Interaction of Dirac and Majorana fermions with δ-shaped defects

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Symanzik approach

The approach proposed by Symanzik for modeling the interaction of a macroscopic material body with quantum fields is considered. Its application in quantum electrodynamics enables one to establish the most general form of the action functional describing the interaction of a 2-dimensional material surface with photon and fermion fields. The models making it possible to calculate the Casimir energy and Casimir-Polder potential for non-perfectly conducting material are presented. Applications of the models to descriptions of other physical phenomena are considered.

Symanzik approach

The proposed by Symanzik action functional describing the interaction of the quantum field with material body has the form:

$$S(\varphi) = S_V(\varphi) + S_{def}(\varphi)$$

where

$$S_V(\varphi) = \int L(\varphi(x)) d^D x, \quad S_{def}(\varphi) = \int_\Gamma L_{def}(\varphi(x)) d^{D'} x,$$

and is a subspace of dimension $D' \leq D$ in D-dimensional space.

Symanzik approach

The quantum electrodynamics (QED) describes the interaction of electromagnetic field $A_\mu(x)$ with spinor Dirac fields $\psi(x)$, $\overline{\psi}(x)$. One defines for them the gauge transformation as

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \varphi(x), \quad \psi(x) \to e^{i\varphi(x)}\psi(x), \quad \overline{\psi}(x) \to e^{-i\varphi(x)}\overline{\psi}(x)$$

with an arbitrary function $\varphi(x)$. The full action of the model, can be presented in the form

$$S(\overline{\psi}, \psi, A) = S_{QED}(\overline{\psi}, \psi, A) + S_{def}(\overline{\psi}, \psi, A)$$

where $S_{QED}(\overline{\psi}, \psi, A)$ satisfies the requirement of locality, renormalizability, gauge and Lorentz invariance.

Symanzik approach

These basic principles define the form of $S_{QED}(\overline{\psi}, \psi, A)$ up to a constant factor:

$$S_{QED}(\overline{\psi}, \psi, A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\hat{\partial} - m + ie\hat{A})\psi$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\hat{A} = A_\mu \gamma^\mu$ with Dirac matrices γ^μ.

Symanzik approach

If it is supposed that the basic principles of QED (gauge invariance, locality, renormalizability) must be fulfilled also for the defect action $S_{def}(\overline{\psi}, \psi, A)$, then for thin film without charges and currents, which shape is defined by equation $\Phi(x) = 0$, $x = (x_0, x_1, x_2, x_3)$, it has the form

$$S_{def}(\overline{\psi}, \psi, A) = S_\Phi(A) + S_\Phi(\overline{\psi}, \psi).$$

Due to the requirements of renormalizability the fields interaction is described by standard contribution $ie\overline{\psi}\hat{A}\psi$ to the QED action.

Casimir effect

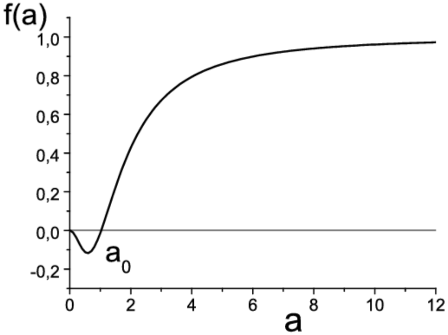
For the simplest case of two plane parallel infinite films the Casimir energy was calculated in V. N. Markov and Yu. M. Pis'mak, J. Phys. A 39:21, 6525-6532 (2006). If the defects are concentrated on planes $x_3 = 0$ and $x_3 = r$, the defect action has the form:

$$S_\Phi = S_{2P} = \frac{1}{2}\int (a_1\delta(x_3) + a_2\delta(x_3 - r))\varepsilon^{3\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x)dx.$$

For this geometry, it is convenient to use notations like $x = (x_0, x_1, x_2, x_3) = (\vec{x}, x_3)$, $\vec{x}^2 = x_0^2 - x_1^2 - x_2^2$, $|\vec{x}| = \sqrt{\vec{x}^2}$.

Casimir effect

Function $f(a)$ determining the Casimir forces between two parallel planes. It is even ($f(a) = f(-a)$), has the minimum $f(a_m) = -0.11723$ at $a_m = 0.5892$, and $f(a_0) = 0$ at $a_0 = 1.03246$.



Dependence of the Casimir force on the interaction constant a

Scattering on a flat surface

For the defect $\Phi(x) = x_3$ the action has a form

$$S(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + S_\Phi(A),$$

where

$$S_\Phi(A) = \frac{a}{2} \int \varepsilon^{3\mu\nu\rho} A_\mu(x) F_{\nu\rho}(x) \delta(\Phi(x)) dx.$$

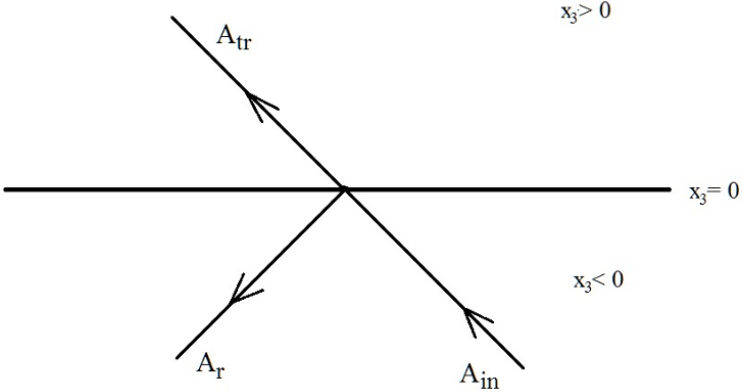
Euler-Lagrange equations are written as a modified Maxwell equations:

$$\frac{\delta S(A)}{\delta A_\nu} = \partial_\mu F^{\mu\nu} + a\varepsilon^{3\nu\sigma\rho} F_{\sigma\rho} \delta(x_3) = 0.$$

Their solutions describe the scattering of electromagnetic waves on the plane $x_3 = 0$.

Scattering on a flat surface

If in the scattering process the incident wave moves to the plane $x_3 = 0$ from the half-space $x_3 < 0$, then there is only the transmitted wave in the half-space $x_3 > 0$, moving away from the plane $x_3 = 0$ in the positive direction of the third axis.



Transmission and reflection coefficients

The coefficients of reflection $K_r \equiv I_r/I_{in}$ and transmission $K_{tr} \equiv I_{tr}/I_{in}$ of a plane wave in its scattering on a plane are independent of the frequency and incidence angle and are expressed in terms of the coupling constant a of the electromagnetic field to the surface, characterizing the properties of its material

$$K_r = \frac{a^2}{1+a^2}, \quad K_{tr} = \frac{1}{1+a^2}.$$

Other results obtained

The explicit expressions for the amplitudes of the electromagnetic field in all three layers obtained for all possible processes of wave propagation. The Chern-Simons interaction does not change Snell's law. However, the reflection and transmission coefficients depend on the strengths of coupling constants. They lead to a mixing between the parallel and perpendicular components of the electromagnetic waves and they change the relation between frequency and wave vector for waves between two totally reflecting media.

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Scattering of Dirac fermions

We will consider the material plane $x_3 = 0$ as a defect. In this case, in the Dirac part of the action

$$S(\overline{\psi}, \psi) = \int \overline{\psi}(x)(i\hat{\partial} - m + \Omega(x_3))\psi(x)dx,$$

the interaction of the spinor field with the plane is described with matrix $\Omega(x_3) = Q\delta(x_3)$. Since $\Omega(x_3)$ and $\delta(x_3)$ have the dimension of mass, the matrix Q is dimensionless. For homogeneous isotropic material plane in more general case, the matrix Q could be presented in the form:

$$Q = r_1 I + ir_2\gamma_5 + r_3\gamma_3 + r_4\gamma_5\gamma_3 + r_5\gamma_0 + r_6\gamma_5\gamma_0 + ir_7\gamma_0\gamma_3 + ir_8\gamma_1\gamma_2$$

with I - identity 4x4 matrix, $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ are Dirac matrices.

Scattering of Dirac fermions

Movement of spinor particle in the field of defect $\Omega(x_3)$ is described by a modified Dirac equation

$$(i\hat{\partial} - m + \Omega(x_3))\psi(x) = 0.$$

It is one of the Euler-Lagrange equations, which is obtained by variational differentiating of the action over $\psi(x)$. Taking the derivative over $\overline{\psi}(x)$ we obtain the second equation

$$(\partial_\mu \overline{\psi}(x))\gamma^\mu + \overline{\psi}(x)(m - \Omega(x_3)) = 0.$$

The condition $\overline{\psi}(x) = \psi^*(x)\gamma_0$ fulfils if $\gamma_0 \Omega^\dagger(x) = \Omega(x)\gamma_0$. It is the case for real values of parameters $r_j, j = 1, \ldots, 8$.

Scattering of Dirac fermions

We denote $\psi(x)$ the solution of the modified Dirac equation, and $\psi_-(x) = \psi(x)$ for $x_3 < 0$, $\psi_+(x) = \psi(x)$ for $x_3 > 0$. The spinors $\psi_\pm(x)$ for $x_3 \neq 0$ satisfy the free Dirac equation and boundary condition

$$\lim_{x_3 \to +0} \psi_+(x) = S \lim_{x_3 \to -0} \psi_-(x),$$

One can choose the regularization procedure for $\delta(x_3)$ in such a way that the matrix S is expressed in terms of Q as

$$S = \exp\{-i\gamma_3 Q\}.$$

Scattering of Dirac fermions

The free Dirac equation in coordinate space reads

$$(i\hat{\partial} - m)\psi(x) = 0.$$

By substitution $\psi(x)$ in the form

$$\psi(x) = \frac{1}{(2\pi)^4}\int e^{-ipx}\psi(\bar{p})d\bar{p}, \quad \bar{p} = (p_0, p_1, p_2)$$

one obtains

$$(\hat{p} - m)\psi(\bar{p}) = 0.$$

For real p_3 the considered spinor $\psi(x)$ describes the scattering state and the imaginary p_3 - the bound state.

Scattering of Dirac fermions

The general solution $\psi(\bar{p})$ of the Dirac equation can be presented as an arbitrary linear combination of linear independent spinors

$$\psi_1(\bar{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{-p_3}{m+p_0} \\ \frac{-p_1+ip_2}{m+p_0} \end{pmatrix}, \quad \psi_2(\bar{p}) = \begin{pmatrix} 0 \\ 1 \\ \frac{-p_1-ip_2}{m+p_0} \\ \frac{p_3}{m+p_0} \end{pmatrix}, \quad p_3 = \pm\sqrt{\bar{p}^2 - m^2}$$

for $p_0 > 0$ and

$$\psi_1'(\bar{p}) = \begin{pmatrix} \frac{p_1-ip_2}{m-p_0} \\ \frac{p_3}{p_0-m} \\ 0 \\ 1 \end{pmatrix}, \quad \psi_2'(\bar{p}) = \begin{pmatrix} \frac{p_3}{m-p_0} \\ \frac{p_1+ip_2}{m-p_0} \\ 1 \\ 0 \end{pmatrix}.$$

for $p_0 < 0$.

Scattering of Dirac fermions

Substituting $p_3 \to \pm i\kappa$ with $\kappa = |\kappa| = \sqrt{m^2 + p_1^2 + p_2^2 - p_0^2}$ we obtain the spinors describing the bound states

$$\psi_\pm(\bar{p}) = \psi(p)|_{p_3 \to \mp i\kappa}.$$

They can be presented as follows

$$\psi_+(\bar{p}) = a_1\psi_{1+}(\bar{p}) + a_2 e^{i\varphi}\psi_{2+}(\bar{p}), \quad \psi_-(\bar{p}) = d_1\psi_{1-}(\bar{p}) + d_2 e^{i\varphi}\psi_{2-}(\bar{p}),$$

$$\psi'_+(\bar{p}) = a'_1\psi'_{1+}(\bar{p}) + a'_2 e^{i\varphi}\psi'_{2+}(\bar{p}), \quad \psi'_-(\bar{p}) = d'_1\psi'_{1-}(\bar{p}) + d'_2 e^{i\varphi}\psi'_{2-}(\bar{p}).$$

Scattering of Dirac fermions

$$\psi_{1\pm}(\bar{p}) = \begin{pmatrix} 1 \\ 0 \\ \pm ik \\ fe^{i\varphi} \end{pmatrix}, \quad \psi_{2\pm}(\bar{p}) = \begin{pmatrix} 0 \\ 1 \\ fe^{-i\varphi} \\ \mp ik \end{pmatrix},$$

$$\psi'_{1\pm}(\bar{p}) = \begin{pmatrix} -fe^{-i\varphi} \\ \pm ik \\ 0 \\ 1 \end{pmatrix}, \quad \psi'_{2\pm}(\bar{p}) = \begin{pmatrix} \mp ik \\ -fe^{i\varphi} \\ 1 \\ 0 \end{pmatrix},$$

$$k = \frac{\kappa}{m + |p_0|}, \quad \frac{p_1 + ip_2}{m + |p_0|} = -\frac{p_1 + ip_2}{m + |p_0|} = fe^{i\varphi}, \quad f = |f|.$$

The spinors $\psi_\pm$, $\psi'_\pm$ fulfill the relations

$$\psi_+(\bar{p}) = S\psi_-(\bar{p}), \quad \psi'_+(\bar{p}) = S\psi'_-(\bar{p}), \quad S = e^{-i\gamma_3 Q}.$$

which can be presented as systems of linear equations for coefficients a_1, a_2, d_1, d_2, a'_1, a'_2, d'_1, d'_2.

Transmission and reflection coefficients

The characteristics of the scattering processes essentially depend on the choice of parameters determining the interaction of the plane with Dirac particles, on their polarization, energy and angle of incidence. The parameters of the model can be chosen so that the transmission coefficient is almost equal to unity at low particle energy and is almost zero for particles with high energy. One can choose the parameters and so that at high energies the particles almost completely pass through the plane, and at low energies they are almost completely reflected.

Transmission and reflection coefficients

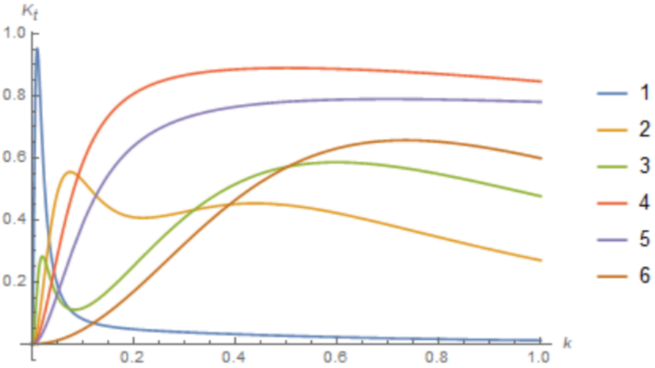

Figure 2. Transmission coefficient $K_t(k) = c_{1+}g(c_{2+};k/c_{3+})\cos(\theta)^2 + c_{1-}g(c_{2-};k/c_{3-})\sin(\theta)^2$ by different values of $c_{1\pm}, c_{2\pm}, c_{3\pm}, \theta$: (1) $c_{1+} = 0.99, c_{2+} = 0.225, c_{3+} = 0.01, c_{1-} = 0.8, c_{2-} = 0.025, c_{3-} = 0.1, \cos(\theta)^2 = 0.95$; (2) $c_{1+} = 0.95, c_{2+} = 0.225, c_{3+} = 0.07, c_{1-} = 0.9, c_{2-} = 0.25, c_{3-} = 0.5, \cos(\theta)^2 = 0.55$; (3) $c_{1+} = 0.8, c_{2+} = 0.25, c_{3+} = 0.02, c_{1-} = 0.9, c_{2-} = 0.2, c_{3-} = 0.6, \cos(\theta)^2 = 0.35$; (4) $c_{1+} = 0.9, c_{2+} = 0.025, c_{3+} = 0.5, c_{1-} = 0.8, c_{2-} = 0.0025, c_{3-} = 0.7, \cos(\theta)^2 = 0.9$; (5) $c_{1+} = 0.8, c_{2+} = 0.025, c_{3+} = 0.7, c_{1-} = 0.7, c_{2-} = 0.0025, c_{3-} = 0.9, \cos(\theta)^2 = 0.9$; (6) $c_{1+} = 0.6, c_{2+} = 0.25, c_{3+} = 0.7, c_{1-} = 0.8, c_{2-} = 0.25, c_{3-} = 0.8, \cos(\theta)^2 = 0.7$.

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Scattering of Majorana fermions

$$S_{Majorana}(\psi) = \psi\bar{\gamma}^0 \left(\hat{\partial} + m\big(\cos\zeta + \bar{\gamma}^5 \sin\zeta\big) + \Omega(x_3)\right)\psi. \quad (1)$$

$$\Omega(x_3) = Q\delta(x_3), \quad (2)$$

Let us note the fundamental difference of this structure from the Dirac case, where the action contains independent fields ψ and $\bar{\psi}$:

$$S_{Dirac}(\overline{\psi}, \psi) = \int \overline{\psi}(x)\big(i\hat{\partial} - m + \Omega(x_3)\big)\psi(x)\, dx. \quad (3)$$

In the general Dirac case, such a matrix can be expanded in a gamma matrix basis with eight independent dimensionless parameters. However, for the Majorana field, additional restrictions are imposed due to the reality of the spinor and the anticommutative nature of fermion fields. In particular, the matrix $\bar{\gamma}^0 Q$ must be real and antisymmetric.
As a result, the admissible form of the matrix Q is significantly simplified and can be written as

$$Q = r_1 I + r_2\bar{\gamma}^5 + r_3\bar{\gamma}^5\bar{\gamma}^3 + r_4\bar{\gamma}^5\bar{\gamma}^0, \quad (4)$$

Summary

- In the framework of the Symanzik approach, we build the model of QED field interaction with 2D material. The action of the model consist of the usual QED action and extra defect contribution. The action contains parameters, that characterize the material property.
- The characteristics of photons, Dirac and Majorana particles scattering on the defect plane can be calculated in the model, also the properties of states localized near the defect plane can be investigated. The model and obtained on its basis results could be used for the theoretical description of the interaction of electrons, positrons and neutrons with two-dimensional materials (graphite, thin films, sputters, sharp boundaries of a solid body).
- We can also compare scenarios for Dirac and Majorana fermions, which is intriguing in the field of neutrino studies.