_{\mu \rightarrow e})$ from T2HK and in the $(\nu_{\mu \rightarrow e}) + (\bar{\nu}_{\mu \rightarrow e}) + (\nu_{e \rightarrow \mu})$ from T2HK+neutrino factory
- In this study, we consider neutrino oscillations in vacuum, but for a more realistic analysis, matter effects must be taken into account.

Back up

Neutrino oscillations in vacuum

Time evolution equation of neutrinos

$$i \frac{d}{dt} \begin{pmatrix} \nu_e(\bar{\nu}_e) \\ \nu_\mu(\bar{\nu}_\mu) \\ \nu_\tau(\bar{\nu}_\tau) \end{pmatrix} = [U^{(*)} \text{diag}(0, \Delta E_{21}, \Delta E_{31}) U^{\dagger(T)}] \begin{pmatrix} \nu_e(\bar{\nu}_e) \\ \nu_\mu(\bar{\nu}_\mu) \\ \nu_\tau(\bar{\nu}_\tau) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \\ \nu_\tau(t) \end{pmatrix} = U \text{diag}(0, e^{-i\Delta E_{21}t}, e^{-i\Delta E_{31}t}) U^\dagger \begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \\ \nu_\tau(0) \end{pmatrix}$$

$$P(\nu_\alpha \rightarrow \nu_\beta) = |A(\nu_\alpha \rightarrow \nu_\beta)|^2 = \left| \sum_j U_{\beta j} e^{-i\Delta E_{j1}t} U_{\alpha j}^* \right|^2$$

Oscillation
probability

Probability
amplitude

$$\Delta E_{jk} = E_j - E_k$$

Neutrino oscillations in matter

When the neutrino beam propagate in matter, we need to consider the matter effect

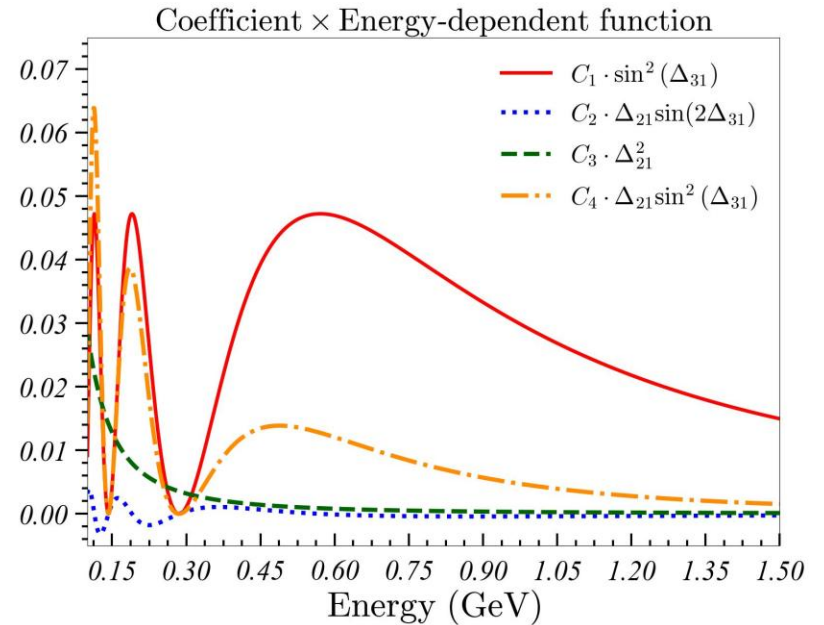
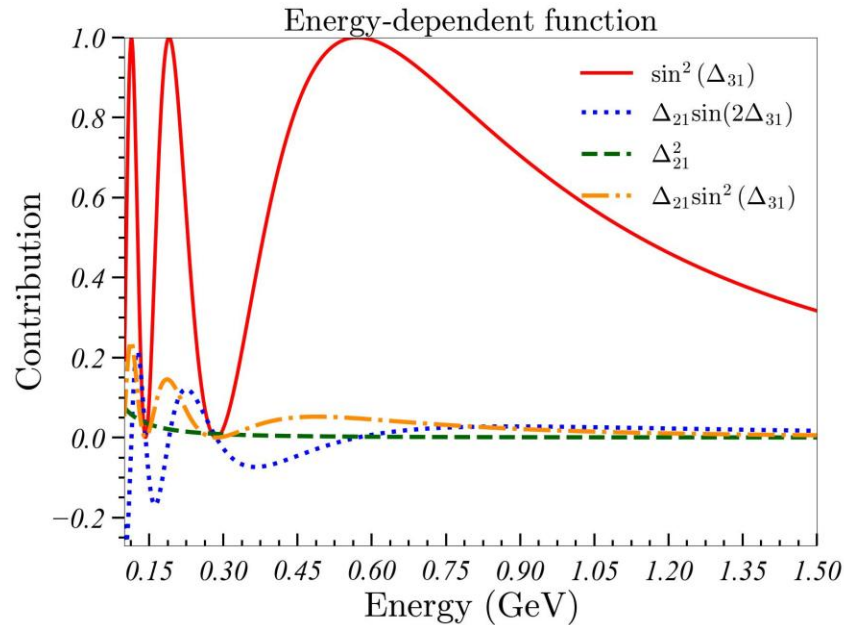
+ for ν , - for $\bar{\nu}$

$$i \frac{d}{dt} \begin{pmatrix} \nu_e(\bar{\nu}_e) \\ \nu_\mu(\bar{\nu}_\mu) \\ \nu_\tau(\bar{\nu}_\tau) \end{pmatrix} = [U^{(*)} \text{diag}(0, \Delta E_{21}, \Delta E_{31}) U^{\dagger(T)} + \text{diag}(\pm A, 0, 0)] \begin{pmatrix} \nu_e(\bar{\nu}_e) \\ \nu_\mu(\bar{\nu}_\mu) \\ \nu_\tau(\bar{\nu}_\tau) \end{pmatrix}$$

$$= \tilde{U}^{(\pm)} \text{diag}(\tilde{E}_1^{(\pm)}, \tilde{E}_2^{(\pm)}, \tilde{E}_3^{(\pm)}) \tilde{U}^{(\pm)\dagger} \begin{pmatrix} \nu_e(\bar{\nu}_e) \\ \nu_\mu(\bar{\nu}_\mu) \\ \nu_\tau(\bar{\nu}_\tau) \end{pmatrix}$$

If the matter effect is constant,
this equation can be easily solved.

The contribution of each energy-dependent function to the oscillation probability



$$\begin{aligned}
 P(\nu_{\mu \rightarrow e}) &= 4|U_{\mu 3} U_{e 3}^*|^2 \sin^2(\Delta_{31}) + 4\text{Re}[U_{\mu 3} U_{e 3}^* U_{\mu 2}^* U_{e 2}] \Delta_{21} \sin(2\Delta_{31}) \\
 &\quad - 4\text{Re}[U_{\mu 2} U_{e 2}^* U_{\mu 1}^* U_{e 1}] \Delta_{21}^2 - 8\text{Im}[U_{\mu 3} U_{e 3}^* U_{\mu 2}^* U_{e 2}] \Delta_{21} \sin^2(\Delta_{31}) \\
 &= C_1 \cdot \sin^2(\Delta_{31}) + C_2 \cdot \Delta_{21} \sin(2\Delta_{31}) \\
 &\quad + C_3 \cdot \Delta_{21}^2 + C_4 \cdot \Delta_{21} \sin^2(\Delta_{31})
 \end{aligned}$$

Background subtraction

At the HK, in principle, ν_μ and $\bar{\nu}_\mu$ are distinguished by neutron tagging method.

We can define the oscillation probability $P(\nu_e \rightarrow \nu_\mu)$ as

$$P(\nu_e \rightarrow \nu_\mu) = \frac{1}{\kappa} \frac{\left(\kappa N_{\text{far}}^{\nu_e \rightarrow \nu_\mu} + (1 - \kappa) N_{\text{far}}^{\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu} \right) - (1 - \kappa) N_{\text{far}}^{\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu} \Big|_{\text{T2HK}}}{N_{\text{near}}^{\nu_e \rightarrow \nu_e}}$$

$$\kappa \equiv \frac{1 + C_{\text{id}}}{2}, \quad C_{\text{id}}: \text{charge identification efficiency}$$

In the case of $C_{\text{id}} = 0.0$, we just do not perform the charge identification analysis and simply add the background events, and subtract the estimated amount by using the T2HK data, i.e. we take $\kappa = 1$ and $1 - \kappa = 0$.

Probability of $\nu_e \rightarrow \nu_\mu$

At the HK, in principle, ν_μ and $\bar{\nu}_\mu$ are distinguished by neutron tagging method.

We can define the oscillation probability $P(\nu_e \rightarrow \nu_\mu)$ as

Perfect charge identification

$$P(\nu_e \rightarrow \nu_\mu) = \frac{N_{\text{far}}^{\nu_e \rightarrow \nu_\mu}}{N_{\text{near}}^{\nu_e \rightarrow \nu_e}}$$

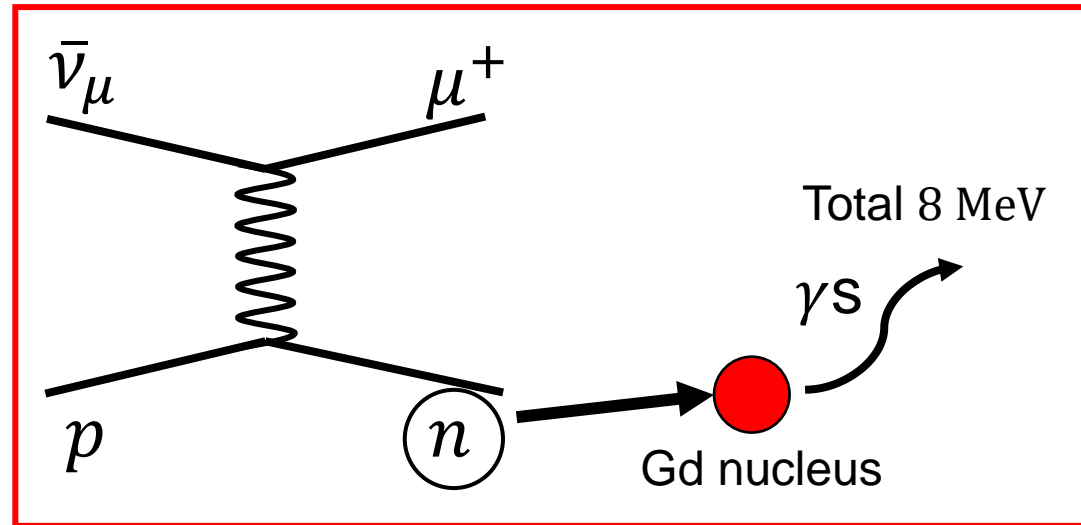
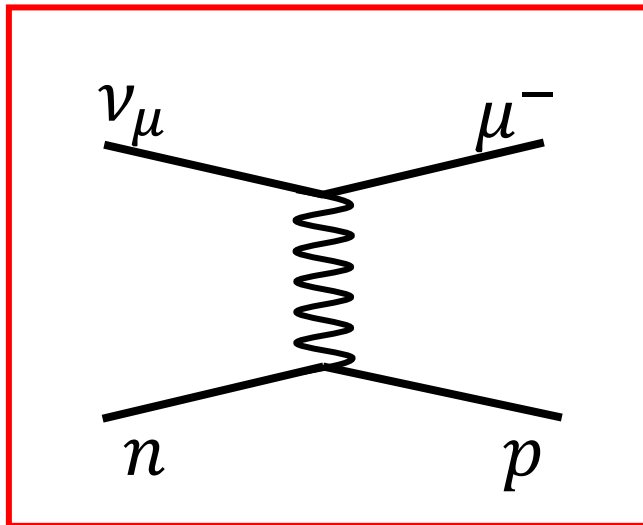
No charge identification

$$P(\nu_e \rightarrow \nu_\mu) = \frac{\left(N_{\text{far}}^{\nu_e \rightarrow \nu_\mu} + N_{\text{far}}^{\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu} \right) - N_{\text{far}}^{\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu} \Big|_{\text{T2HK}}}{N_{\text{near}}^{\nu_e \rightarrow \nu_e}}$$

Neutron tagging

SK-Gd : efficiency $\sim 70\%$
Hyper-K : efficiency $\geq 70\%$

R. Akutsu. Ph.D thesis, Tokyo University, 2019.

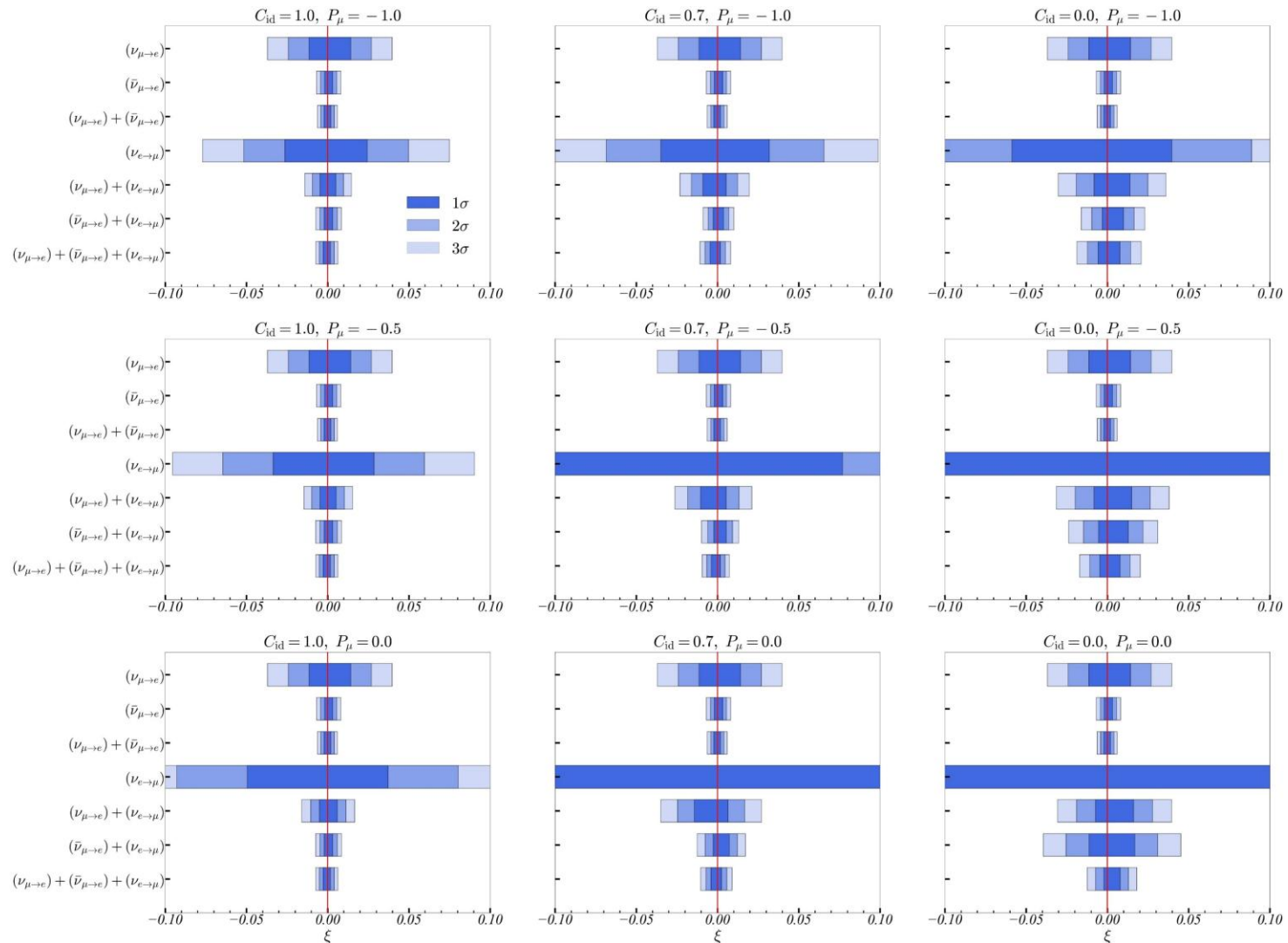


Derivation of χ^2

$$\begin{aligned}
 \chi^2 &\equiv \sum_j \frac{\left(N_{\text{far},j}^{\text{obs}} - N_{\text{far},j}^{\text{exp}}\right)^2}{\left(\Delta N_j^{\text{obs}}\right)^2} \\
 &= \sum_j \frac{\left(N_{\text{far},j}^{\text{obs}} - \tilde{N}_{\text{near},j}^{\text{obs}} \cdot P(E_j)\right)^2}{\left(\Delta N_{\text{far},j}^{\text{obs}}\right)^2 + \left(\Delta \tilde{N}_{\text{near},j}^{\text{obs}} \cdot P_j^{\text{obs}}\right)^2} \\
 &= \sum_j \frac{\left[\frac{N_{\text{far},j}^{\text{obs}}}{\tilde{N}_{\text{near},j}^{\text{obs}}} - (\mathcal{BC})_j\right]^2}{(P_j^{\text{obs}})^2 \cdot \left[\left(\frac{\Delta N_{\text{far},j}^{\text{obs}}}{N_{\text{far},j}^{\text{obs}}}\right)^2 + \left(\frac{\Delta N_{\text{near},j}^{\text{obs}}}{N_{\text{near},j}^{\text{obs}}}\right)^2\right]} \\
 &= \sum_j \left[\frac{P^{\text{obs}}(E_j) - \sum_{k=1}^4 C_k \cdot B_k}{\Delta P(E_j)}\right]^2
 \end{aligned}$$

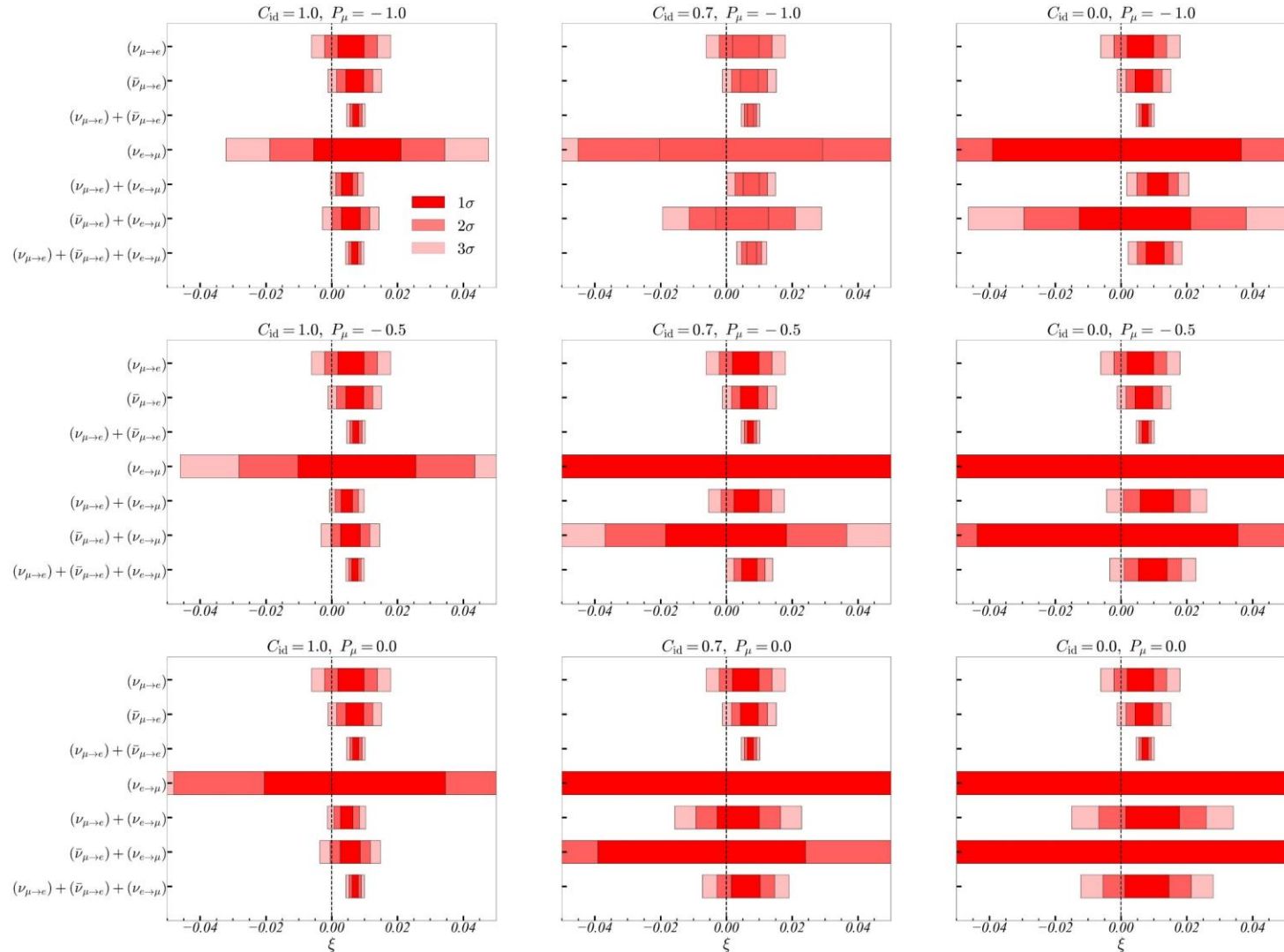
Unitarity Test

Fitting 3-gen model to 3-gen events



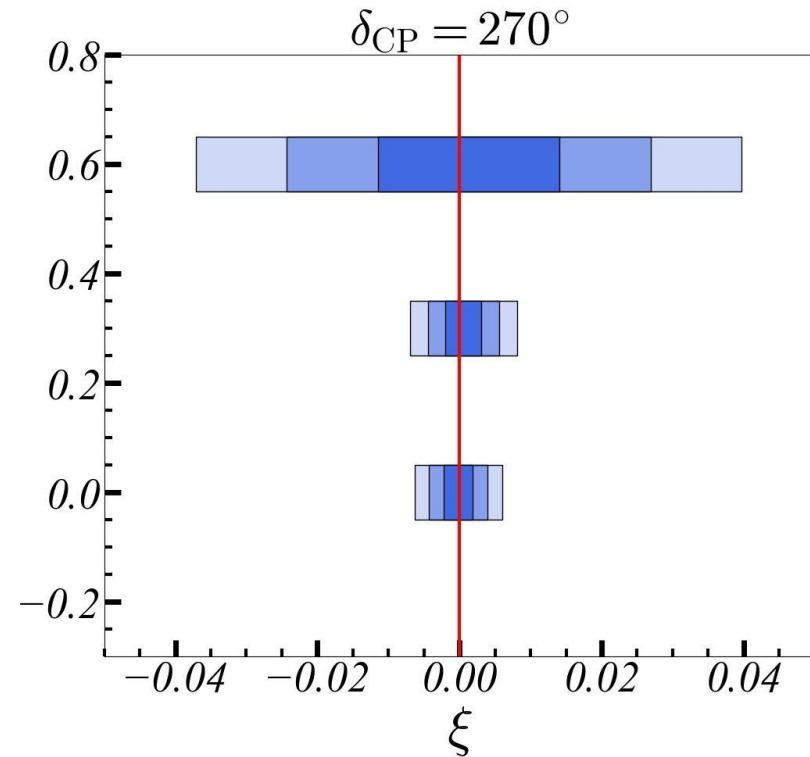
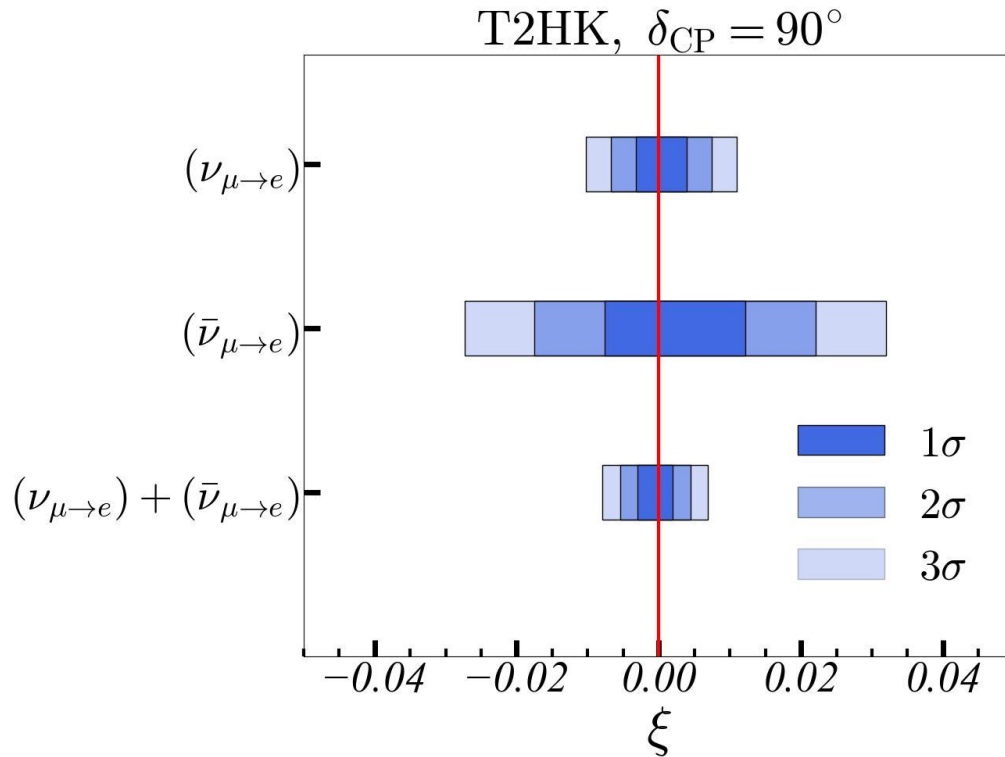
Unitarity Test

Fitting 3-gen model to 4-gen events



Unitarity Test

Fitting 3-gen model to 3-gen events



Opposite behavior arises from
 $\delta \rightarrow -\delta \iff (\nu_{\mu \rightarrow e}) \rightarrow (\bar{\nu}_{\mu \rightarrow e})$

Which uncertainty is smaller depends on δ

Virtual-experiments

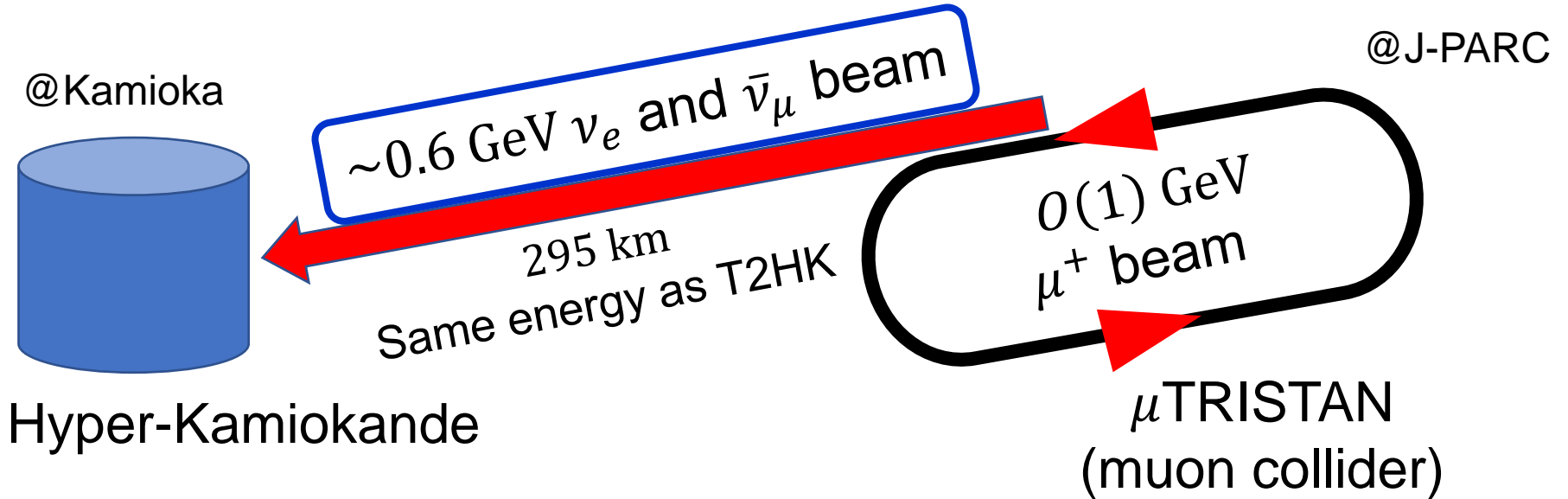
Assume a binomial distribution $B(n, p)$,
since the observed number of events is an integer

$$B(n, p) \left\{ \begin{array}{l} n: \text{Number of non-oscillated events} \\ p: \text{Oscillation probability} \end{array} \right.$$

n, p can be assigned to each energy bin

→ the virtual-experiments are randomly generated
according to the binomial distribution in each bin

Future neutrino factory



Number of events

$$N_j^{\nu_e \rightarrow \nu_\mu} = \int_{E_j}^{E_{j+1}} \frac{dE_\nu}{E_\mu} \times \frac{12N_\mu \cdot V \cdot n_N}{\pi L^2} \times \gamma^2 \left(\frac{E_\nu}{E_\mu} \right)^2 \times \left[\left(1 - \frac{E_\nu}{E_\mu} \right) - P_\mu \left(1 - \frac{E_\nu}{E_\mu} \right) \right] \\ \times P_j^{\nu_e \rightarrow \nu_\mu}(E_\nu) \times \sigma_{\nu_\mu}(E_\nu)$$

$N_j^{\nu_e \rightarrow \nu_\mu}$: number of events

N_μ : number of muons

V : detector volume

n_N : number density of the nucleon in water

$\gamma = \frac{E_\mu}{m_\mu}$: boost factor

E_ν : neutrino beam energy

E_μ : muon beam energy

P_μ : polarization of anti-muon beam

σ_{ν_μ} : cross section

Number of events

$$N_j^{\bar{\nu}\mu\rightarrow\mu} = \int_{E_j}^{E_{j+1}} \frac{dE_\nu}{E_\mu} \times \frac{2N_\mu \cdot V \cdot n_N}{\pi L^2} \times \gamma^2 \left(\frac{E_\nu}{E_\mu}\right)^2 \times \left[\left(3 - \frac{2E_\nu}{E_\mu}\right) - P_\mu \left(1 - \frac{2E_\nu}{E_\mu}\right) \right] \\ \times P_j^{\bar{\nu}\mu\rightarrow\mu}(E_\nu) \times \sigma_{\bar{\nu}\mu}(E_\nu)$$

$N_j^{\bar{\nu}\mu\rightarrow\mu}$: number of events

N_μ : number of muons

V : detector volume

n_N : number density of the nucleon in water

$\gamma = \frac{E_\mu}{m_\mu}$: boost factor

E_ν : neutrino beam energy

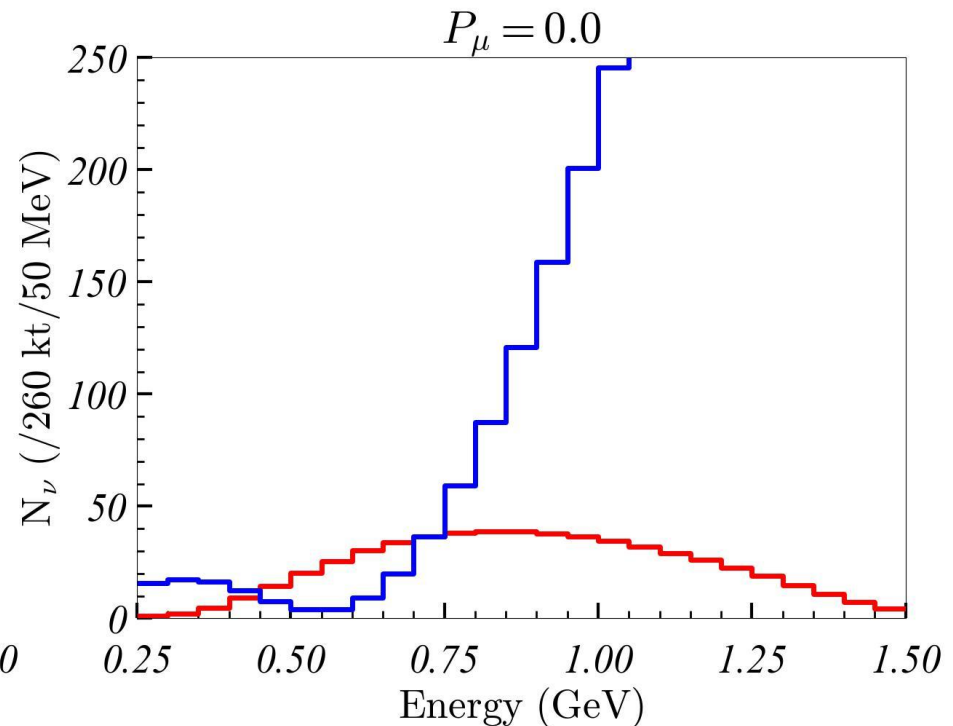
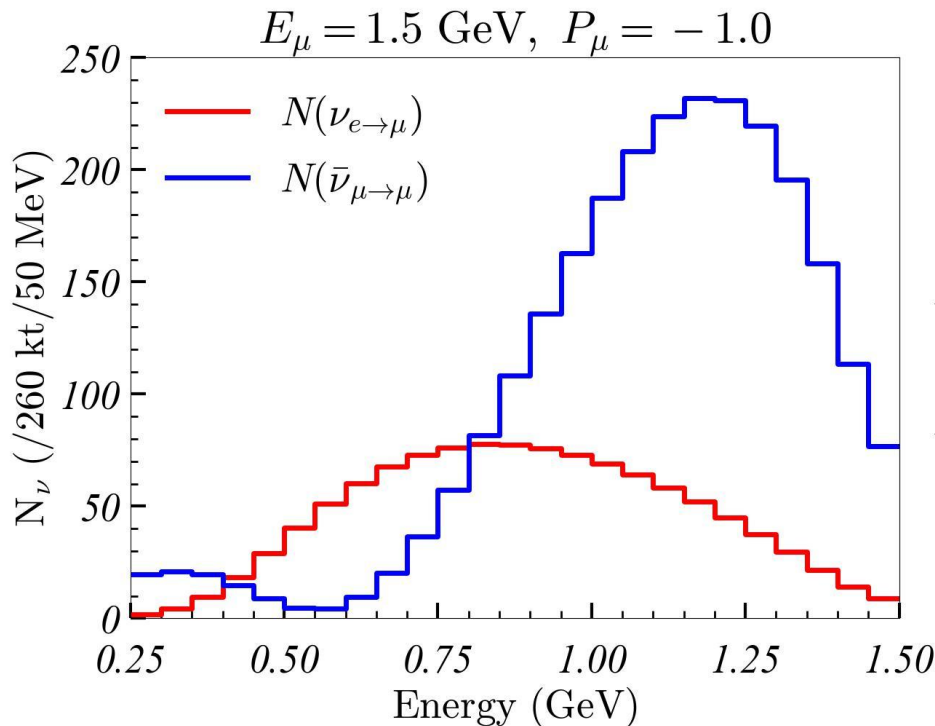
E_μ : muon beam energy

P_μ : polarization of anti-muon beam

$\sigma_{\bar{\nu}\mu}$: cross section

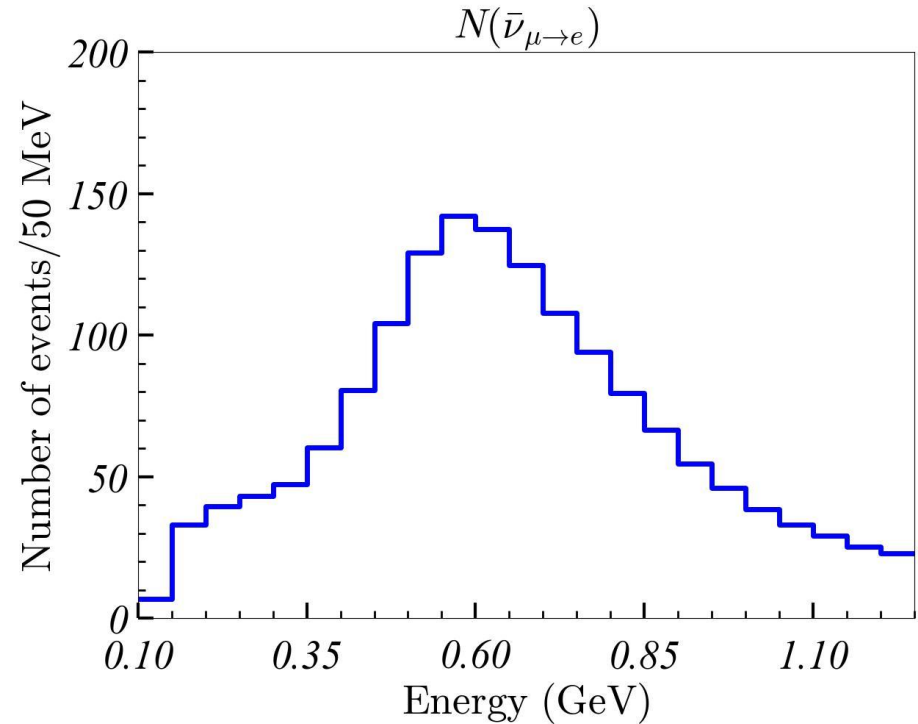
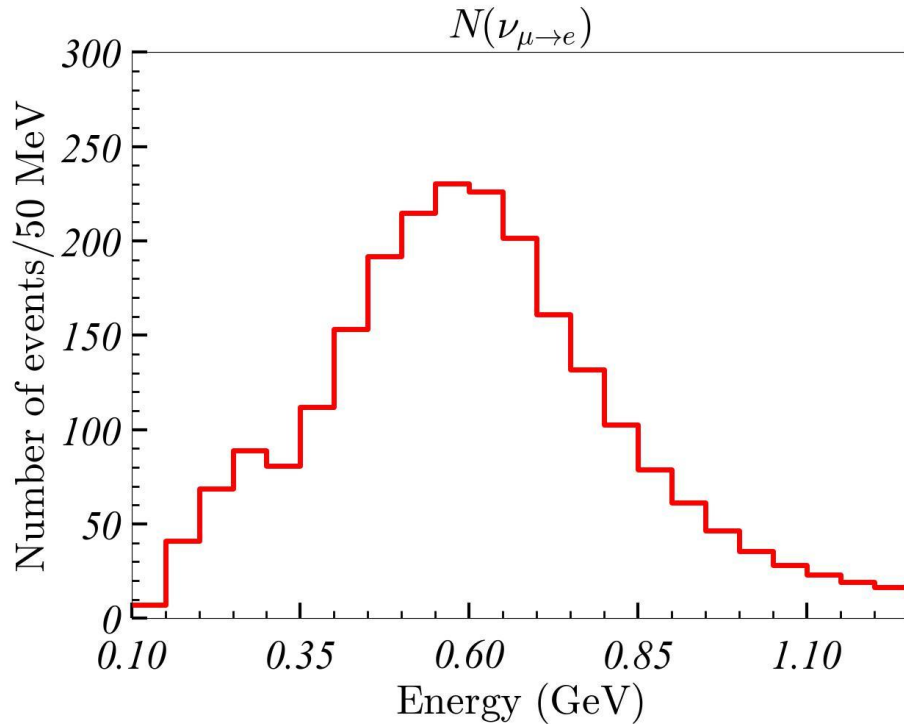
Neutrino flux in neutrino factory

Total number of muons (which decay toward HK) is 10^{22}

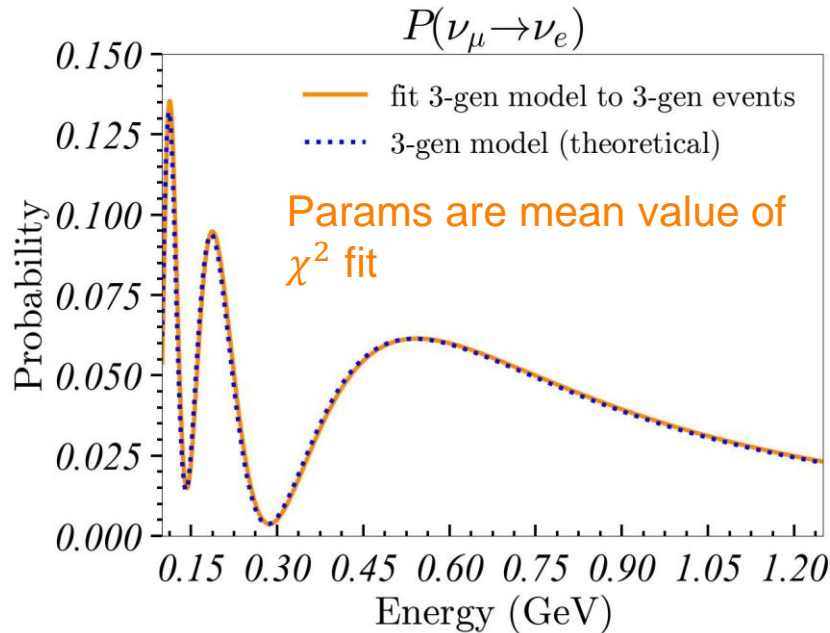


Neutrino flux in T2HK

2.7×10^{22} POT ($\nu:\bar{\nu} = 1:3$)
calculated based on K. Abe et al. 2018

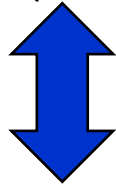


Oscillation probability using the results of fit

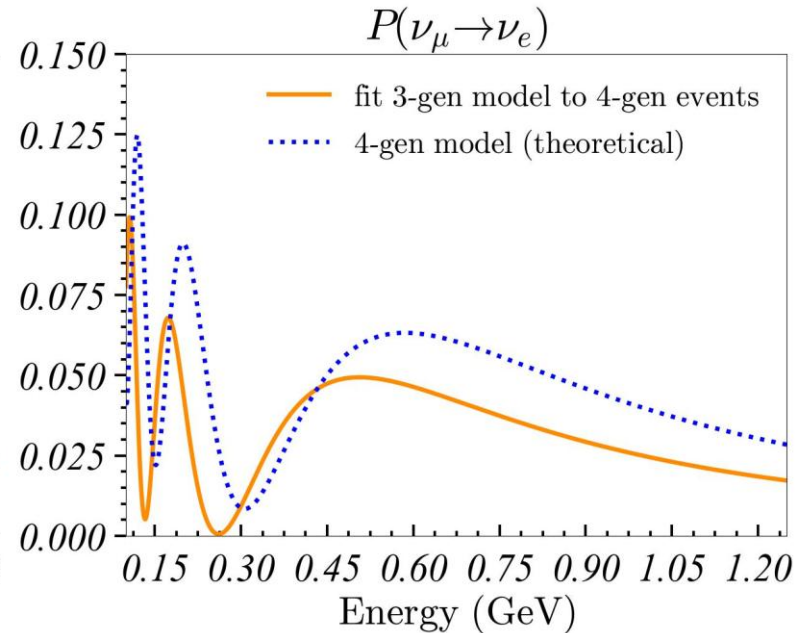


Fitting the 3-gen model to 3-gen events (orange solid)

Consistent

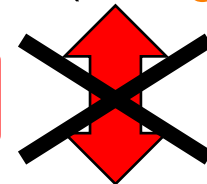


Theoretical curve of the 3-gen model (blue dotted)



Fitting the 3-gen model to 4-gen events (orange solid)

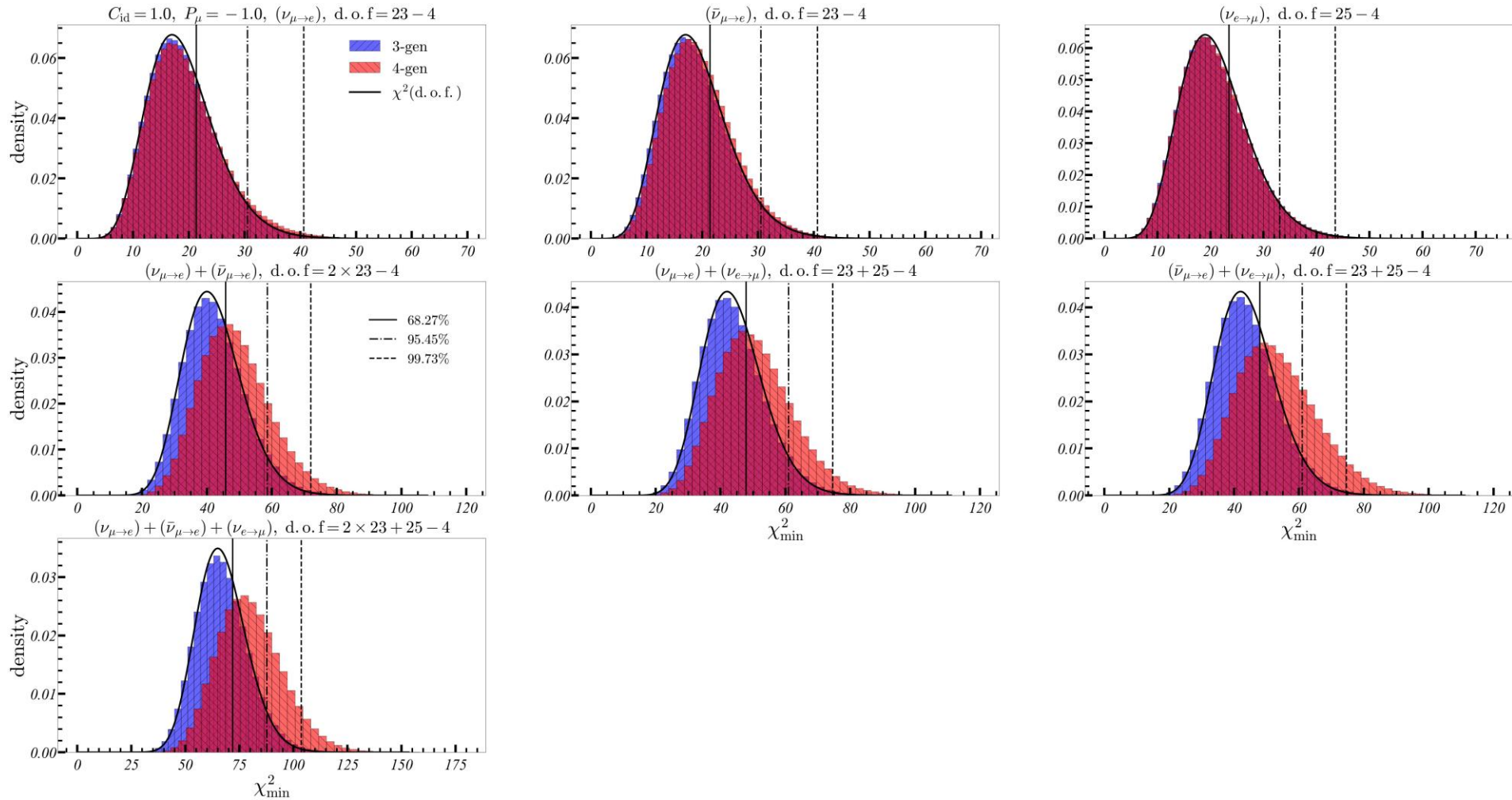
Inconsistent



Theoretical curve of the 4-gen model (blue dotted)

Distribution of $\chi_{\min}^2 = \chi^2(\mathbf{C}_{\text{b.f.p.}}^L)$, ($C_{id} = 1.0, P_{\mu} = -1.0$)

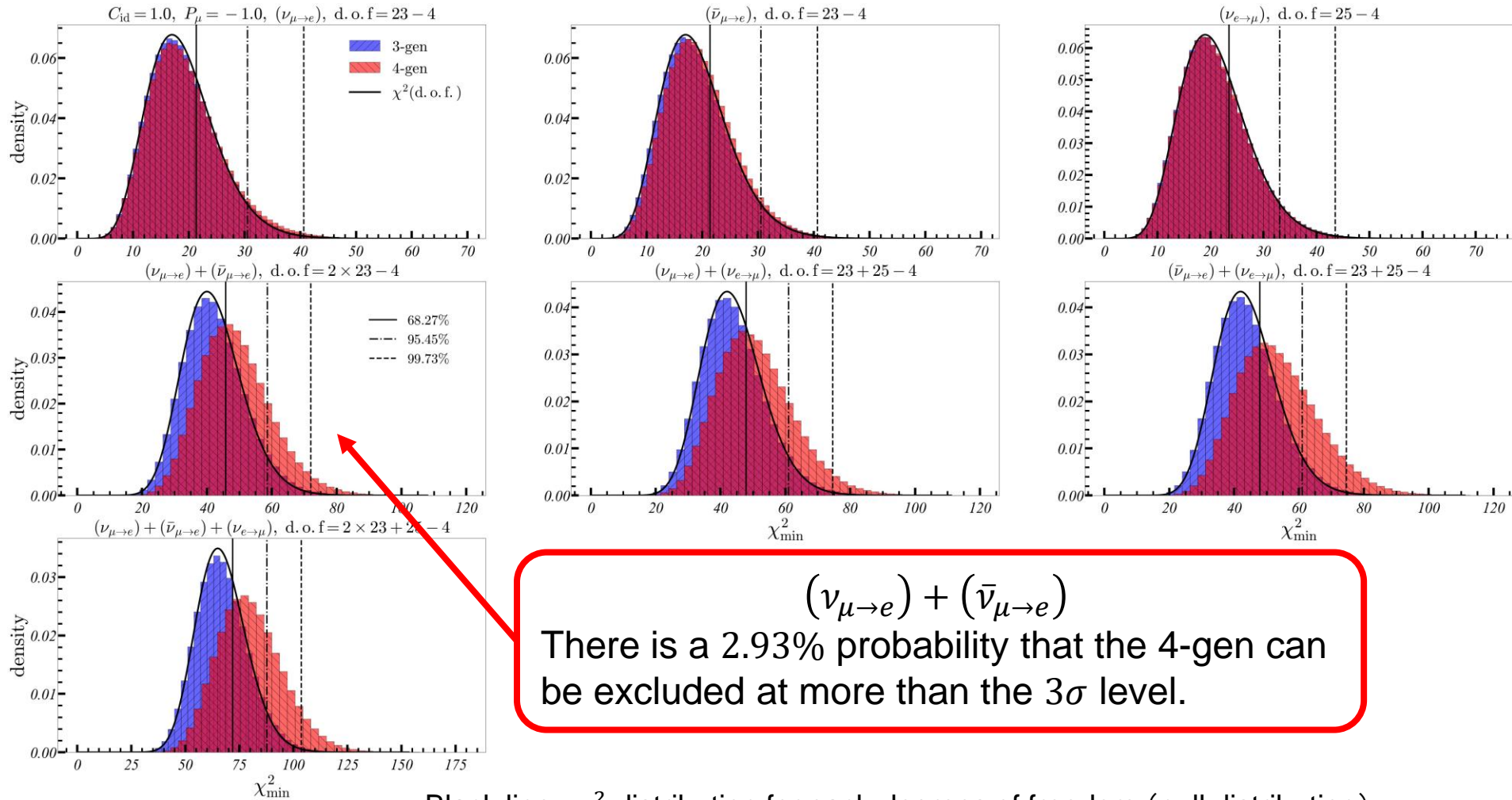
Fit 3-gen model in vacuum to 3-, 4-gen events in vacuum



Distribution of $\chi^2_{\min} = \chi^2(\mathbf{C}_{\text{b.f.p.}}^L)$, ($C_{id} = 1.0, P_{\mu} = -1.0$)

Fit 3-gen model in vacuum to 3-, 4-gen events in vacuum

When considering the distribution of χ^2_{\min} , only the statistical power can be evaluated.



Black line: χ^2 distribution for each degrees of freedom (null distribution)

Vertical line: Percentile points of the χ^2 distribution shown by the black curve