



Accelerator RF Cavities



PARTICLE ACCELERATORS
— AND —
LASER TECHNOLOGY
— RESEARCH GROUP —



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Outline

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Charged Particle Acceleration and RF Gaps

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Rectangular & Circular Waveguides

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Resonant RF Cavities

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Cavity optimization

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Summary

Charged Particle Acceleration

Lorentz force: $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$

q = Charge of the particle

v = Velocity of the particle

Energy gain: $W = \int (\vec{F} \cdot \vec{v}) dt = \int \vec{F} \cdot d\vec{s}$

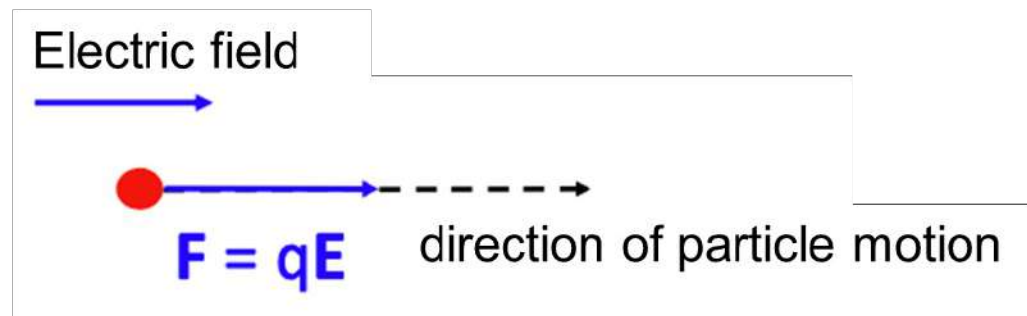
E = Electric field

B = Magnetic field

$$\begin{aligned} W &= \int \vec{F} \cdot d\vec{s} \\ &= q \int \vec{E} \cdot d\vec{s} + q \int (\vec{v} \times \vec{B}) \cdot d\vec{s} \\ &= q \int \vec{E} \cdot d\vec{s} + q \int (\vec{v} \times \vec{B}) \cdot \vec{v} dt \end{aligned} \rightarrow (\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

$$\therefore W = q \int \vec{E} \cdot d\vec{s}$$

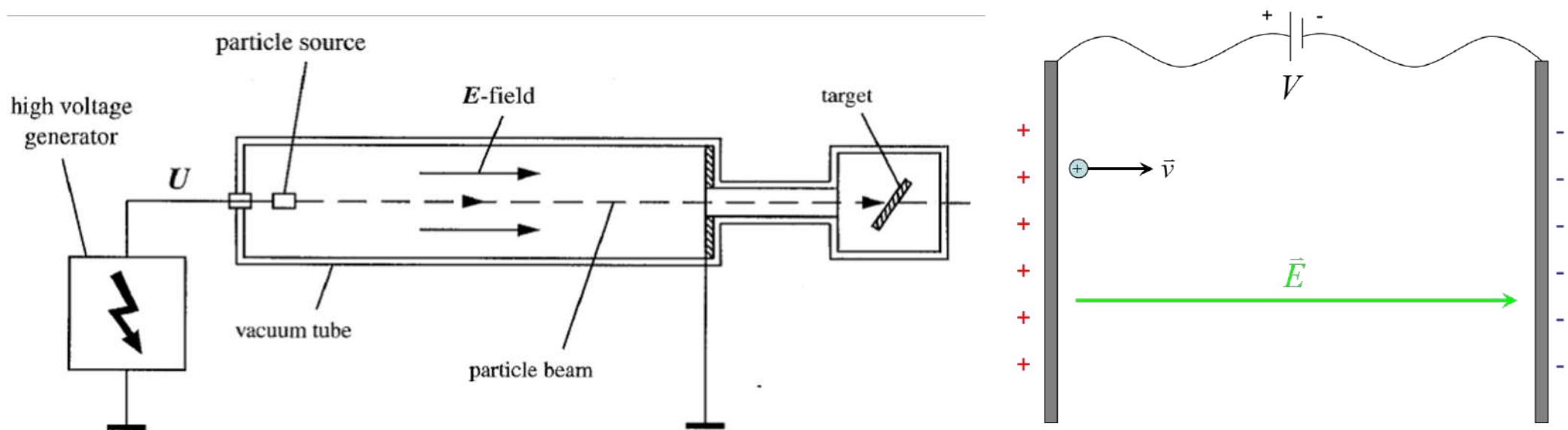
Requires an electric field in the direction of particle motion for acceleration.



Acceleration with Electrostatic Field

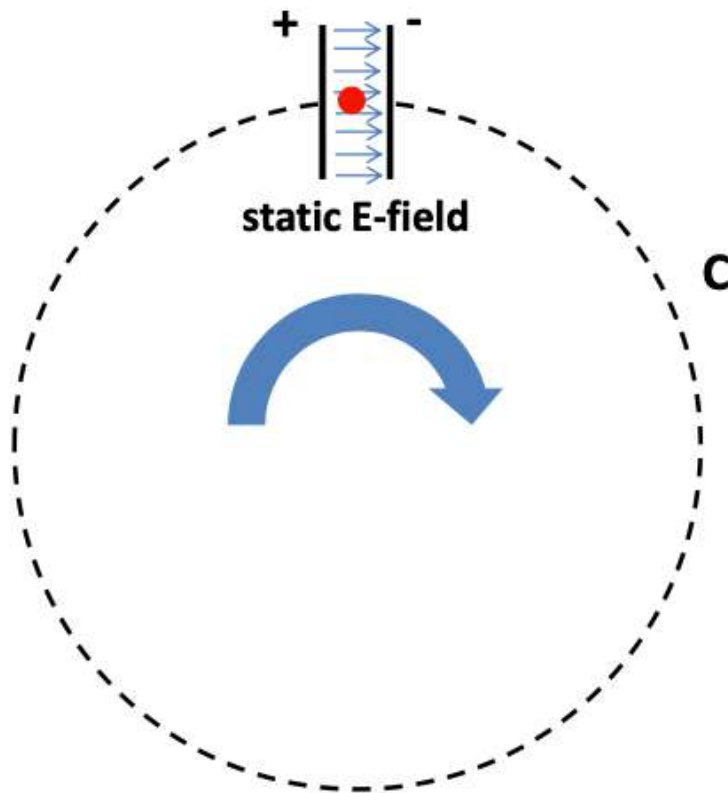
In linear accelerator, when a positive charge q is accelerated with electric potential V

- It gains a kinetic energy of $W = qV$
- DC voltage is **limited to a few MV**.



K. Wille, The Physics of Particle Accelerators: An Introduction, Oxford University Press, 2000

Question: Can we use static electric field to accelerate charged particles for repetitive acceleration in circular accelerator?



Faraday's law: $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

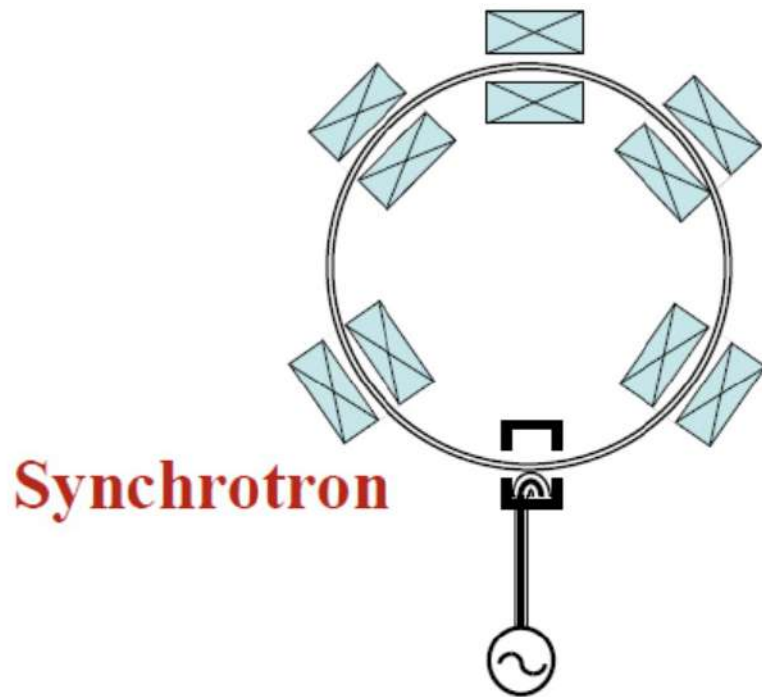
For static electric field: $\nabla \times \vec{E} = 0$

$$\oint \vec{E} \cdot d\vec{s} = 0$$

$$W = q \oint \vec{E} \cdot d\vec{s} = 0 \rightarrow \text{No energy gain!}$$

Courtesy: W.-K. Lau, ISBA22

Acceleration with Time-varying EM Field



Faraday's law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Energy gain: $W = q \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi}{dt}$

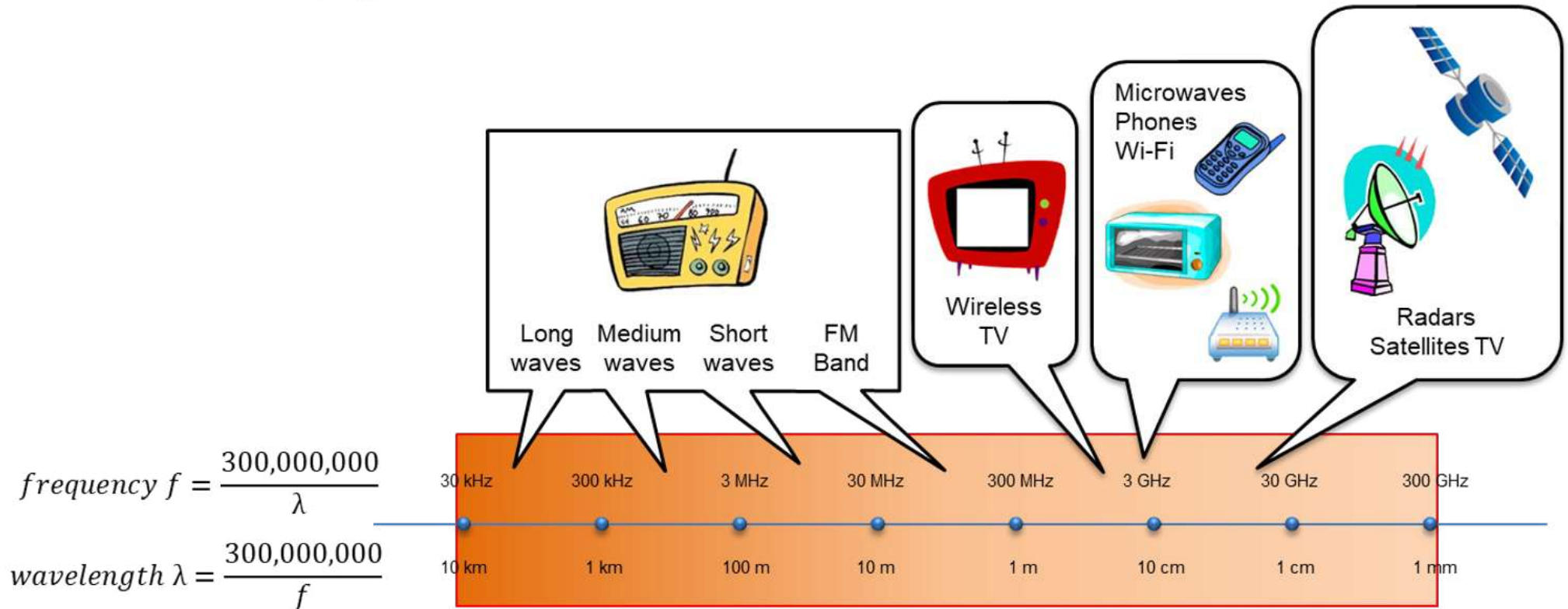
- To accelerate charged particles in circular accelerator, time-varying magnetic flux through the surface enclosed by the orbit is required.

$$\vec{E} = \vec{E}(\vec{r}, t) \quad \Rightarrow \quad \vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r}) \cos \omega t \quad \text{or} \quad \vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r}) \sin \omega t$$

$$\vec{B} = \vec{B}(\vec{r}, t) \quad \Rightarrow \quad \vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}) \cos \omega t \quad \text{or} \quad \vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}) \sin \omega t$$

Particle Acceleration with Radio-frequency (RF) Waves

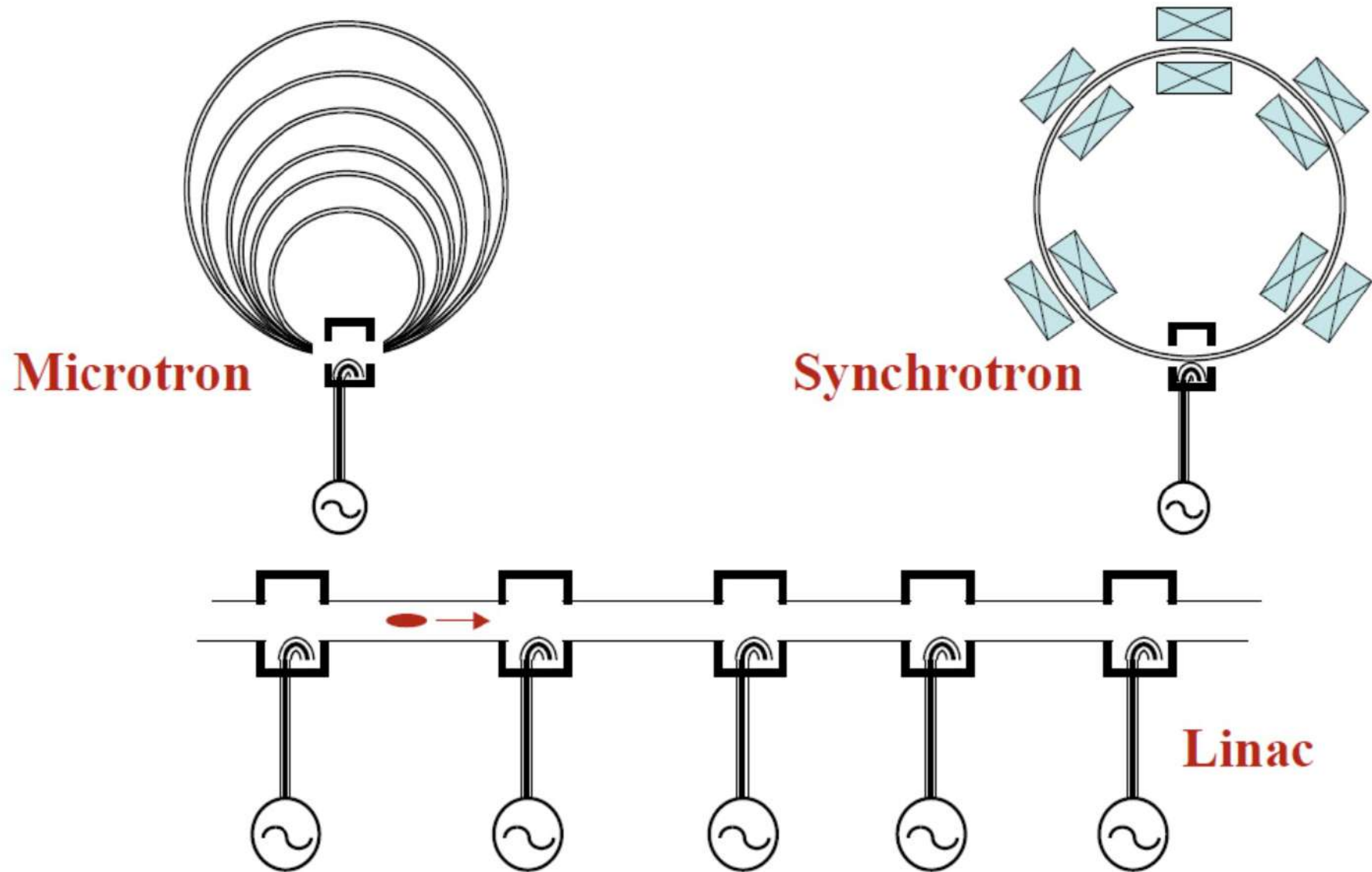
- Frequency range: 10 MHz - 30 GHz
- Accelerating gradient: 1 - 100 MV/m



With $\epsilon = \sim 1.0$ (dielectric constant of air at 20°C)

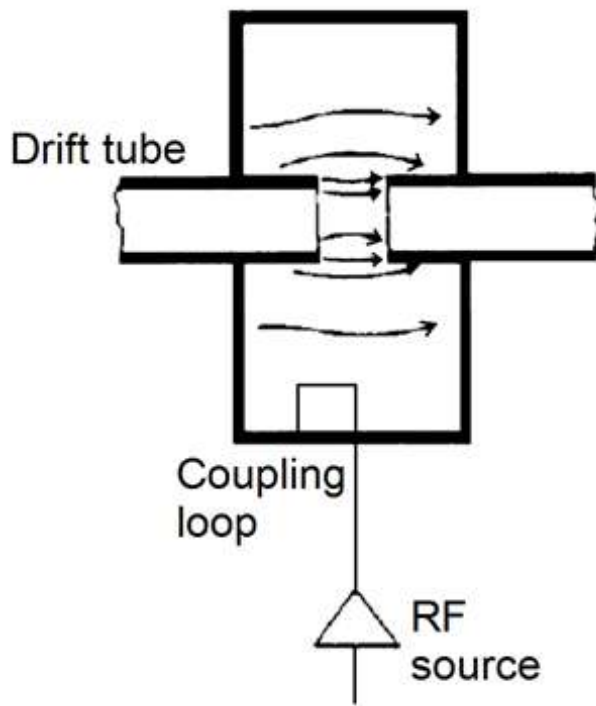
Courtesy: E. Montesinos, RF powering, CAS 2015

RF Cavities in Circular & Linear Accelerators



Single RF Gap Acceleration

Instead of using two electric plates, we apply a **RF wave** to a gap between two drift tubes. A particle gains kinetic energy of



$$W = q \int \vec{E} \cdot d\vec{s} = qE_0 \int \cos(\omega t) ds$$

Consider $E_0 = V_0 / L = \text{constant}$

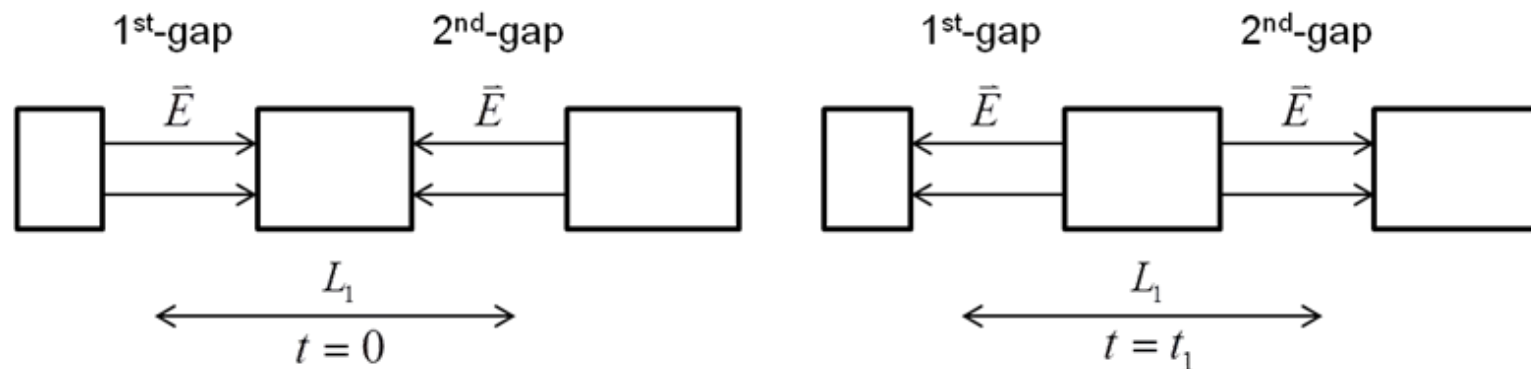
$$W = \frac{qV_0}{L} \int_{-L/2}^{+L/2} \cos(\omega s / v) ds = qV_0 T$$

$$T = \frac{\Delta E_k}{qV_0} = \frac{\sin(\omega L / 2v)}{\omega L / 2v} \equiv \text{Transit-time factor}$$

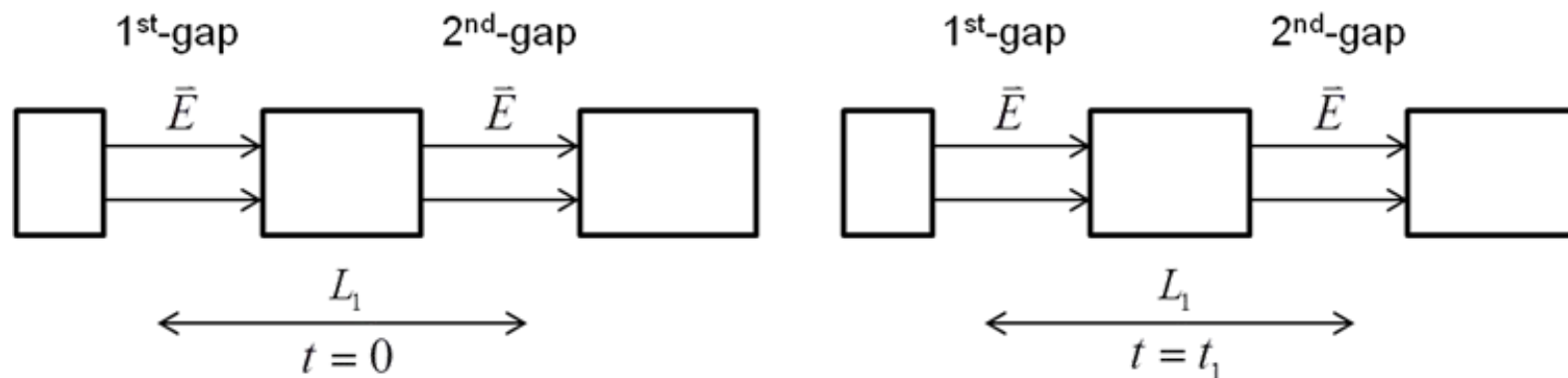
Two RF Gaps Acceleration

For two RF gaps, a particle experiences the RF wave at certain selected points and times with two types of acceleration:

➤ π -mode acceleration:



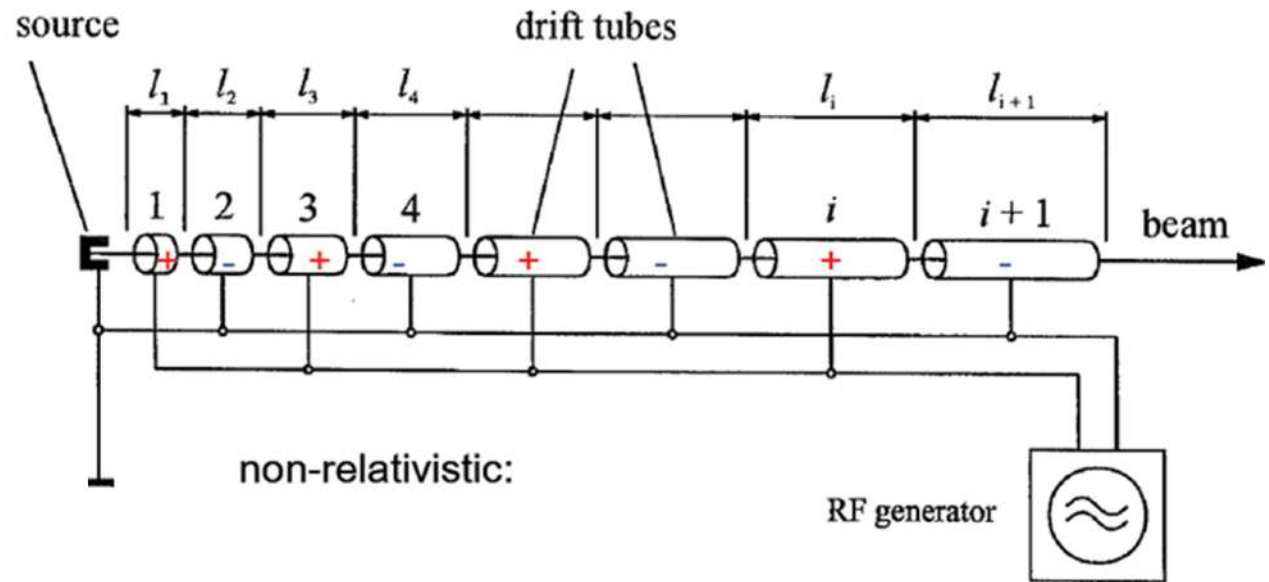
➤ 0-mode or 2π -mode acceleration:



Multi RF Gaps Acceleration: π -mode

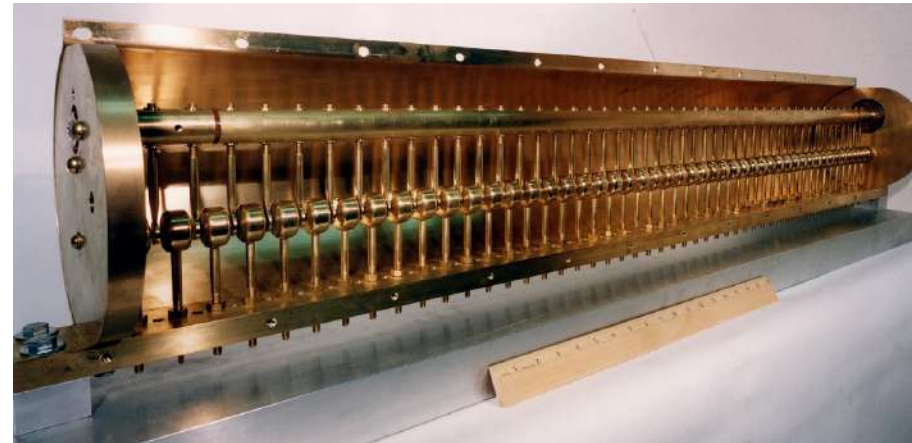
π -mode acceleration
synchronicity condition:

$$l_i = \frac{vT_{rf}}{2} = \beta \frac{\lambda_{rf}}{2}$$



Example: Wideröe linacs

- The first RF linac proposed by Ising in 1925 & built by Wideröe in 1928



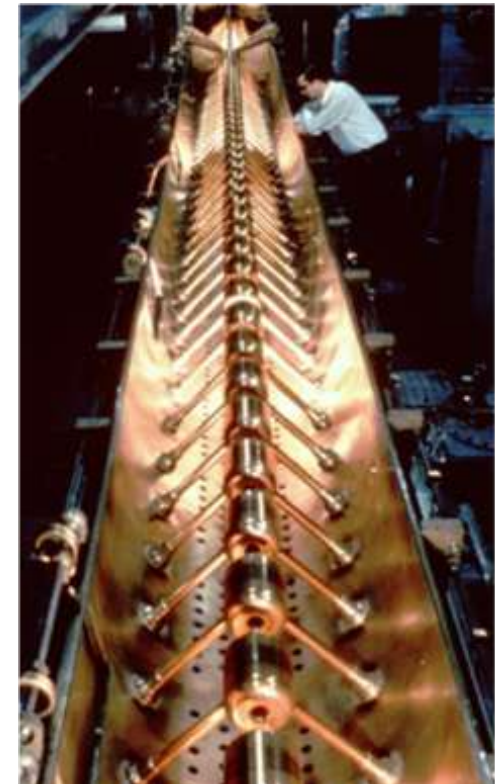
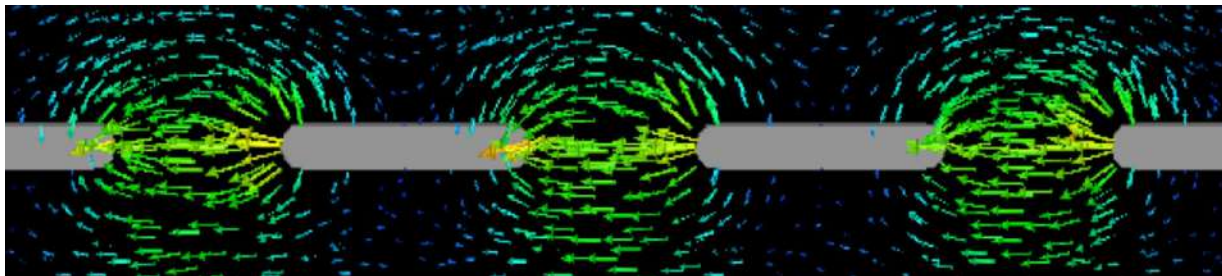
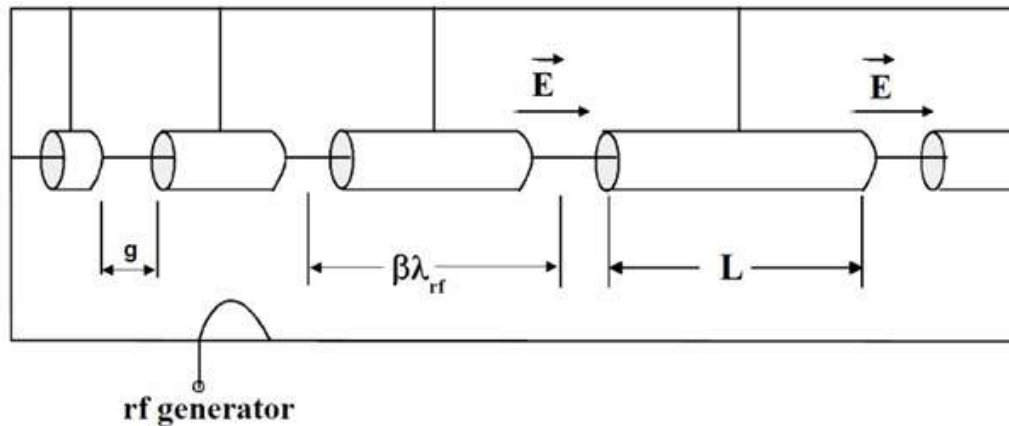
G. Hoffstaetter, U.S. Particle Accelerator School, June 2010; www.flickr.com/photos/departmentofenergy/16520269905

Multi RF Gaps Acceleration: 0-mode (or 2π -mode)

0-mode acceleration synchronicity condition:

$$l_i = vT_{rf} = \beta\lambda_{rf}$$

Example: Alvarez linacs (Developed in 1947)

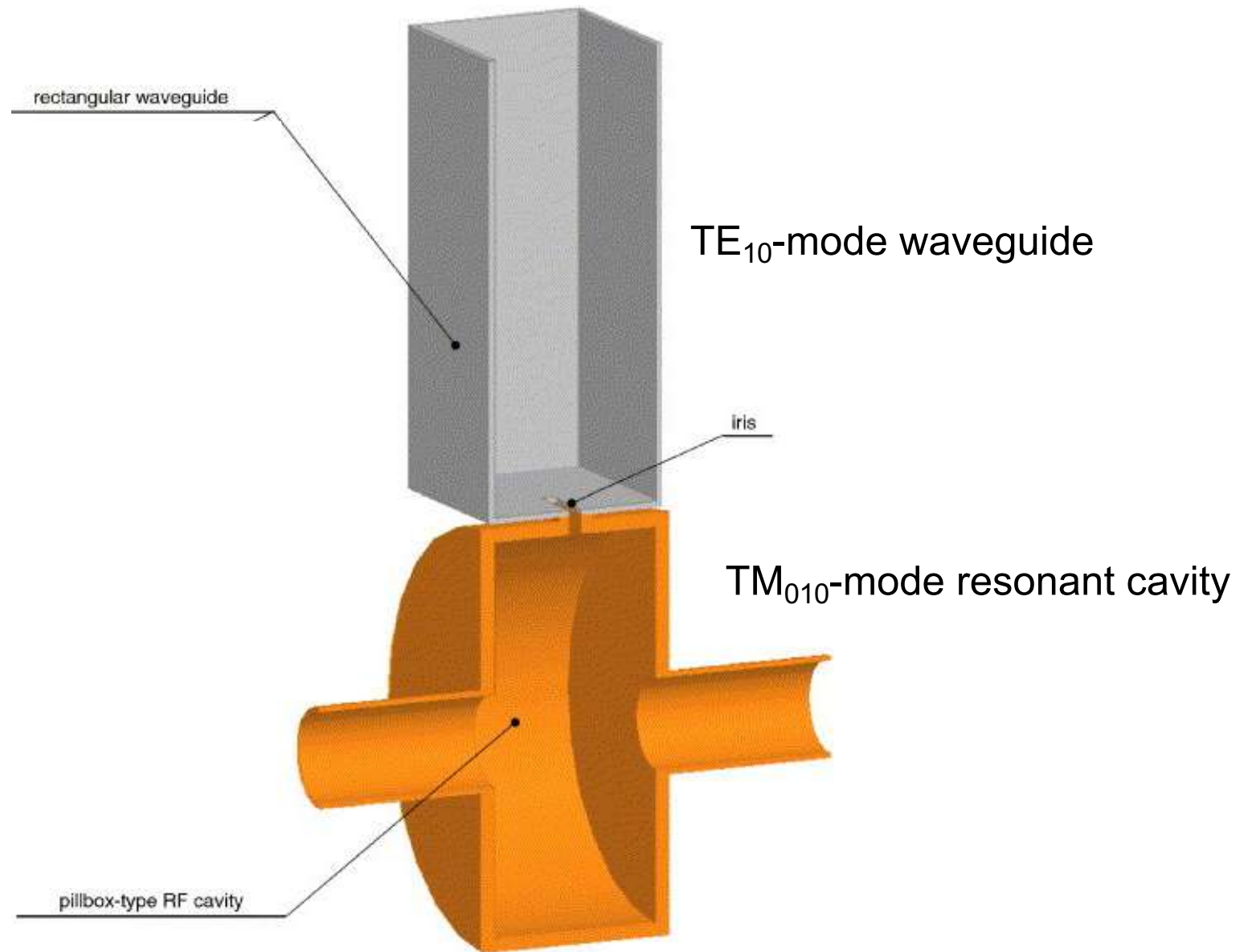


G. Hoffstaetter, U.S. Particle Accelerator School, June 2010;
E. Jensen, CERN Accelerator School, Divonne 2009

Multi RF Gaps Acceleration: Key Differences in Acceleration Modes

Feature	Wideröe Linac (π -mode)	Alvarez Linac (0-mode)
Phase Difference	π mode (180° phase shift between gaps)	0-mode (0° phase shift; fields point in the same direction)
Drift Tube Length	Designed so particles traverse a drift tube in half an RF period ($L = \beta\lambda / 2$)	Particles traverse a tube in a full RF period ($L = \beta\lambda$)
Field Behavior	The field reverses polarity while the particle is shielded inside the drift tube.	The field points in the same accelerating direction across all gaps.

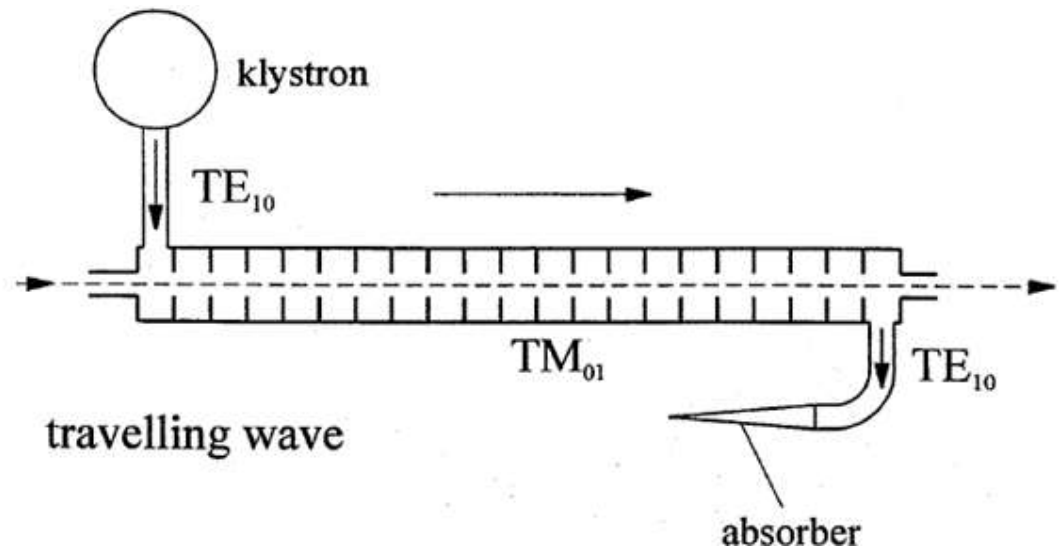
Present RF Cavity: Single Cavity



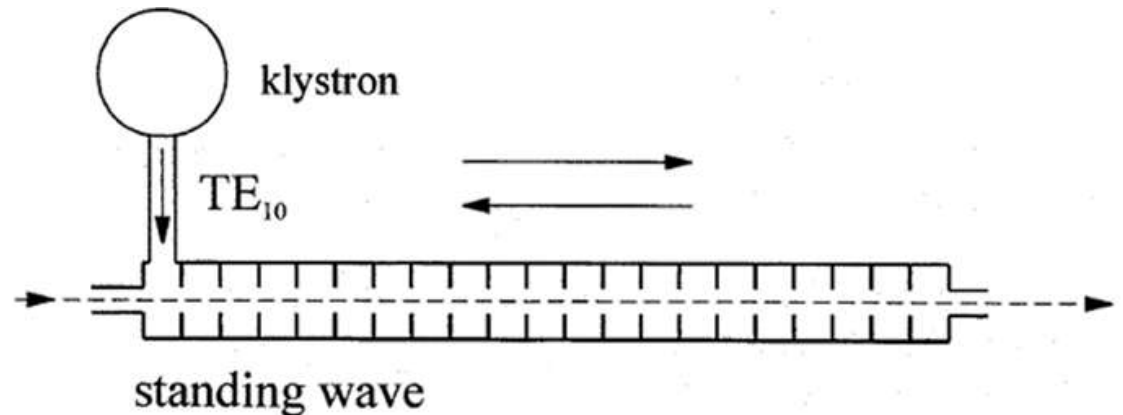
Present RF Linac Structures: Multi Cavities

Two principal types of RF linac:

- **Travelling wave structures:**
Disk loaded waveguides (TM_{01})



- **Standing wave structures:**
Resonant cavities (TM_{010})



Wille, Klaus. The Physics of Particle Accelerators: An Introduction. Oxford University Press, 2000.

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Maxwell's Equations

Differential form:

$$\nabla \cdot \mathbf{D} = \rho \quad \leftarrow \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \leftarrow \text{Faraday's law}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \leftarrow \text{Ampere-Maxwell's law}$$

Electric displacement field: $\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E}$

↑
Electric permittivity

Magnetic field: $\mathbf{B} = \mu \mathbf{H} = \mu_r \mu_o \mathbf{H}$

↑
Magnetic permeability

Wave Equations in Nonconducting Medium

Maxwell's equations in an infinite nonconducting medium and absence of sources

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Using vector identity: $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\mu \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{D}}{\partial t}\right) = -\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Wave equations:

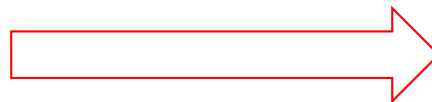
$$\nabla^2 \mathbf{E} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Harmonic waves

$$\mathbf{E} \propto e^{j\omega t}$$

$$\mathbf{B} \propto e^{j\omega t}$$



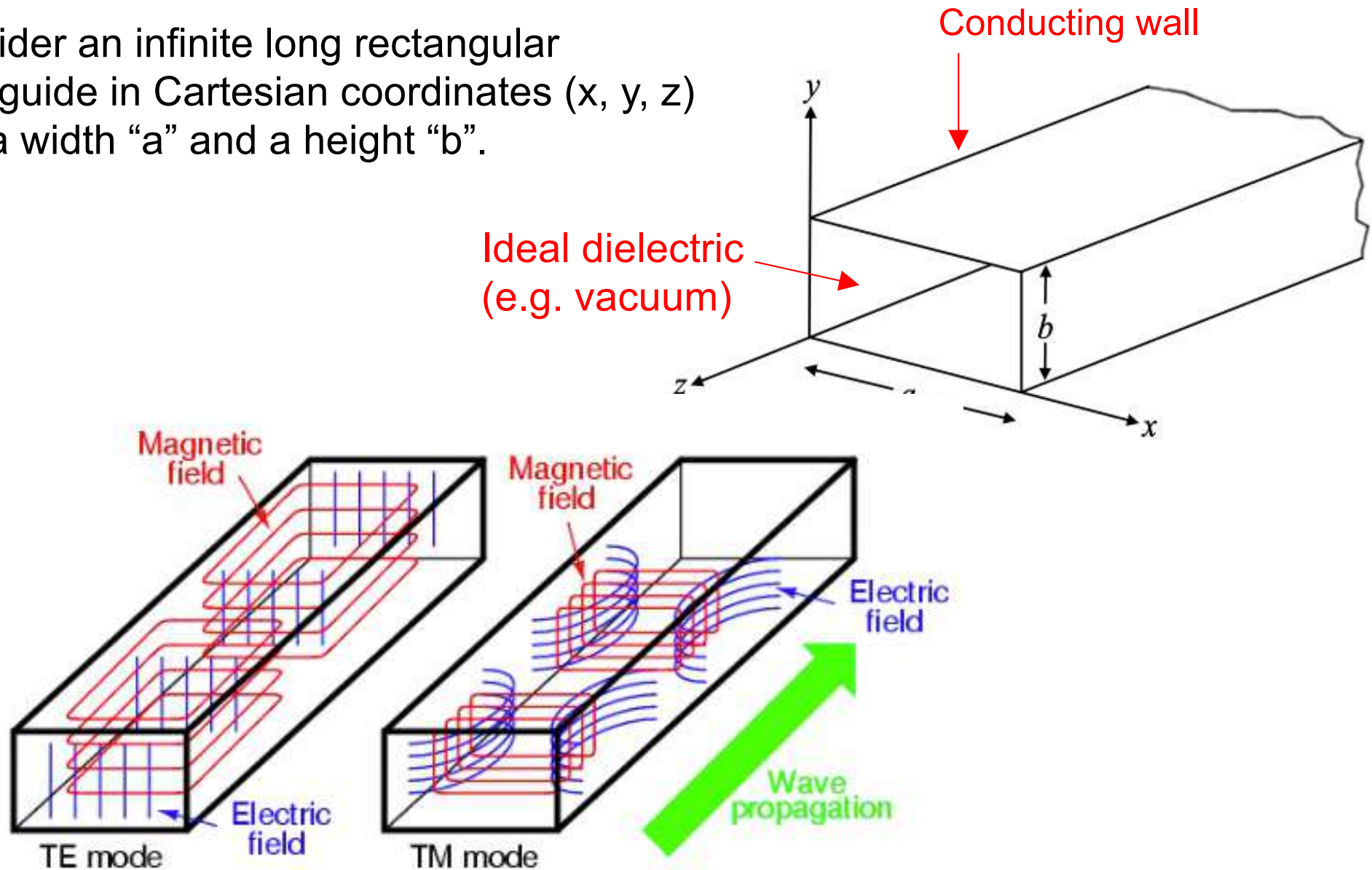
Helmholtz equation

$$\nabla^2 \mathbf{E} + \omega^2 \mu \epsilon \mathbf{E} = 0$$

$$\nabla^2 \mathbf{H} + \omega^2 \mu \epsilon \mathbf{H} = 0$$

Rectangular Waveguides

Consider an infinite long rectangular waveguide in Cartesian coordinates (x, y, z) with a width “ a ” and a height “ b ”.



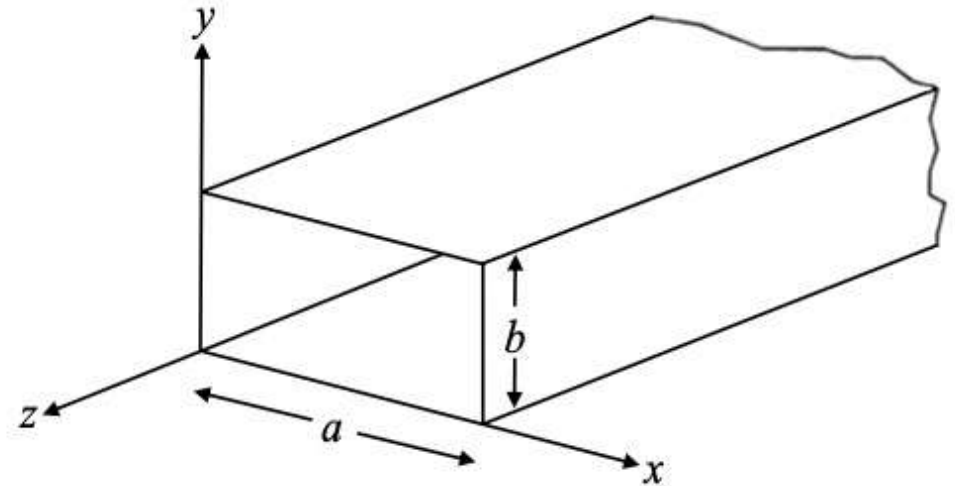
Rectangular Waveguides: TE Modes

TE waves: $E_z = 0$

Wave equations:

$$\nabla^2 \mathbf{E}(\mathbf{r}) + \omega^2 \mu \epsilon \mathbf{E}(\mathbf{r}) = 0$$

$$\nabla^2 \mathbf{H}(\mathbf{r}) + \omega^2 \mu \epsilon \mathbf{H}(\mathbf{r}) = 0$$



Infinite long waveguide $\rightarrow$ 2D boundary condition problem (x, y)

$$\nabla_{\perp}^2 H_z(x, y) = \frac{\partial^2 H_z(x, y)}{\partial x^2} + \frac{\partial^2 H_z(x, y)}{\partial y^2} = -k_c^2 H_z(x, y)$$

$$k_c^2 \equiv \gamma^2 + k^2 = k^2 - k_z^2 = \omega^2 \mu \epsilon - k_z^2$$

Using separation of variables technique: $H_z(x, y) = g_x(x)g_y(y)$

$$H_z(x, y) = (C_1 \sin k_x x + C_2 \cos k_x x)(D_1 \sin k_y y + D_2 \cos k_y y)$$

$$k_c^2 = k_x^2 + k_y^2$$

Boundary condition:

$$\left. \frac{\partial H_z}{\partial n} \right|_S = 0$$

$$H_z(x, y) = (C_1 \sin k_x x + C_2 \cos k_x x)(D_1 \sin k_y y + D_2 \cos k_y y)$$

$$E_x = -\frac{j}{k_c^2} \left(k_z \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) = -\frac{j\omega \mu}{k_c^2} \frac{\partial H_z}{\partial y}$$

$$E_y = \frac{j}{k_c^2} \left(-k_z \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right) = \frac{j\omega \mu}{k_c^2} \frac{\partial H_z}{\partial x}$$

$$E_x = -\frac{j\omega \mu k_y}{k_c^2} (C_1 \sin k_x x + C_2 \cos k_x x)(D_1 \cos k_y y - D_2 \sin k_y y)$$

$$E_y = \frac{j\omega \mu k_x}{k_c^2} (C_1 \cos k_x x - C_2 \sin k_x x)(D_1 \sin k_y y + D_2 \cos k_y y)$$

$$H_z(x, y) = H_0 \cos k_x x \cos k_y y$$

$$k_x = \frac{m\pi}{a}, \quad m = 0, 1, 2, \dots$$
$$k_y = \frac{n\pi}{b}, \quad n = 0, 1, 2, \dots$$

Electric and magnetic field of TE_{mn} modes

$$E_x = \frac{j\omega\mu k_y}{k_c^2} H_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$E_y = -\frac{j\omega\mu k_x}{k_c^2} H_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$E_z = 0$$

$$H_x = -\frac{jk_z}{k_c^2} \frac{\partial H_z}{\partial x} = \frac{jk_z k_x}{k_c^2} H_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$H_y = -\frac{jk_z}{k_c^2} \frac{\partial H_z}{\partial y} = \frac{jk_z k_y}{k_c^2} H_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$H_z = H_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$k_x = \frac{m\pi}{a}, \quad m = 0, 1, 2, \dots$$
$$k_y = \frac{n\pi}{b}, \quad n = 0, 1, 2, \dots$$

Cut-off wave number: $k_c^2 = k_x^2 + k_y^2$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Cut-off frequency

$$\omega_c = \frac{k_c}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Phase velocity:

$$v_p = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{k^2 - k_c^2}}$$

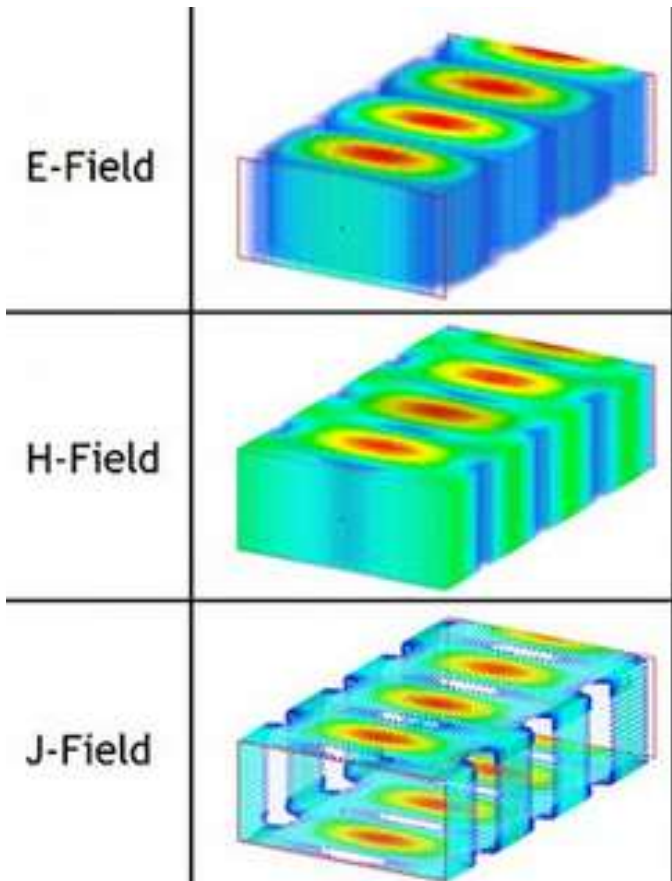
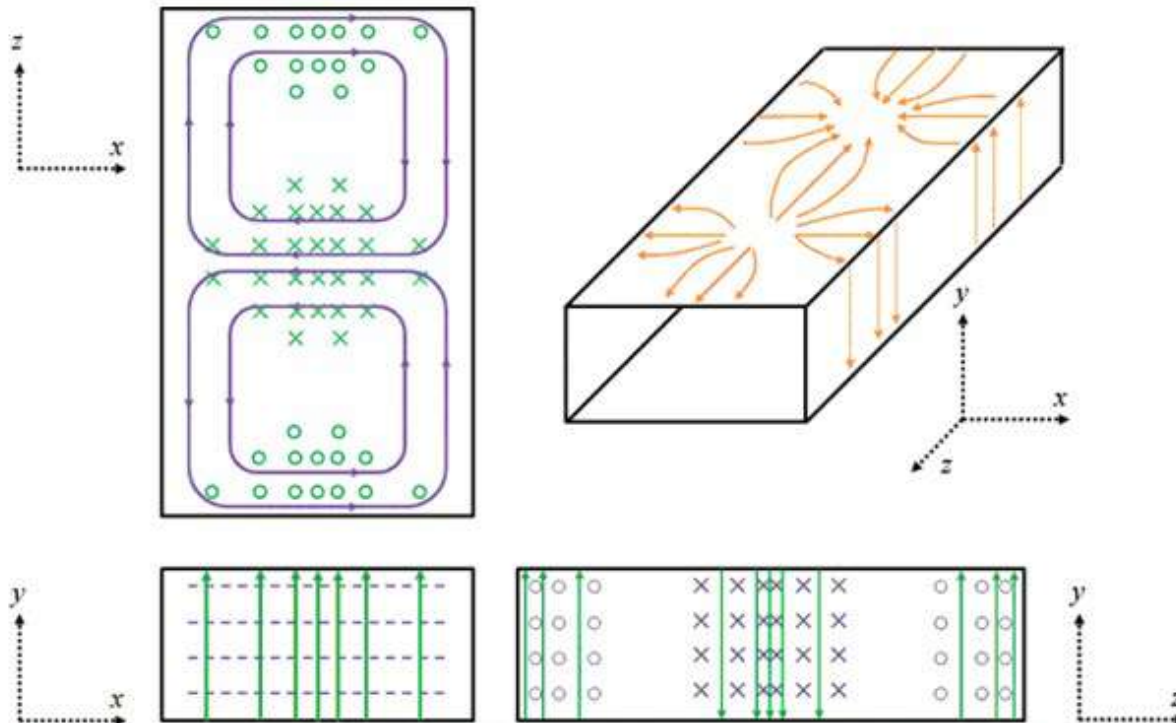
In a conducting waveguides, $k_c \neq 0$ and the wave can propagate without any attenuation only when $k > k_c$

Rectangular Waveguide Fundamental Mode: TE₀₁

$$E_x = 0, \quad E_y = E_0 \sin \frac{\pi x}{a}, \quad E_z = 0$$

$$k_c = \frac{\pi}{a}, \quad \lambda_c = 2a, \quad f_c = \frac{1}{2a\sqrt{\mu\epsilon}}$$

$$H_x = H_0 \cos \frac{\pi x}{a}, \quad H_y = 0, \quad H_z = H_0 \cos \frac{\pi x}{a}$$



Green lines: E-field, Purple lines: B-field, Orange lines: J-field

www.3ds.com/products-services/simulia/resources/hollow-rectangular-waveguide/

Cylindrical Waveguides

Consider an infinite long circular waveguide in cylindrical coordinates (r, φ, z)

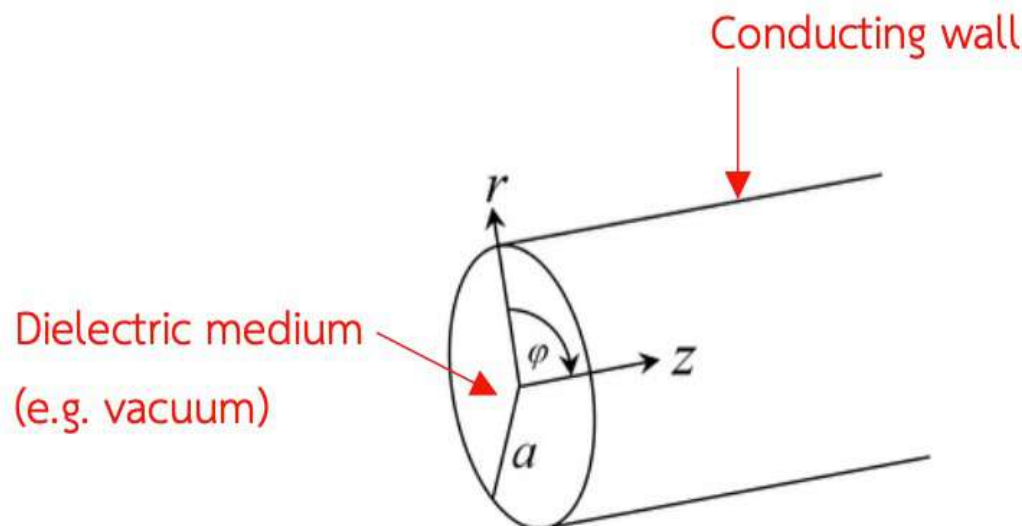
Electric and magnetic fields:

$$\mathbf{E} = \mathbf{E}_o(r, \varphi)e^{j(\omega t - k_z z)}$$

$$\mathbf{H} = \mathbf{H}_o(r, \varphi)e^{j(\omega t - k_z z)}$$

Taking derivative w.r.t to time:

$$\frac{\partial}{\partial t} = j\omega, \quad \frac{\partial}{\partial t^2} = -\omega^2$$



Wave equations:

$$\nabla^2 \mathbf{E} = -k^2 \mathbf{E}$$

$$\nabla^2 \mathbf{H} = -k^2 \mathbf{H}$$

Wave number:

$$k = \omega \mu \epsilon$$

Express Laplacian in transverse and longitudinal planes: $\nabla^2 \mathbf{E} = \nabla_{\perp}^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{\partial z^2}$
 $\nabla^2 \mathbf{H} = \nabla_{\perp}^2 \mathbf{H} + \frac{\partial^2 \mathbf{H}}{\partial z^2}$

Taking derivative w.r.t to z-coordinates: $\frac{\partial}{\partial z} = -jk_z$, $\frac{\partial^2}{\partial z^2} = -k_z^2$

Wave equations of longitudinal E-field and B-field: $\nabla_{\perp}^2 E_z = -k_c^2 E_z$
 $\nabla_{\perp}^2 H_z = -k_c^2 H_z$

$k_c^2 = \gamma^2 + k^2 = k^2 + k_z^2$ where k_c is the cutoff frequency of cylindrical waveguide.

Solutions of wave equations for longitudinal E-field and B-field:

$$E_z(r, \varphi, z, t) = E_{o,z}(r, \varphi) e^{j(\omega t - k_z z)}$$

$$H_z(r, \varphi, z, t) = H_{o,z}(r, \varphi) e^{j(\omega t - k_z z)}$$

Wave equations of longitudinal E-field and B-field: in cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \varphi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_c^2 E_z = 0 \quad (\text{Eq. 1})$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial H_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 H_z}{\partial \varphi^2} + \frac{\partial^2 H_z}{\partial z^2} + k_c^2 H_z = 0 \quad (\text{Eq. 2})$$

Using “Separation of variable technique” and substitute. $E_{o,z} = R(r)\Phi(\varphi)$ in (Eq. 1), we get

$$\frac{r}{R} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} + k_c^2 r^2 = 0 \quad \longrightarrow \quad R(r) \text{ and } \Phi(\varphi) \text{ are independent.}$$

$$\frac{r}{R} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + k_c^2 r^2 = -\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} = \underset{\substack{\uparrow \\ \text{constant}}}{n^2}$$

$\frac{r}{R} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + k_c^2 r^2 = n^2$

$\frac{\partial^2 \Phi}{\partial \varphi^2} + n^2 \Phi = 0$

Longitudinal E-field and B-field

Radial dependent solution:

$$\frac{r}{R} \frac{\partial}{\partial r} \left(r \frac{\partial R}{\partial r} \right) + k_c^2 r^2 = n^2 \quad \longrightarrow \quad r^2 R'' + r R' + (k_c^2 r^2 - n^2) R = 0 \quad \text{“Bessel’s equation”}$$

$$R(r) = \underset{\substack{\nearrow \\ \text{Bessel's function} \\ \text{of the first kind}}}{A J_n(k_c r)} + \underset{\substack{\nearrow \\ \text{Bessel's function} \\ \text{of the second} \\ \text{kind}}}{B Y_n(k_c r)} \quad \text{“Bessel function”}$$

Azimuthal angle dependent solution:

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + n^2 \Phi = 0 \quad \longrightarrow \quad \Phi(\varphi) = C e^{\pm j n \varphi} \quad \text{“Single-value function”}$$

$$n = 0, 1, 2, \dots$$

Solutions for longitudinal electric and magnetic fields:

$$E_z(r, \varphi, z, t) = [A J_n(k_c r) + B Y_n(k_c r)] e^{j(\omega t - n \varphi - k_z z)}$$

$$H_z(r, \varphi, z, t) = [A' J_n(k_c r) + B' Y_n(k_c r)] e^{j(\omega t - n \varphi - k_z z)}$$

Transverse E-field and B-field

Using curl equations: $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$ and $\nabla \times \mathbf{E} = -j\omega\epsilon\mathbf{H}$

$$\frac{1}{r} \frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = j\omega\epsilon E_r$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j\omega\epsilon E_\varphi$$

$$\frac{1}{r} H_\varphi + \frac{\partial H_\varphi}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \varphi} = j\omega\epsilon E_z$$

$$\frac{1}{r} \frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = -j\omega\epsilon H_r$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -j\omega\epsilon H_\varphi$$

$$\frac{1}{r} E_\varphi + \frac{\partial E_\varphi}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \varphi} = -j\omega\epsilon H_z$$

Transverse E-field and B-field components in terms of longitudinal fields:

$$E_r = -\frac{j}{k_c^2} \left(k_z \frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r} \frac{\partial H_z}{\partial \varphi} \right)$$

$$E_\varphi = \frac{j}{k_c^2} \left(-\frac{k_z}{r} \frac{\partial E_z}{\partial \varphi} + \omega\mu \frac{\partial H_z}{\partial r} \right)$$

$$H_r = \frac{j}{k_c^2} \left(\frac{\omega\epsilon}{r} \frac{\partial E_z}{\partial \varphi} - k_z \frac{\partial H_z}{\partial r} \right)$$

$$H_\varphi = -\frac{j}{k_c^2} \left(\omega\epsilon \frac{\partial E_z}{\partial r} + \frac{k_z}{r} \frac{\partial H_z}{\partial \varphi} \right)$$

TM Modes in Cylindrical Waveguides

TM wave: $H_z = 0$

Longitudinal E-field: $E_z(r, \varphi) = [AJ_n(k_c r) + BY_n(k_c r)]e^{-jn\varphi}$

$$E_z(r, \varphi) = [AJ_n(k_c r) + BY_n(k_c r)][C \cos n\varphi + D \sin n\varphi]$$

$$E_z(r, \varphi) = E_o J_n(k_c r) \cos n\varphi$$

Transverse fields: $E_r = \frac{-jk_z}{k_c^2} \frac{\partial E_z}{\partial r} = \frac{k_z}{\omega \epsilon} H_\varphi = -\frac{jk_z}{k_c} E_o J'_n(k_c r) \cos n\varphi$

$$E_\varphi = \frac{-jk_z}{k_c^2 r} \frac{\partial E_z}{\partial \varphi} = -\frac{k_z}{\omega \epsilon} H_r = \frac{jk_z m}{k_c^2 r} E_o J_n(k_c r) \sin n\varphi$$

$$H_r = \frac{j\omega \epsilon}{k_c^2 r} \frac{\partial E_z}{\partial \varphi} = -\frac{j\omega \epsilon n}{k_c^2 r} E_o J_n(k_c r) \sin n\varphi$$

$$H_\varphi = -\frac{j\omega \epsilon}{k_c^2} \frac{\partial E_z}{\partial r} = -\frac{j\omega \epsilon}{k_c} E_o J'_n(k_c r) \cos n\varphi$$

TM_{nl} Modes

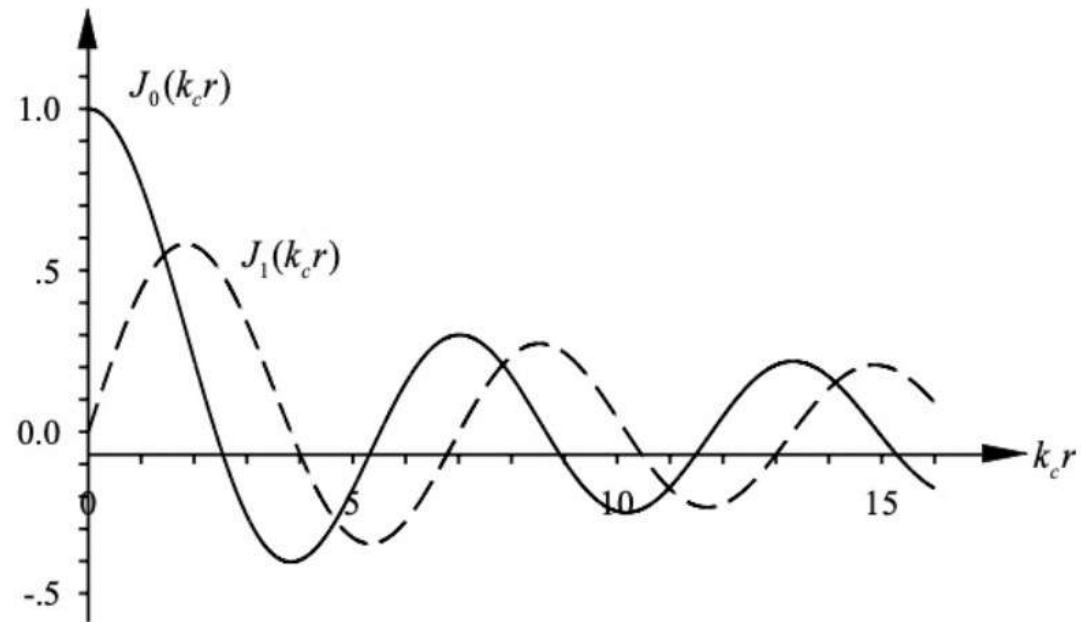
Consider boundary conditions at the cylindrical waveguide's wall at $r = a$

➤ At $r = a$, $E_z(r = a) = 0$

$$E_z(r, \varphi) = E_0 J_n(k_c r) \cos n\varphi = 0$$

$$E_0 J_n(k_c a) = 0$$

$$k_c a = \omega_c \sqrt{\mu \epsilon} a = \frac{2\pi a}{\lambda_c} = x_{nl}$$



Cutoff wavenumber: $k_c = \frac{x_{nl}}{a}$

Cutoff frequency: $\omega_c = \frac{k_c}{\sqrt{\mu \epsilon}} = \frac{x_{nl}}{a \sqrt{\mu \epsilon}}$

n	x_{n1}	x_{n2}	x_{n3}	x'_{n1}	x'_{n2}	x'_{n3}
0	2.405	5.520	8.654	3.832	7.016	10.174
1	3.832	7.016	10.173	1.841	5.331	8.536
2	5.136	8.417	11.620	3.054	6.706	9.970

The Lowest Mode: TM_{01}

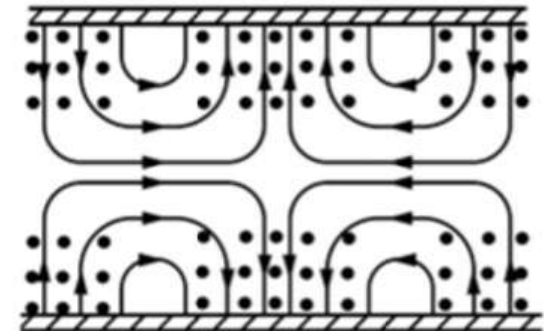
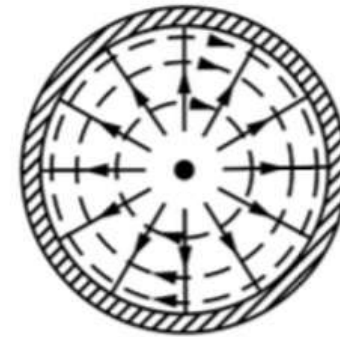
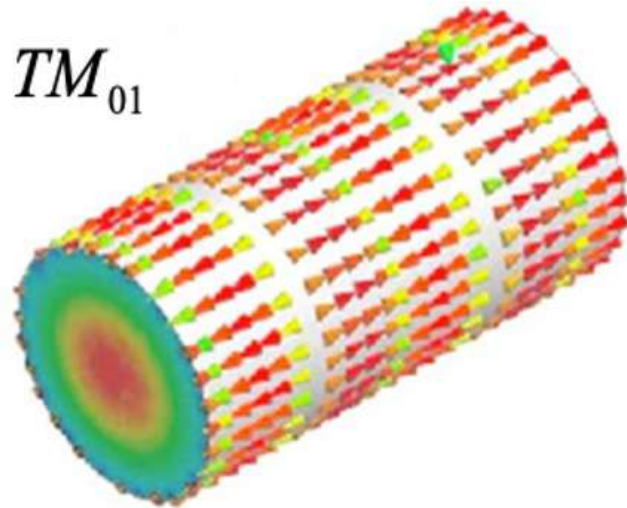
The **simplest and dominant propagation mode** is the TM_{01} mode.

$$E_r = -\frac{jk_z}{k_c} E_o J_1'(k_c r) \cos \varphi, \quad E_\varphi = \frac{jk_z}{k_c^2 r} E_o J_1(k_c r) \sin \varphi, \quad E_z = E_o J_1(k_c r) \cos \varphi$$

$$H_r = -\frac{j\omega\epsilon m}{k_c^2 r} E_o J_1(k_c r) \sin \varphi, \quad H_\varphi = -\frac{j\omega\epsilon}{k_c} E_o J_1'(k_c r) \cos \varphi, \quad H_z = 0$$

$$k_c = \frac{2\pi}{\lambda_c} = \frac{2.405}{R}$$

TM_{01}



Can we use uniform waveguide to accelerate particle?

From propagation constant in terms of cutoff frequency, we get the phase constant as

$$\gamma = jk_z = j\omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} \longrightarrow k_z = \frac{\omega\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}{c}$$

For uniform waveguide

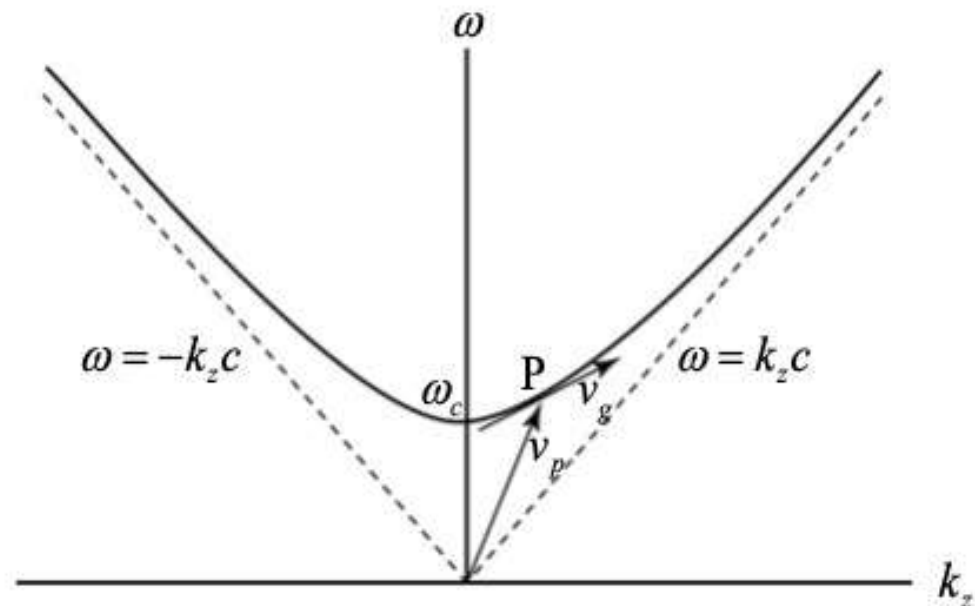
$$\omega = \sqrt{c^2 k_z^2 + \omega_c^2} \quad \text{“Dispersion relation”}$$

$$v_g = \frac{\partial\omega}{\partial k_z} = c^2 \frac{k_z}{\omega}$$

$$v_p = \frac{\omega}{k_z} = \frac{c\lambda_g}{\lambda} = \frac{\omega\lambda_g}{2\pi} > c$$

$$v_p v_g = c^2$$

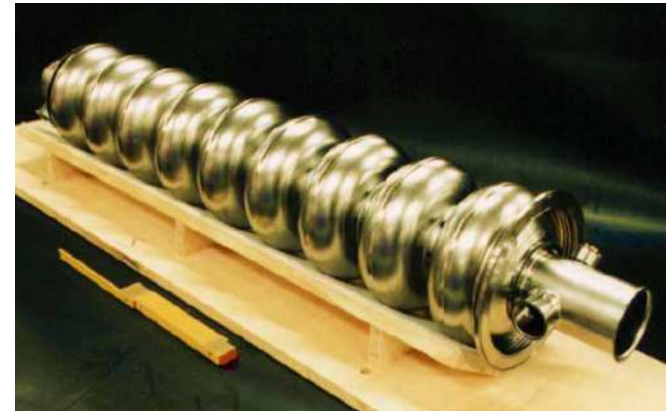
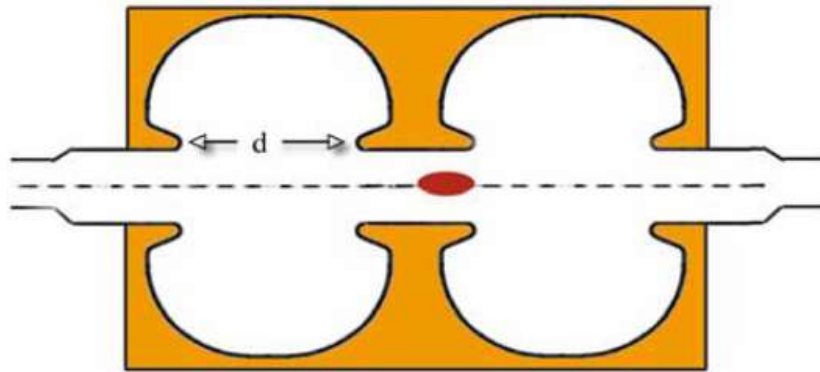
$$v_g < c \text{ and } v_p > c$$



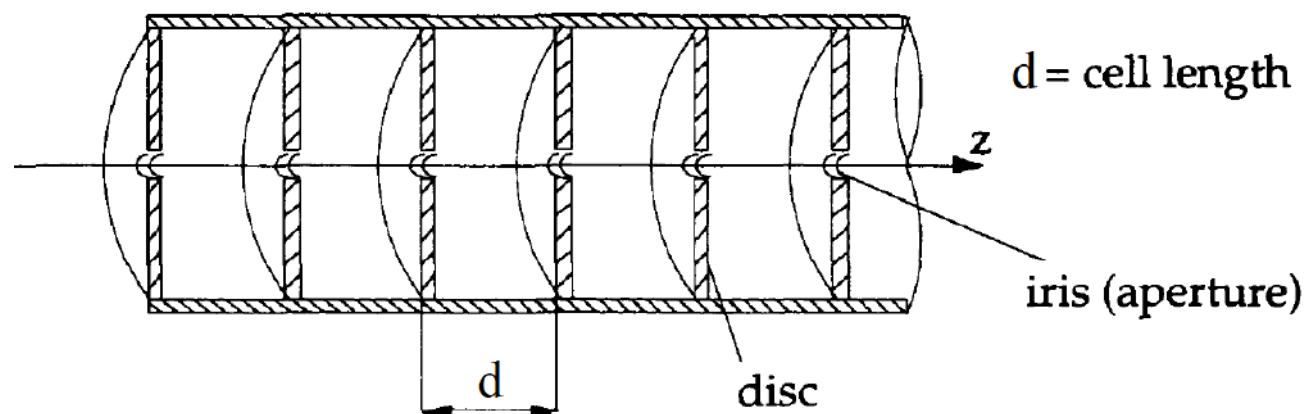
Particle and wave **“do not have the same velocity”** to maintain the synchronism.

How to solve this problem? → Periodic Accelerating Structures

➤ Periodic array of coupled resonant cavities



➤ Periodic accelerating structure: e.g. Disc-loaded structure



Outline

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Charged Particle Acceleration and RF Gaps

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Rectangular & Circular Waveguides

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Resonant RF Cavities

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Cavity optimization

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Disc-loaded Structure

6

Summary

Resonant RF Cavity

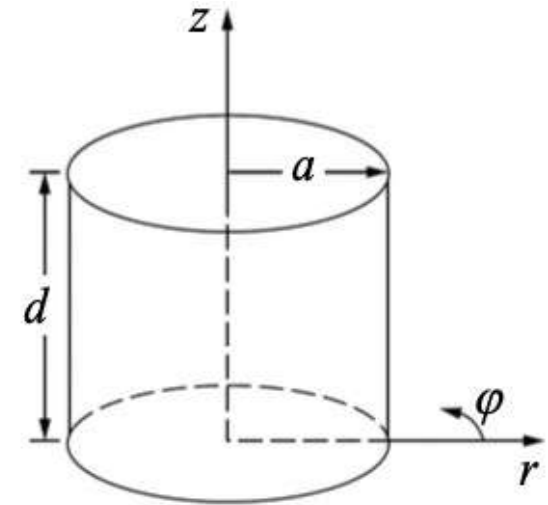
Consider superposition of longitudinal E-field of a standing wave in end-closed waveguide

$$\mathbf{E}(z, t) = \mathbf{E}_o e^{j(\omega t - kz)} + \mathbf{E}_o e^{j(\omega t + kz)} = 2\mathbf{E}_o \cos(kz) e^{j\omega t}$$

Amplitude of standing wave: $2E_o \cos(kz)$

Max. amplitude occurs at $\cos(kz) = 1$

Min. amplitude occurs at $\cos(kz) = 0 \longrightarrow kz = (p + \frac{1}{2})\pi$



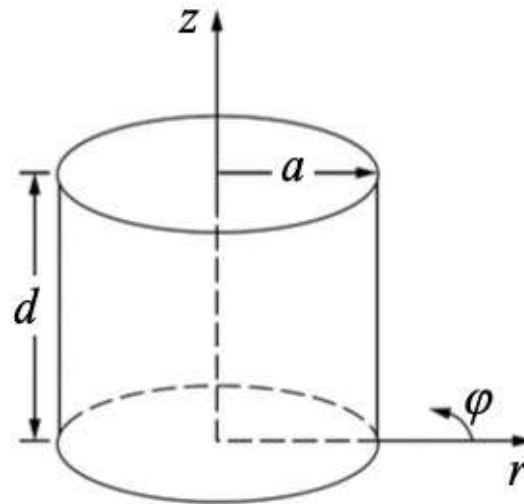
Resonant condition occurs when the length between conducting plates is a multiple of a half-guide wavelength for each mode

$$d = p \frac{\lambda_g}{2} \quad p = 0, 1, 2, 3, \dots$$

Resonant wavelength: $\frac{1}{\lambda^2} = \frac{1}{\lambda_c^2} + \frac{1}{4} \left(\frac{p}{d}\right)^2$

Cylindrical Resonant Cavity

Cylindrical resonant cavity with a width a and a length d



Longitudinal fields in TM_n waveguide mode:

$$E_z(r, \varphi) = E_o J_n(k_c r) \cos n\varphi$$

$$H_z = 0$$

Superposition of longitudinal E-field: $E_z(r, \varphi, z, t) = (Ae^{-jk_z z} + Be^{jk_z z})J_n(k_c r) \cos n\varphi e^{j\omega t}$

Boundary condition: $E_z|_s = 0$

Consider boundary conditions at $z = 0$ and $z = d$

$$E_z = (Ae^{-jk_z z} + Be^{jk_z z})J_n(k_c r) \cos n\varphi$$

We get $A = B$ and $k_z d = p\pi$

$$p = 0, 1, 2, 3, \dots$$

Longitudinal E-field:

$$\begin{aligned} E_z &= B(e^{-jk_z z} + e^{jk_z z})J_n(k_c r) \cos n\varphi \\ &= 2BJ_n(k_c r) \cos n\varphi \cos \frac{p\pi z}{d} \end{aligned}$$

E-field and B-field of TM_{np} Cavity Modes

$$E_z = 2BJ_n(k_cr) \cos n\varphi \cos \frac{p\pi z}{d}$$

$$H_z = 0$$

$$E_r = -\frac{j}{k_c^2} \left(k_z \frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r} \frac{\partial H_z}{\partial \varphi} \right) = -\frac{2B}{k_c} \left(\frac{p\pi}{d} \right) J'_n(k_cr) \cos n\varphi \sin \frac{p\pi z}{d}$$

$$H_r = \frac{j}{k_c^2} \left(\frac{\omega\varepsilon}{r} \frac{\partial E_z}{\partial \varphi} - k_z \frac{\partial H_z}{\partial r} \right) = -j \frac{2B\omega\varepsilon n}{k_c^2 r} J_n(k_cr) \sin n\varphi \cos \frac{p\pi z}{d}$$

$$E_\varphi = \frac{j}{k_c^2} \left(-\frac{k_z}{r} \frac{\partial E_z}{\partial \varphi} + \omega\mu \frac{\partial H_z}{\partial r} \right) = \frac{2Bn}{k_c^2 r} \frac{p\pi}{l} J_n(k_cr) \sin n\varphi \sin \frac{p\pi z}{d}$$

$$H_\varphi = -\frac{j}{k_c^2} \left(\omega\varepsilon \frac{\partial E_z}{\partial r} + \frac{k_z}{r} \frac{\partial H_z}{\partial \varphi} \right) = -j \frac{2B\omega\varepsilon}{k_c} J'_n(k_cr) \cos n\varphi \cos \frac{p\pi z}{d}$$

Cutoff wavenumber: $k_c = \frac{x_{nl}}{a} = \sqrt{\frac{\omega^2 \mu \epsilon}{c^2} + k_z^2} = \sqrt{\frac{\omega^2 \mu \epsilon}{c^2} + \left(\frac{p\pi}{d}\right)^2}$

Phase constant: $k_z = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 + k_c^2} = \frac{p\pi}{d}$

Resonant wavelength: $\lambda = \left[\left(\frac{x_{nl}}{2\pi a}\right)^2 + \left(\frac{p}{2d}\right)^2 \right]^{-1/2}$

Resonant frequency: $f = \frac{c}{2\pi\sqrt{\mu\epsilon}} \sqrt{k_c^2 + k_z^2} = \frac{c}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x_{nl}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$

$$n = 0, 1, 2, \dots$$

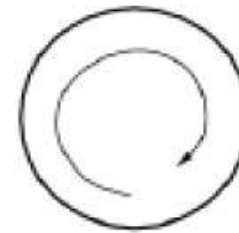
$$l = 1, 2, 3, \dots$$

$$p = 0, 1, 2, 3, \dots$$

The modes are described by 3 indices:

$$n = 0, 1, 2, \dots$$

No. of variation of field of the azimuthal variations around theta



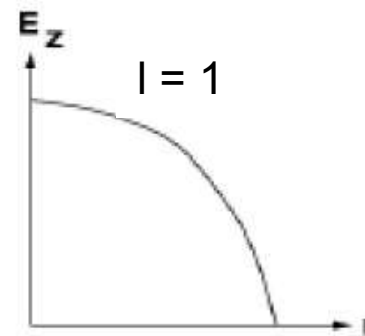
$n = 0$



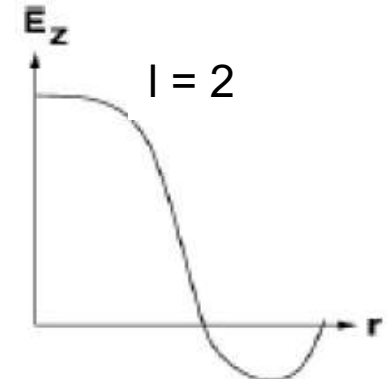
$n = 1$

$$l = 1, 2, 3, \dots$$

No. of nulls in E_z along the radial direction



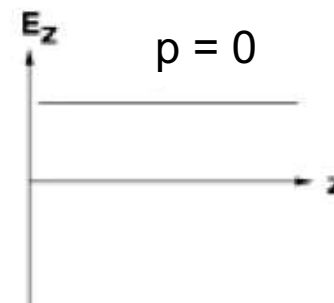
$l = 1$



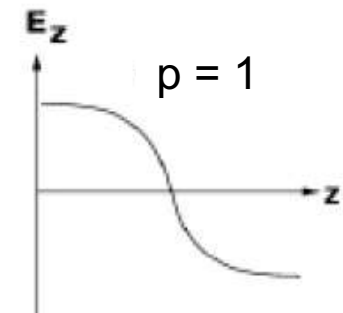
$l = 2$

$$p = 0, 1, 2, 3, \dots$$

No. of nodes of E_z along z-axis



$p = 0$



$p = 1$

The Lowest Cavity Mode: TM_{010} pillbox cavity

➤ TM_{010} : $n = 0, l = 1$ and $p = 0$

$$x_{01} = 2.405$$

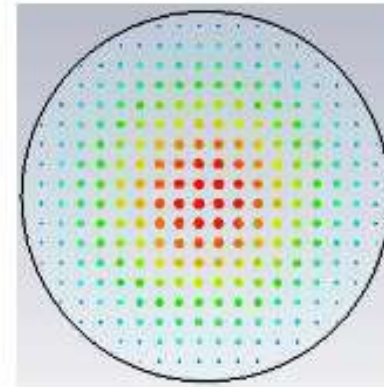
$$k_c = 2.405/a$$

$$E_z = E_0 J_0(k_c r) \cos \omega t$$

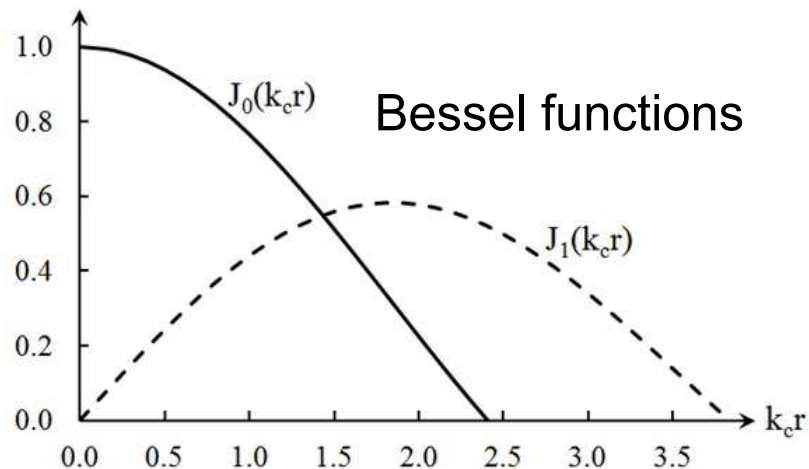
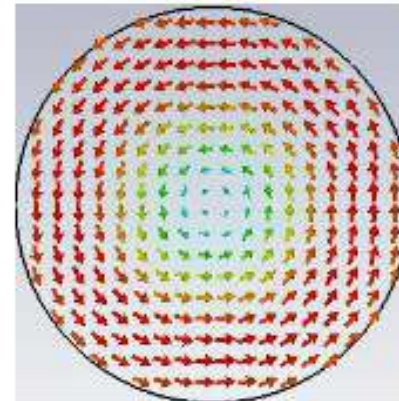
$$H_\phi = j E_0 \frac{\omega \epsilon}{k_c} J_1(k_c r) \sin \omega t$$

$$E_r = E_\theta = H_r = H_z = 0$$

E-field



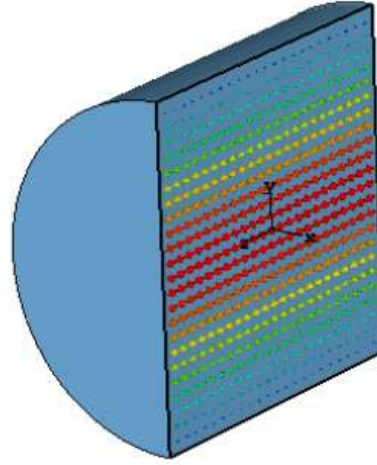
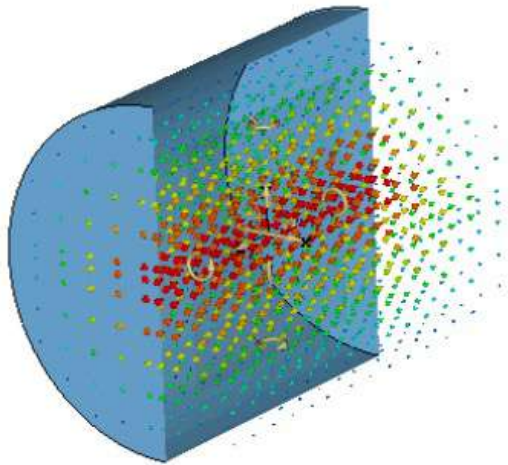
H-field



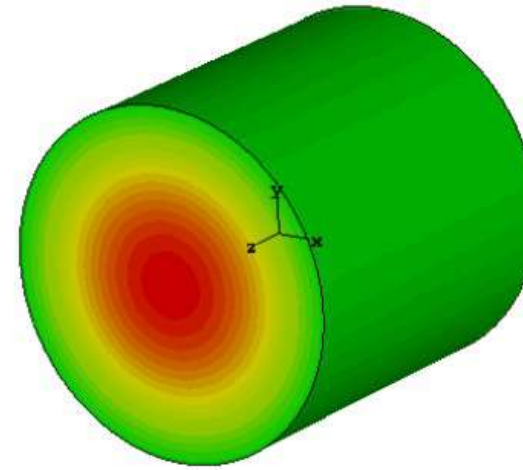
Courtesy: Dr. Sang-hoon Kim, ISBA23

3D Simulation with CST MWS: TM_{010} pillbox cavity

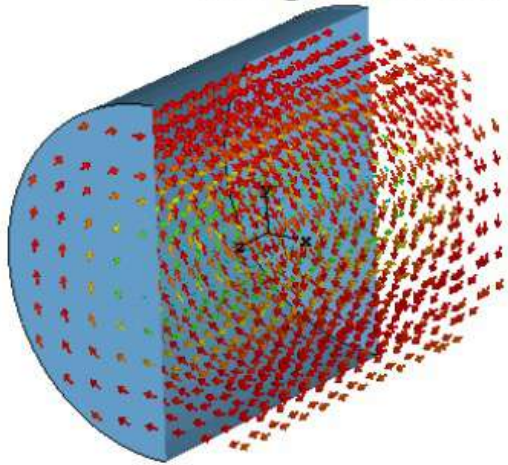
Electric Field



Surface Electric Field

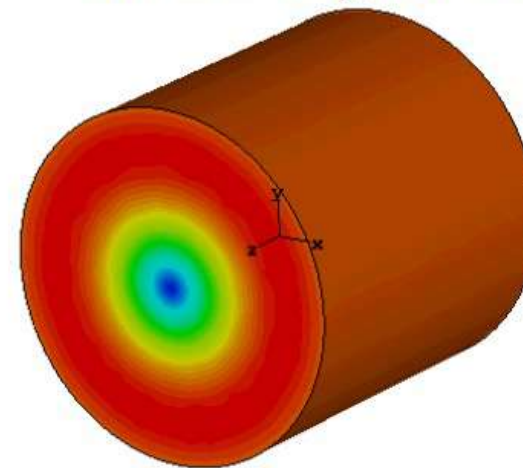


Magnetic Field



- Peak surface electric field at end plates
- Peak surface magnetic field at outer cylindrical surface

Surface Magnetic Field



Modes in a Pill Box Cavity

- **TM₀₁₀**
 - Electric field is purely longitudinal
 - Electric and magnetic fields have no angular dependence
 - Frequency depends only on radius, independent of length
- **TM_{0lp}**
 - Monopole modes that can couple to the and exchange energy
- **TM_{1lp}**
 - Dipole modes that can deflect the beam
- **TE modes**
 - No longitudinal electric field
 - Cannot couple to the beam
 - Can deflect the beam

Cavity Parameters (1)

Fundamental frequency: $f_0 = \frac{\omega}{2\pi} = \frac{ck_c}{2\pi\sqrt{\mu\epsilon}} = \frac{x_{nl}c}{2\pi\sqrt{\mu\epsilon}a}$

Radius of resonant cavity: $a = \frac{x_{nl}c}{2\pi\sqrt{\mu\epsilon}f_0}$

Stored energy: $U = \frac{\epsilon}{2} \int_V |E|^2 dV = \frac{\mu}{2} \int_V |H|^2 dV$

Skin depth: $\delta = \sqrt{\frac{2}{\mu\omega\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}}$

Cavity power wall loss: $\frac{dP_w}{dS} = \frac{R_s}{2} |\mathbf{H}_{\parallel}|^2$

Cavity Parameters (2)

- **Q-factor** (cavity quality factor): determines **maximum energy** the cavity can fill to for a given input power

$$Q_0 = \frac{2\pi}{T_{rf}} \frac{U}{P_w} = \omega \frac{U}{P_w}$$

- **r/Q (R-upon-Q)**: surface-independent parameter as a measure of **acceleration** efficiency

$$\frac{R}{Q_0} = \frac{V_c^2}{2\omega U}$$

- **Geometric factor (G)**: surface-independent parameter as a measure of **power** efficiency:

$$G = Q_0 R = \frac{\omega U}{\frac{1}{2} \int_s |\vec{H}|^2 dS}$$

Figure of Merit

Accelerator operation requirement

$$P_c = \frac{V_c^2}{R} = \frac{V_c^2}{\frac{R_s}{Q_0} \cdot Q_0} = \frac{V_c^2}{\frac{R_s}{Q_0} \cdot G} \cdot R_s$$

Cavity material

Cavity geometry

Equivalent Circuit of Resonant Cavity

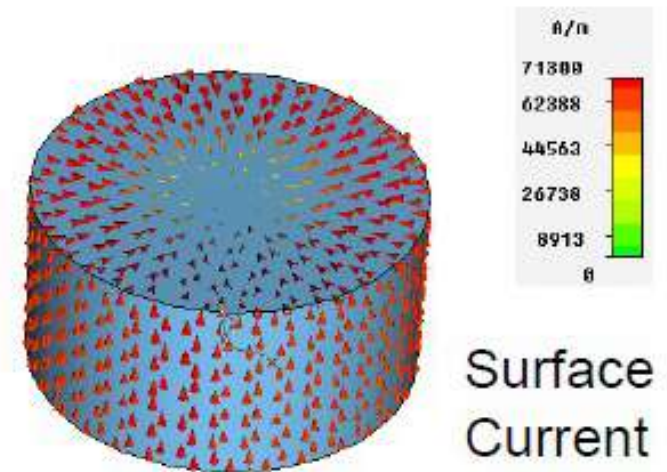
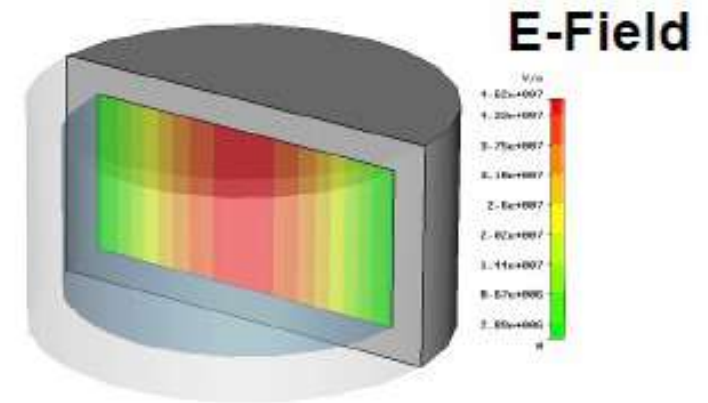
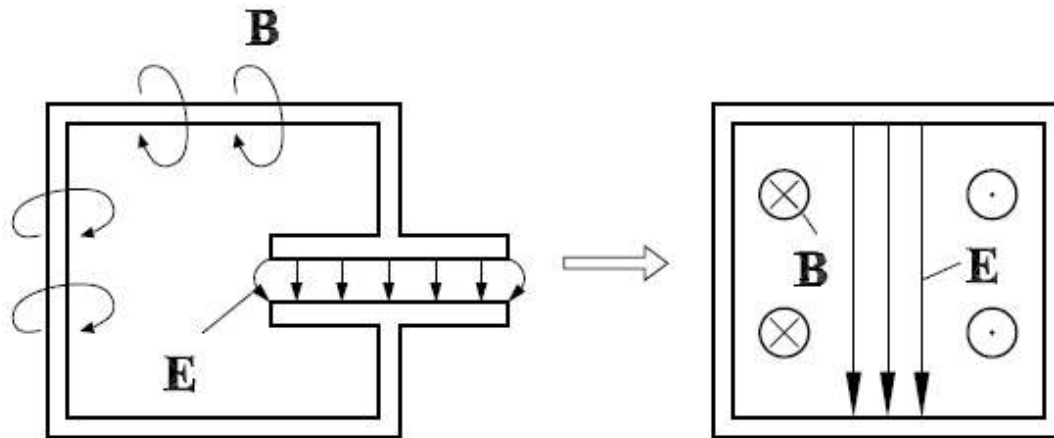
For simplicity, consider TM_{010} pill-box cavity

- E-field is contained between two metal plates.

→ **Capacitance C**

- Surface current circulates around the outside of the cavity and induces the B-field.

→ **Inductance L**

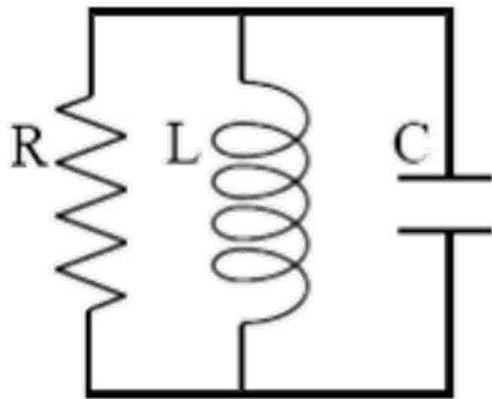
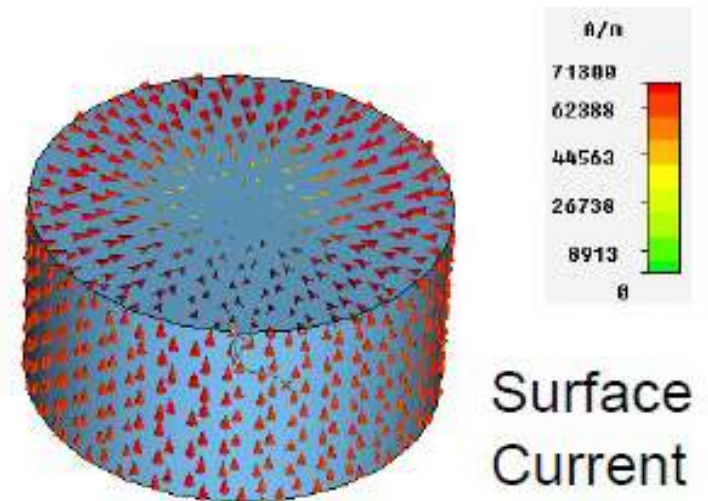


Courtesy: G. Burt, Introduction to RF for Accelerators, Lancaster University

- For cavity with finite conductivity wall, the surface current flows in the **skin depth** causing ohmic heating and power loss P_c .

→ **Cavity shunt impedance R**

- In this model, we assume the voltage across the resistor is the cavity voltage V_c .



$$\omega = \frac{1}{\sqrt{LC}}$$

$$P_c = \frac{V_c^2}{2R}$$

$$U = \frac{CV_c^2}{2}$$

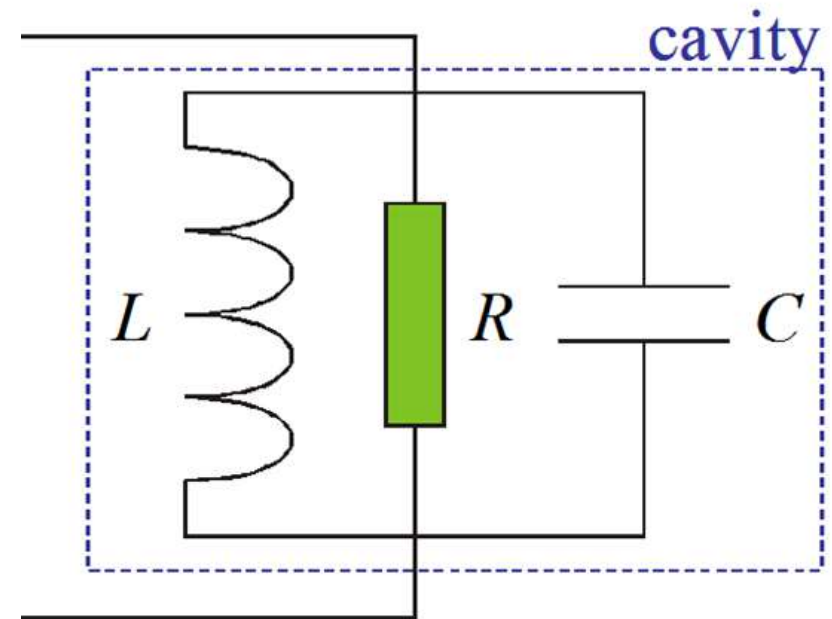
Courtesy: G. Burt, Introduction to RF for Accelerators, Lancaster University

- Simple equivalent circuit can be used to calculate the cavity parameters

$$V_c = V_0 T$$

$$Q_0 = \frac{\omega U}{P_c} = \sqrt{\frac{C}{L}} R$$

$$\frac{R}{Q_0} = \frac{V_c^2}{2\omega U} = \frac{1}{\omega C} = \sqrt{\frac{L}{C}}$$



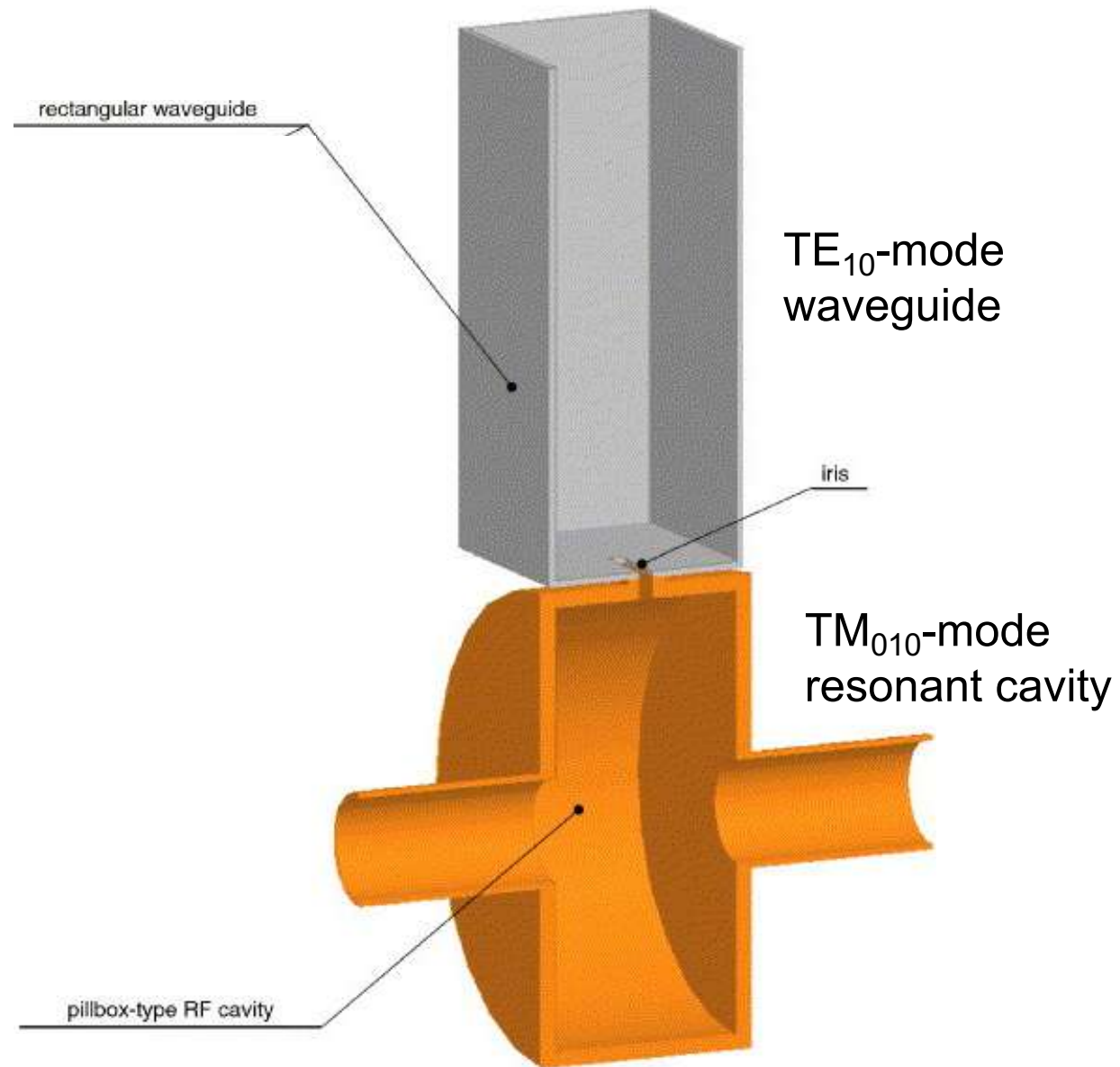
Courtesy: G. Burt, Introduction to RF for Accelerators, Lancaster University]

Practical Accelerating Cavity

- RF coupling from waveguide to cavity

TE₁₀-mode waveguide → TM₀₁₀-mode cavity

- The beam enters and exits the cavity through **beam apertures**.
- Cavity optimization



Outline

1

Charged Particle Acceleration and RF Gaps

2

Rectangular & Circular Waveguides

3

Resonant RF Cavities

4

Cavity optimization

5

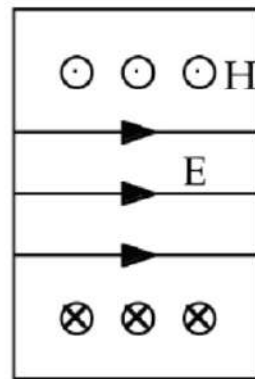
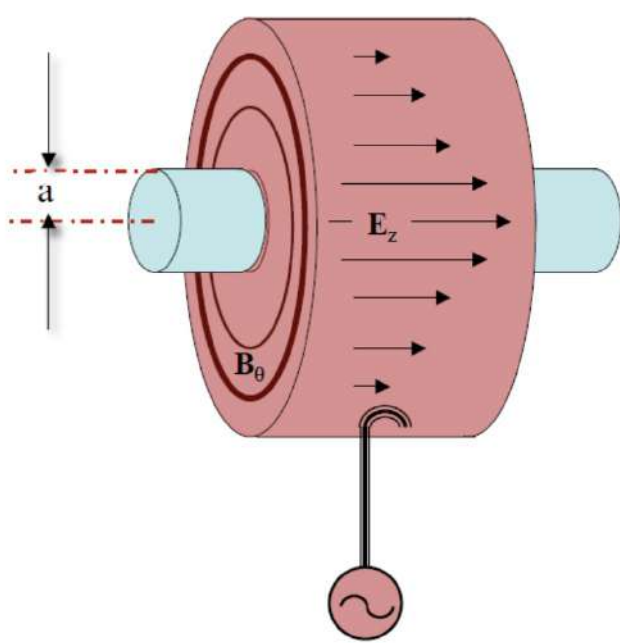
Disc-loaded Structure

6

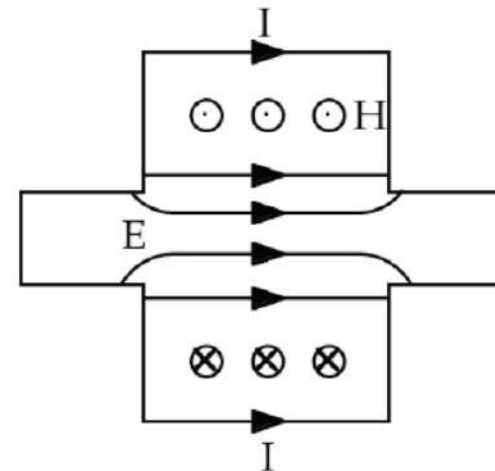
Summary

Introduction of Beam Apertures

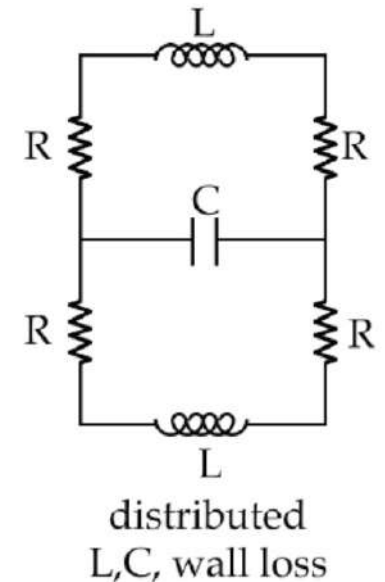
- Beam tubes add length w/o acceleration
 - Makes field modes in cavity more complicate
- Peak E-field is no longer on axis
- Resonant frequency is more sensitive to cavity dimensions
 - Mechanical tuning & detuning are needed.



pillbox

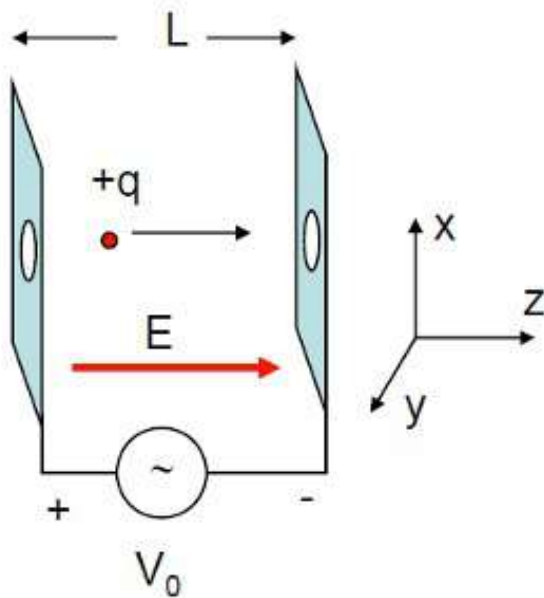


pillbox + beam pipes



Acceleration between Infinite Parallel Plates (1)

Consider infinite parallel plates separated by a distance L with time-varying voltage applied and assume uniform E-field in the gap (**neglect holes**).



Accelerating field: $E_z = E_z(t) = E_0 \cos(\omega t + \phi)$

ϕ is the synchronous phase of the field relative to the crest.

Energy gain in the accelerating gap:

$$\Delta W = q \int_{-L/2}^{L/2} E_z dz = q E_0 \int_{-L/2}^{L/2} \cos(\omega t(z) + \phi) dz$$

Assume the velocity change through the gap is small (**$v \sim \text{constant}$**):

$$\omega t \approx \omega \frac{z}{v} = \frac{2\pi c}{\lambda} \frac{z}{c\beta} = \frac{2\pi z}{\beta \lambda}$$

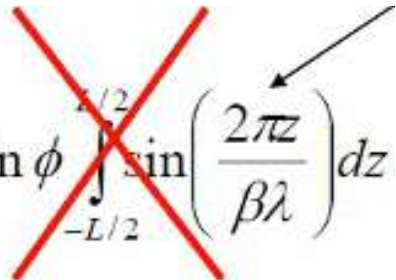
Acceleration between Infinite Parallel Plates (2)

Energy gain:

$$\Delta W = qE_0 \int_{-L/2}^{L/2} (\cos \omega t \cos \phi - \sin \omega t \sin \phi) dz$$

$$= qE_0 \cos \phi \int_{-L/2}^{L/2} \cos\left(\frac{2\pi z}{\beta\lambda}\right) dz - qE_0 \sin \phi \int_{-L/2}^{L/2} \sin\left(\frac{2\pi z}{\beta\lambda}\right) dz$$

Odd-function of z



$$= qE_0 \cos \phi \frac{\beta\lambda}{2\pi} \left[\sin \frac{2\pi z}{\beta\lambda} \right]_{-L/2}^{L/2}$$

$$= qE_0 \frac{\sin(\pi L / \beta\lambda)}{\pi L / \beta\lambda} L \cos \phi$$

Acceleration between Infinite Parallel Plates (3)

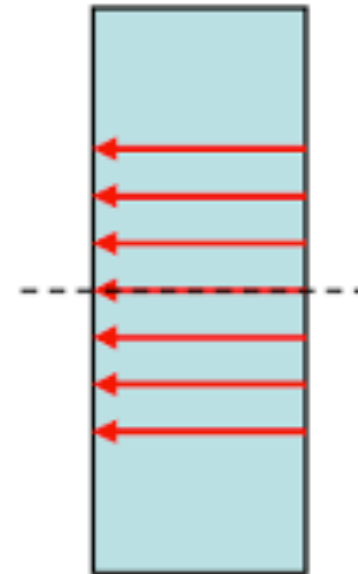
Energy gain: $\Delta W = qE_0 \frac{\sin(\pi L / \beta\lambda)}{\pi L / \beta\lambda} L \cos \phi$

Transit time factor: a factor that takes into account the time variation of the field during particle transit through the gap

$$T = \frac{\sin(\pi L / \beta\lambda)}{\pi L / \beta\lambda}$$

Energy gain:

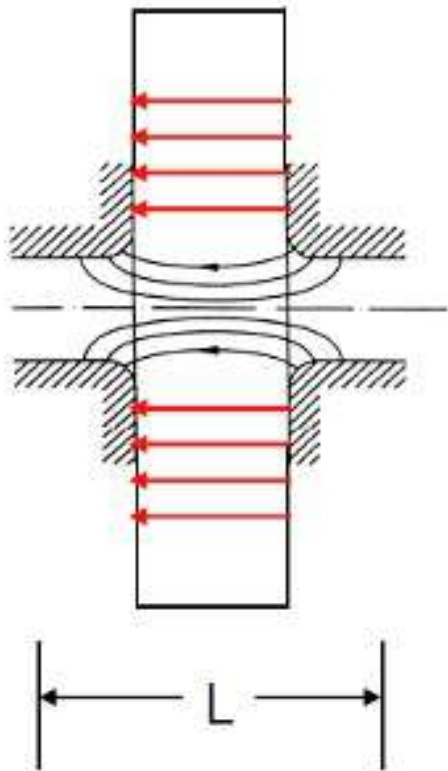
$$\Delta W = qV_0 T \cos \phi$$



We get energy gain for an on-axis particle acceleration in a pill box cavity neglecting the beam holes.

Effects of Beam Apertures: Energy gain and transit time factor (1)

Taking into account of beam apertures, the accelerating field depends on ***r*** and ***z***:



Accelerating field: $E_z = E_z(r, z, t) = E(r, z) \cos(\omega t + \phi)$

Energy gain:

$$\Delta W = q \int_{-L/2}^{L/2} E_z dz = q \int_{-L/2}^{L/2} E(0, z) \cos(\omega t(z) + \phi) dz$$

$$\Delta W = q \int_{-L/2}^{L/2} E(0, z) (\cos \omega t \cos \phi - \sin \omega t \sin \phi) dz$$

Choose the origin at the center of the gap, defined as

$$\int_{-L/2}^{L/2} E(0, z) \sin \omega t(z) dz = 0$$

Effects of Beam Apertures: Energy gain and transit time factor (2)

Energy gain:

$$\Delta W = qV_0 T \cos \phi = q \left[\int_{-L/2}^{L/2} E(0, z) dz \right] \left[\frac{\int_{-L/2}^{L/2} E(0, z) \cos \omega t dz}{\int_{-L/2}^{L/2} E(0, z) dz} \right] \cos \phi$$

Transit time factor **in general form** can be defined as

$$T = \frac{\int_{-L/2}^{L/2} E(0, z) \cos \omega t dz}{\int_{-L/2}^{L/2} E(0, z) dz}$$

Effects of Beam Apertures: Energy gain and transit time factor (3)

Assume the velocity change through the gap is small ($v \sim \text{constant}$):

$$\omega t \approx \omega \frac{z}{v} = \frac{2\pi z}{\beta\lambda} = kz$$

Wave number: $k = \frac{2\pi}{\beta\lambda}$

Transit time factor for an on-axis particle acceleration:

$$T(k) \equiv T(0, k) = \frac{1}{V_0} \int_{-L/2}^{L/2} E(0, z) \cos(kz) dz$$

→ The same as the case w/o holes.

Maximum accelerating voltage: $V_0 = \int_{-L/2}^{L/2} E(0, z) dz$

Effects of Beam Apertures: Energy gain and transit time factor (4)

Transit time factor for **an off-axis particle acceleration**:

$$T(r, k) = \frac{1}{V_0} \int_{-L/2}^{L/2} E(r, z) \cos(kz) dz \quad \Longrightarrow \quad T(r, k) = T(k) I_0(Kr)$$

I_0 is the modified Bessel function of order zero.

$K = \frac{2\pi}{\gamma\beta\lambda}$ is the wave number in the case of off-axis acceleration.

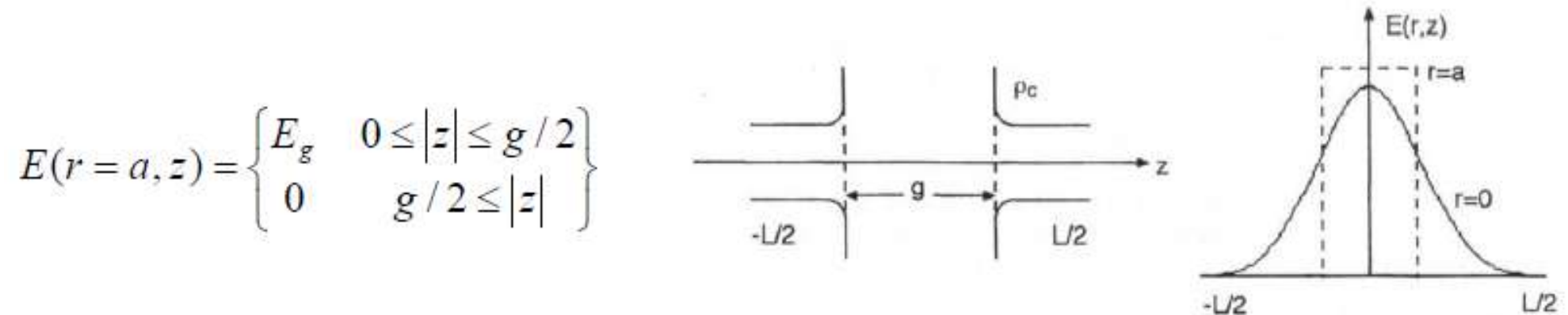
Energy gain in case of off-axis particle acceleration (**r-dependent acceleration**):

$$\Delta W = qV_0 T(k) I_0(Kr) \cos \phi$$

Compare to the case of on-axis acceleration: $\Delta W = qV_0 T \cos \phi$

Effects of Beam Apertures: Energy gain and transit time factor (5)

Assume the accelerating field at the drift-tube bore radius ($r = a$) is constant within the gap and zero outside the gap.



Using the definition of r - and z -dependent transit-time factor :

$$T(r, k) = \frac{1}{V_0} \int_{-L/2}^{L/2} E(r, z) \cos(kz) dz$$

$$V_0 T(k) = \frac{E_g g}{I_0(Ka)} \frac{\sin(kg/2)}{kg/2} = \frac{E_g g}{J_0(2\pi a / \lambda)}$$

Effects of Beam Apertures: Energy gain and transit time factor (6)

Transit-time factor for the RF gap with drift tube:

$$T(r, k) = T(k)I_0(Kr) = I_0(Kr) \frac{J_0(2\pi a / \lambda)}{I_0(Ka)} \frac{\sin(\pi g / \beta \lambda)}{\pi g / \beta \lambda}$$

Energy gain:

$$\Delta W = qV_0 T(k)I_0(Kr) \cos \phi = qV_0 T(r, k) \cos \phi$$

J_0 is the Bessel function of order zero.

I_0 is the modified Bessel function of order zero.

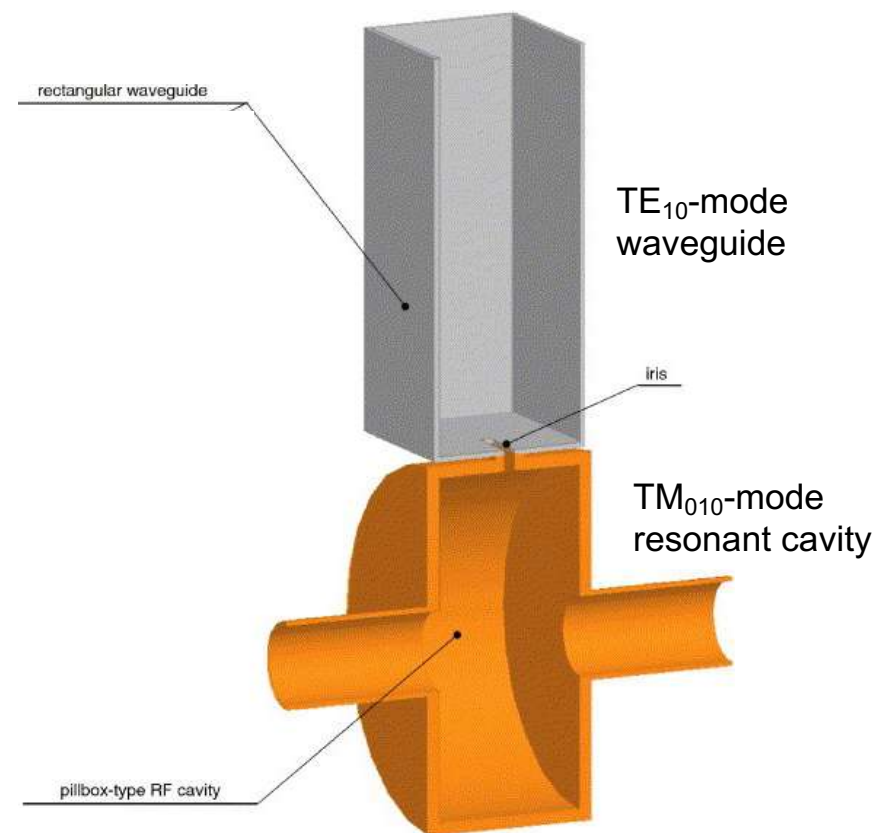
$K = \frac{2\pi}{\gamma\beta\lambda}$ is the wave number in the case of off-axis acceleration.

Effects of Beam Apertures: Conclusion

Beam apertures allow particles to pass through accelerating cavity, while influencing the cavity's fields and accelerating performance.

Main effects:

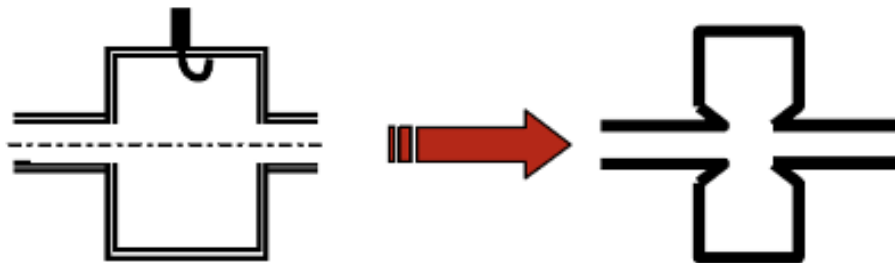
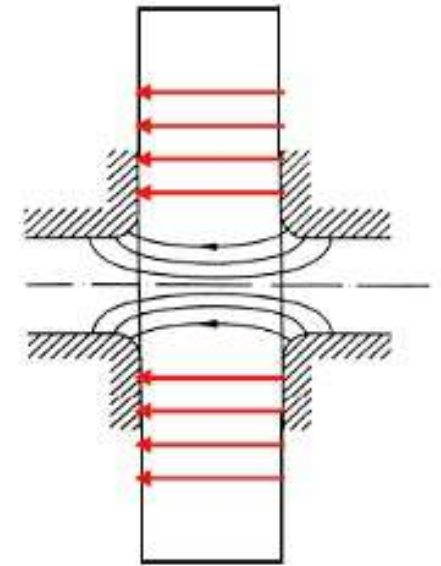
- Shift the resonant frequency
- Perturb the E-field and B-field distributions
- Modify the axial accelerating field profile and transit time factor.
- Affect the effective accelerating voltage and particle energy gain.
- Larger apertures may increase RF field leakage and reduce accelerating efficiency.
- Potentially reduce the quality factor (Q)



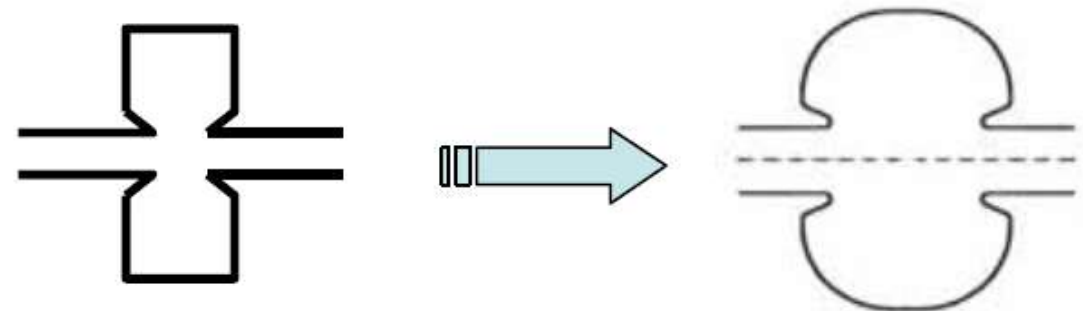
Cavity optimization is needed !

Cavity Optimization: Normal conducting cavity (1)

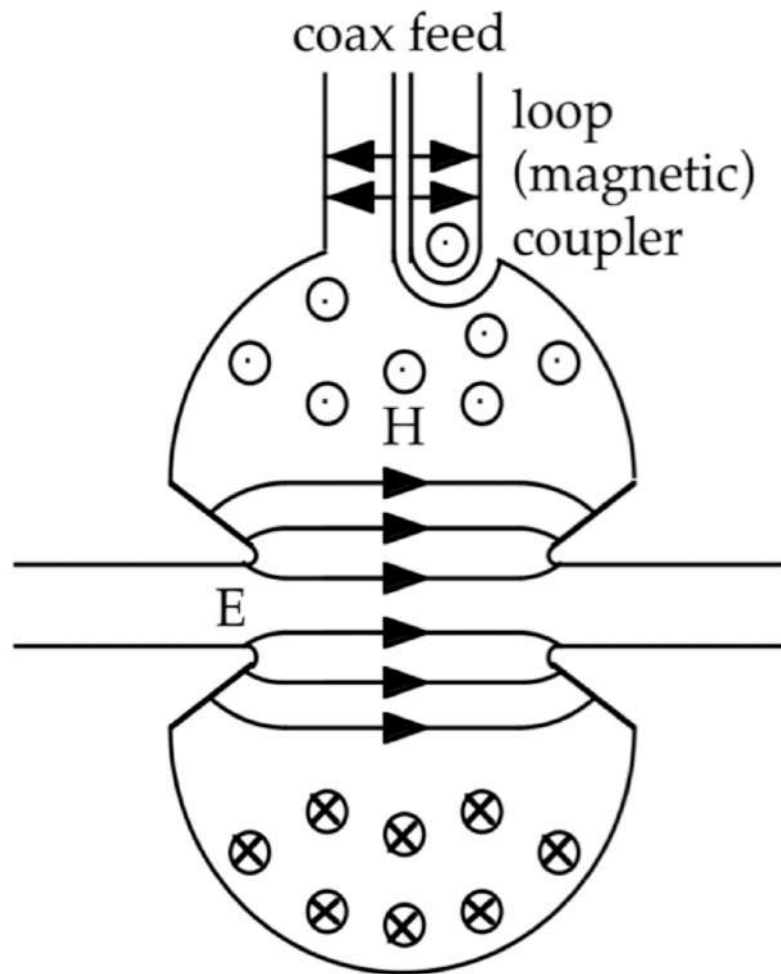
1. Adding beam apertures causes E_z at the nose distort because the condensation of E-field lines.
2. Modify the nose cone of pill-box cavity to be a **re-entrant cavity** for maximizing transit-time factor and shunt impedance.



3. Modify the wall of the pill-box cavity (which corresponds to B-field) to increase the Q-factor and reduces the power loss.



Cavity Optimization: Normal conducting cavity (2)



"nose-cone" copper cavity

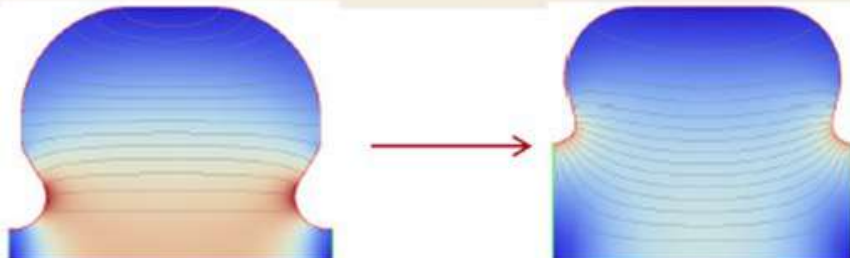
$$\frac{R}{Q} = \sqrt{\frac{L}{C}}$$

- **Goal:** To maximize L and minimize C , while keeping the optimum interaction length (maximum T) and maximum Q .
- **Nose-cone (re-entrant) cavity:**
 - Increases the volume occupied by the B-field and the surface area carrying the surface current.
 - Surface area in the capacitive region (nose tips) decreases.
 - Wall-power dissipation and electric field strength at the nose-tips limit achievable gradient.

Courtesy: John Byrd, RF cavities and linacs, Lawrence Berkeley National Laboratory

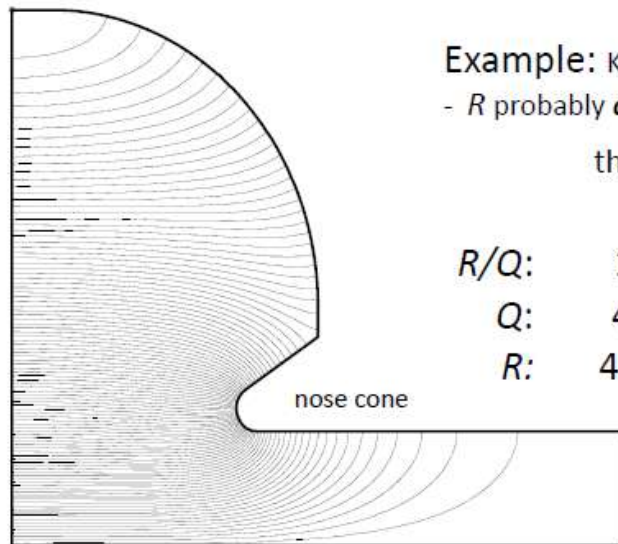
Cavity Optimization: Normal conducting cavity (3)

The shunt impedance is strongly dependent on aperture.



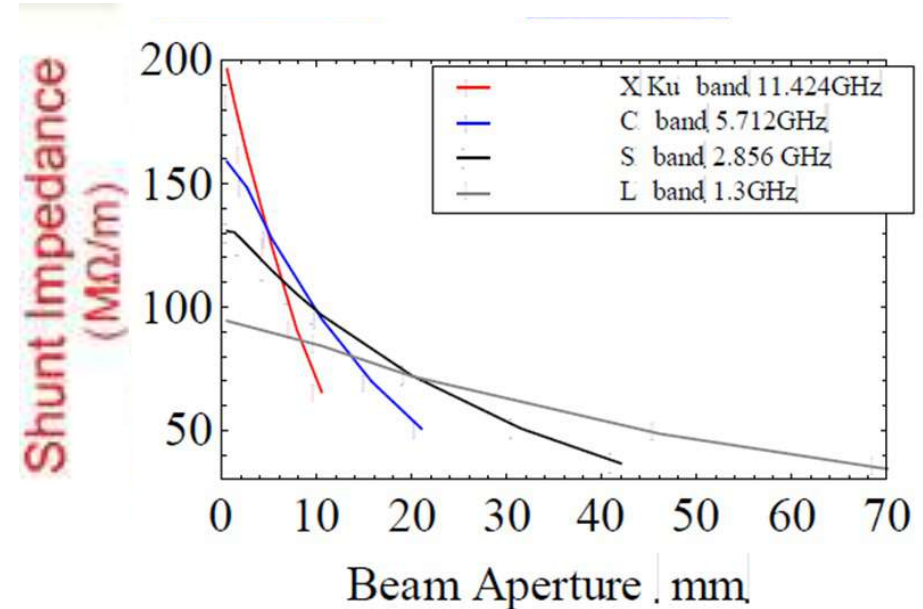
**Higher frequencies
need smaller apertures !**

Nose cone example Freq. = 500.003



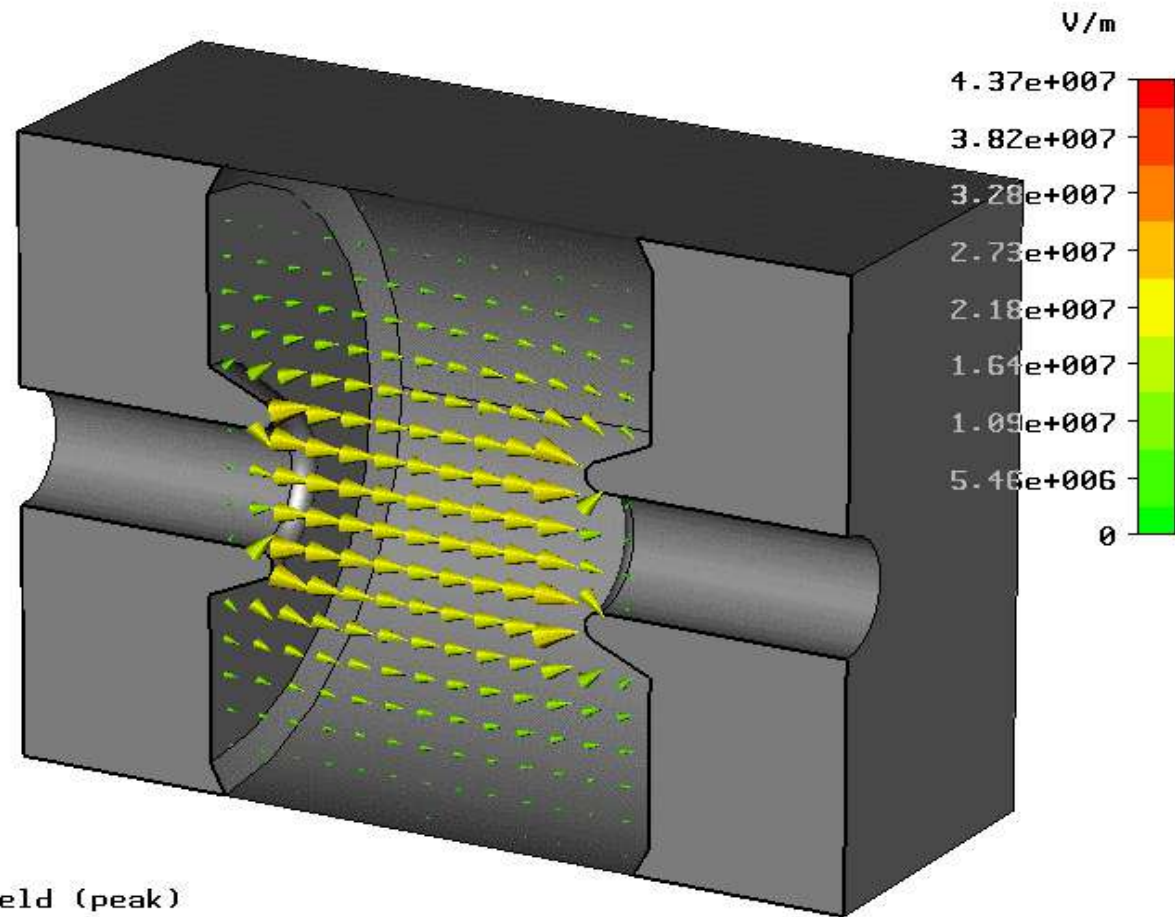
Example: KEK photon factory 500 MHz
- *R* probably **as good as it gets** -

	this cavity	optimized pillbox
R/Q :	111 Ω	107.5 Ω
Q :	44270	41630
R :	4.9 M Ω	4.47 M Ω



Cavity Optimization: Normal conducting cavity (3)

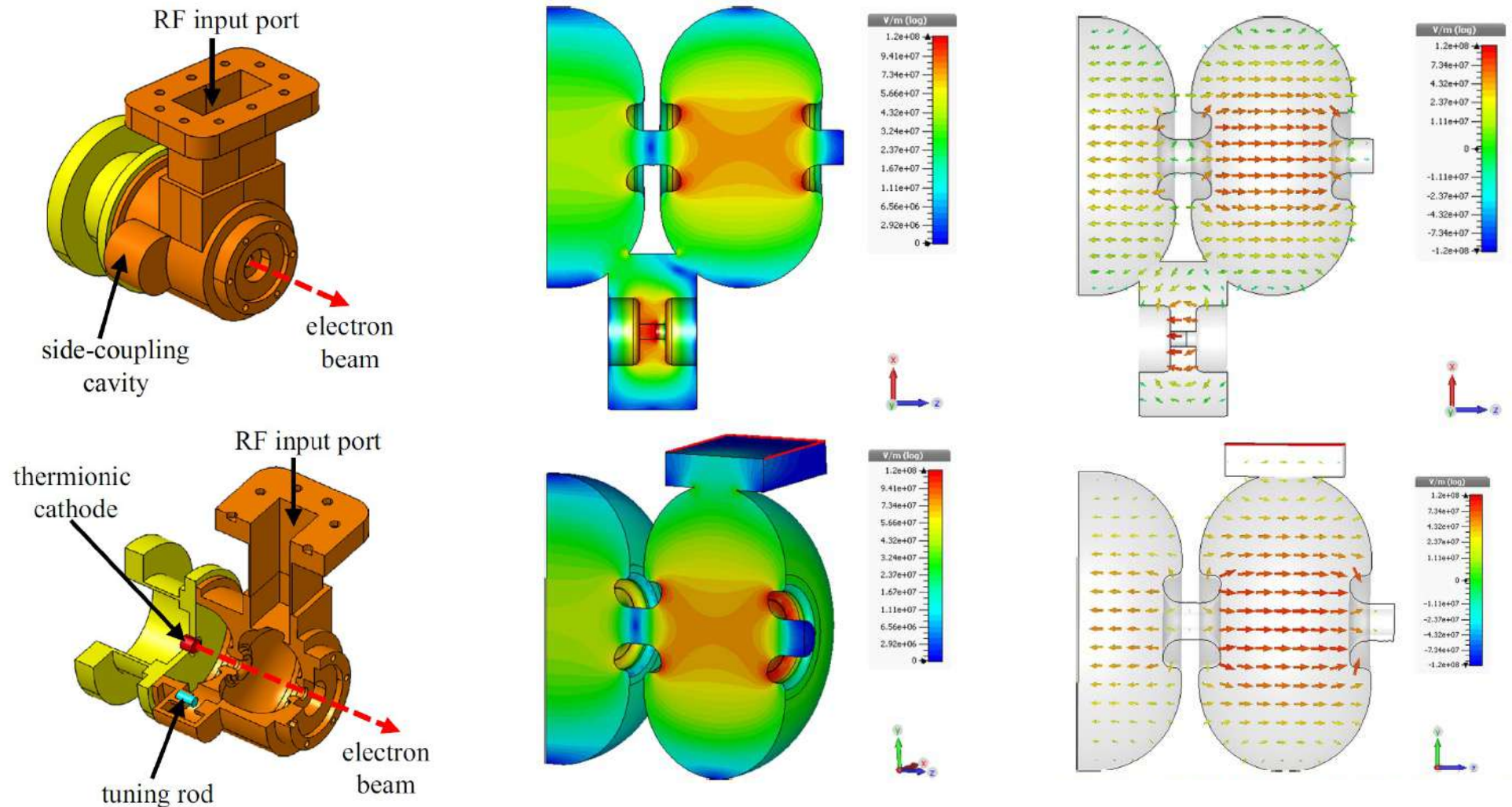
CST Microwave Studio model of nose-cone (re-entrant) normal conducting cavity



Type = E-Field (peak)
Monitor = Mode 1
Plane at z = 0
Frequency = 1.28858
Phase = 0 degrees
Maximum-2d = 4.36809e+007 V/m at -41.0665 / 19.37 / -2.80202e-015

Cavity Optimization: Normal conducting cavity (4)

Example: Optimization of beam apertures, nose-cone, waveguide port, side-coupling cavity.

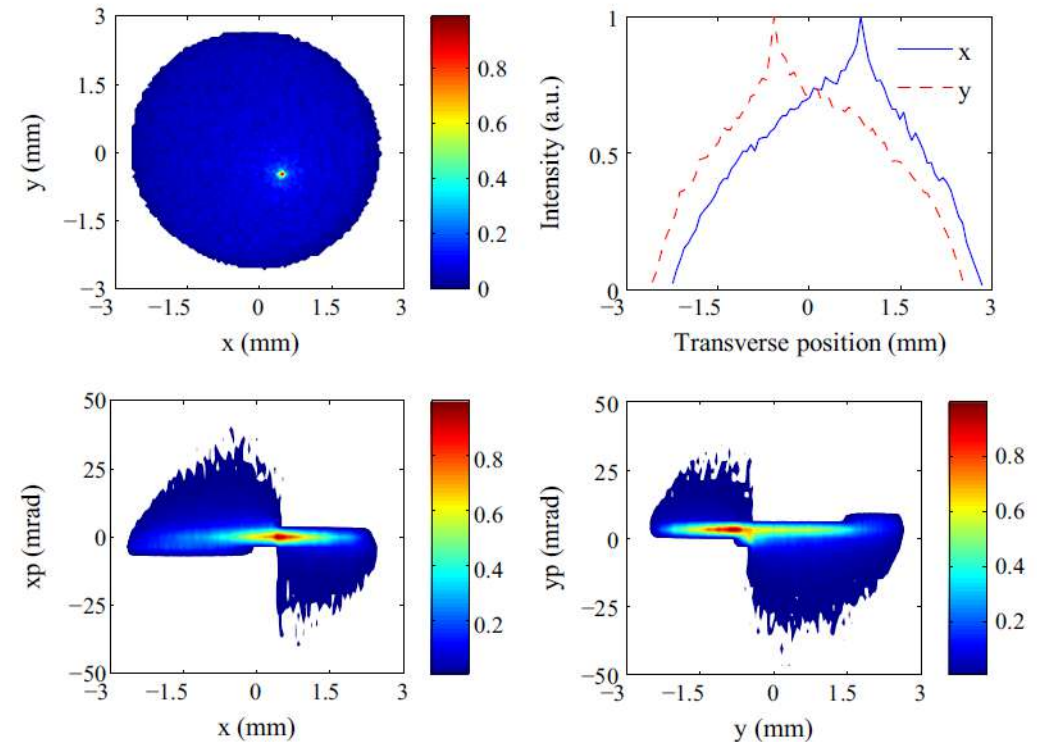
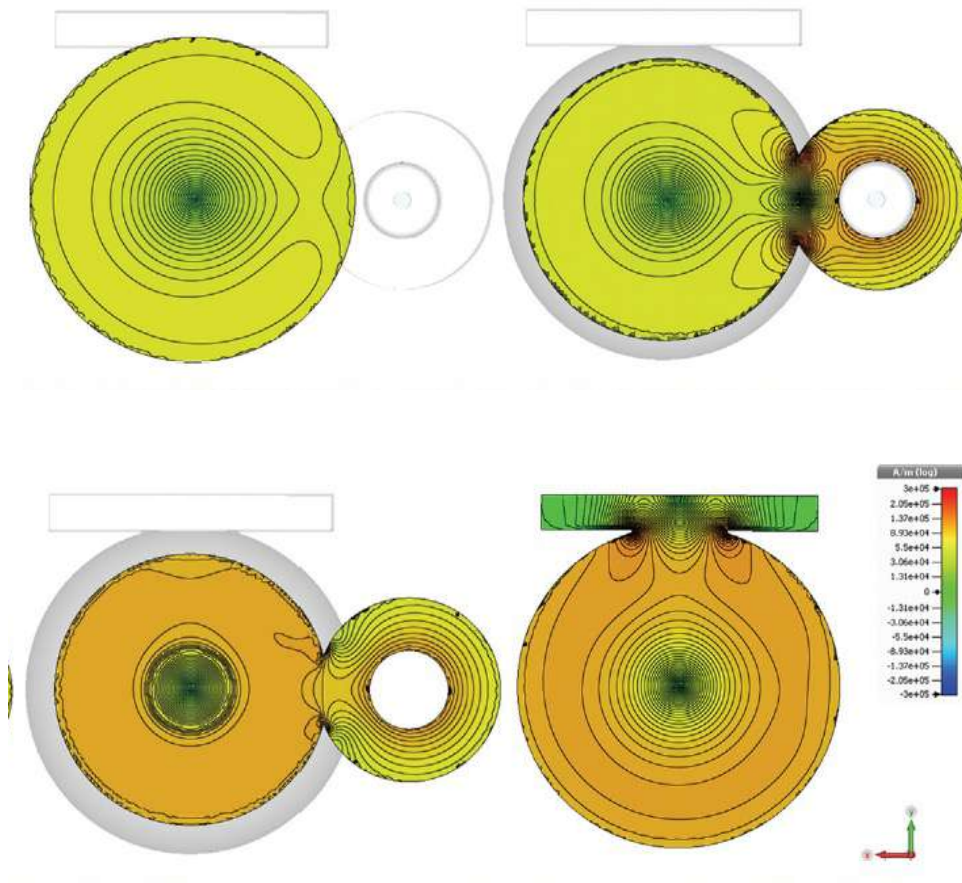


S. Rimjaem et al. Nuclear Instruments and Methods in Physics Research A 736 (2014) 10–21.

Cavity Optimization: Normal conducting cavity (5)

RF waveguide in-input port and side-coupling cavity cause asymmetric azimuthal B-field distribution.

Asymmetric beam distribution!

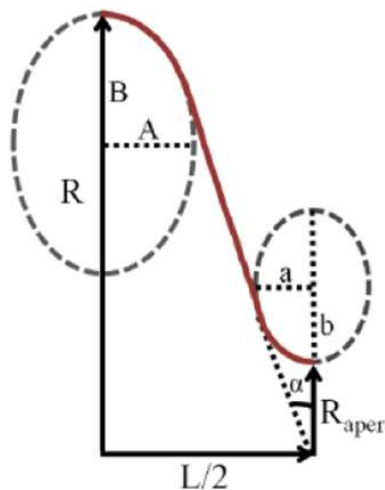
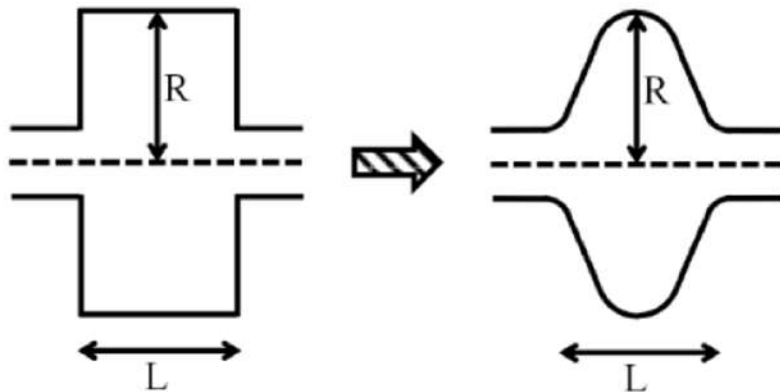


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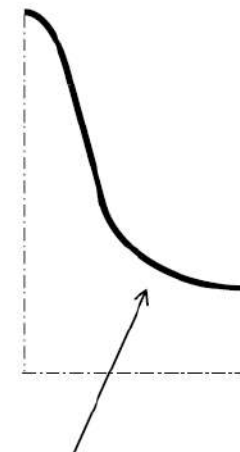
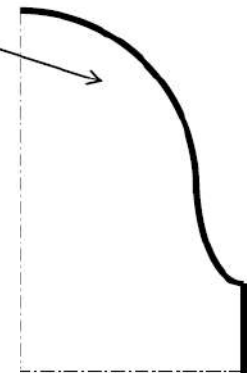
Cavity Optimization: Superconducting cavity (1)

Optimization of superconducting cavity → Optimization of peak surface fields

Bell-shape Cavity



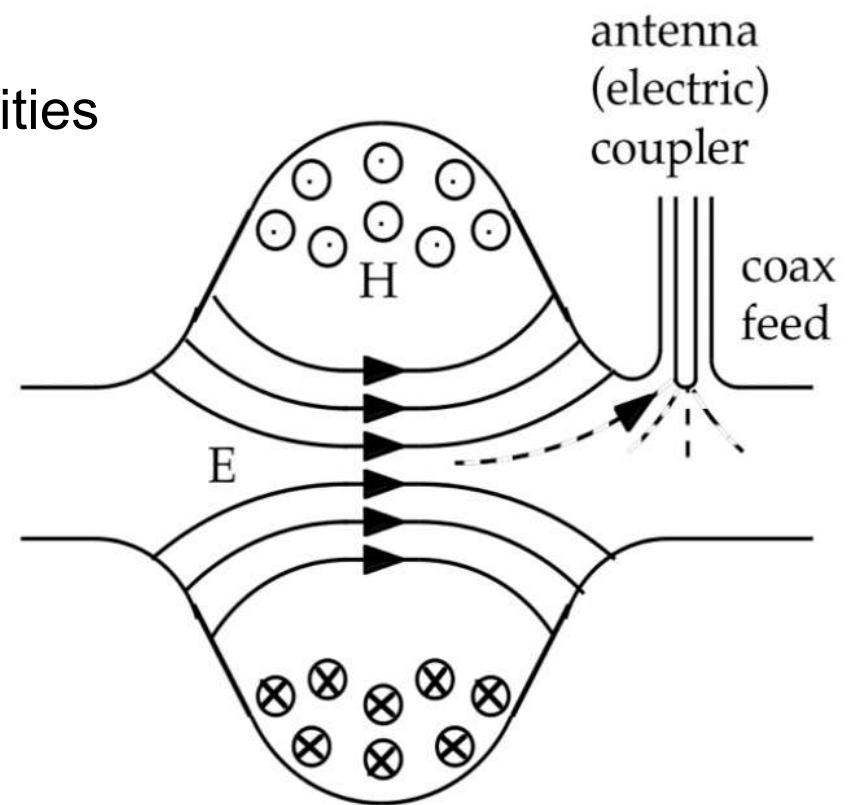
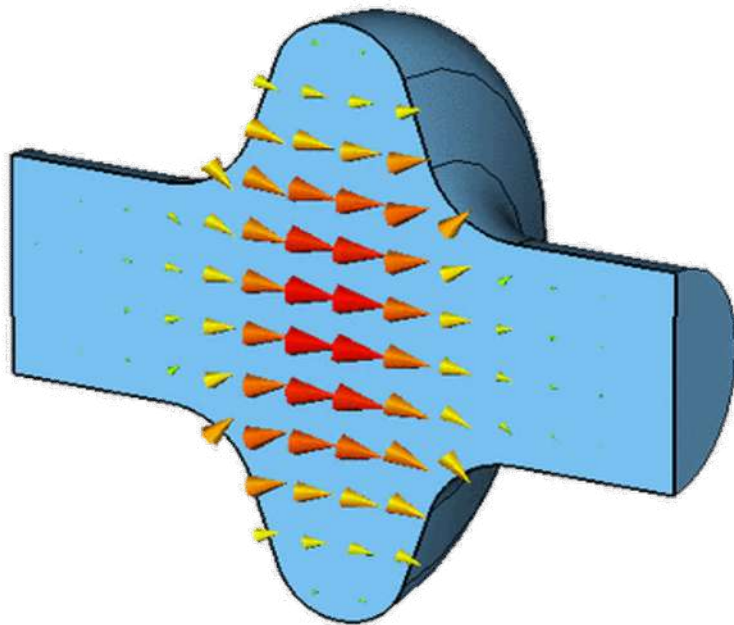
Add “*magnetic volume*” at the equator to reduce B_{peak}



Add “*electric volume*” at the iris to reduce E_{peak}

Cavity Optimization: Superconducting cavity (2)

- Smooth shape to **avoid field emission from the surface**.
 - R/Q is low but Q is very high
- Gradient is limited by
 - Surface magnetic field
 - Field emission from surface impurities

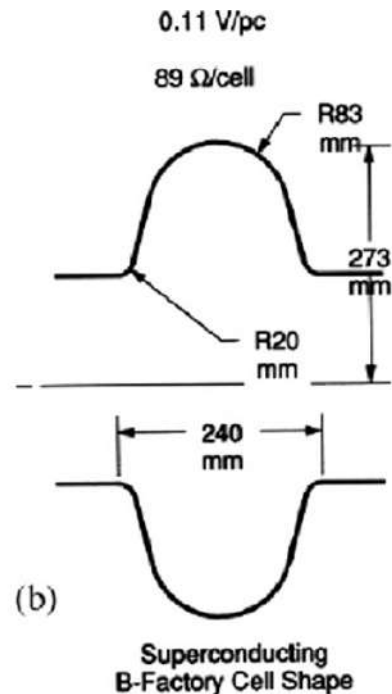


"bell-shaped" superconducting cavity

Courtesy: John Byrd, RF cavities and linacs, Lawrence Berkeley National Laboratory

Cavity Optimization: Superconducting cavity (3)

Example: Cornell 500 MHz superconducting cavity

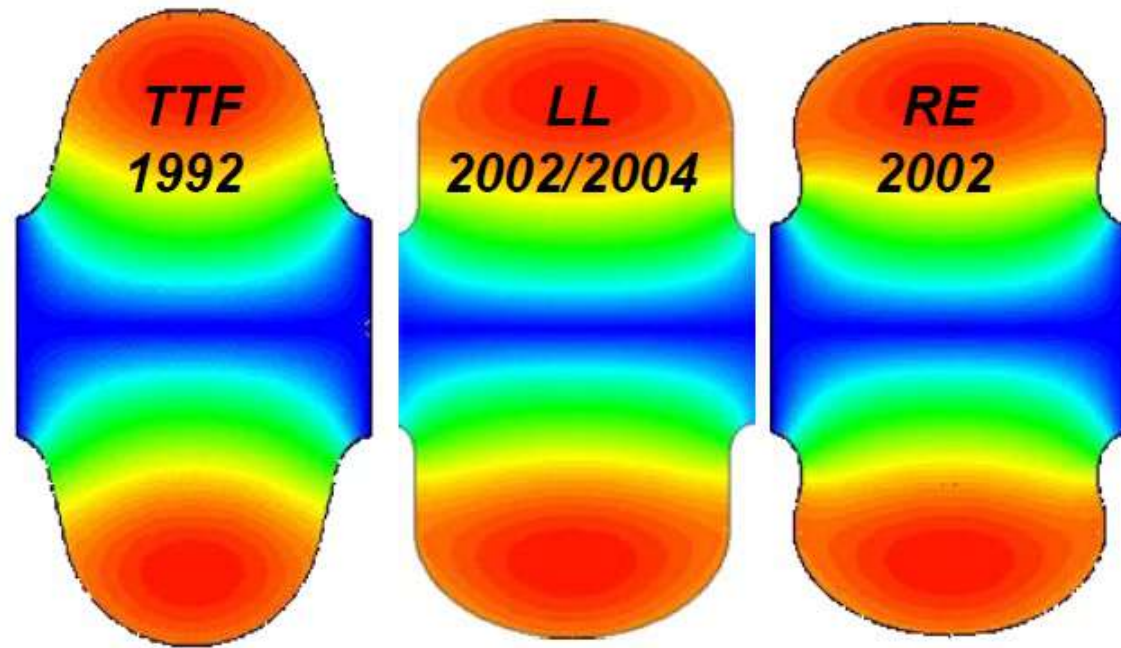


Quantity	Cornell SC 500 MHz	Pillbox
G	270 Ω	257 Ω
R_a/Q_0	88 Ω /cell	196 Ω /cell
E_{pk}/E_{acc}	2.5	1.6
H_{pk}/E_{acc}	52 Oe/MV/m	30.5 Oe/(MV/m)

Cavity Optimization: Superconducting cavity (4)

Example:

$f = 1300 \text{ MHz}$



r_{iris}	[mm]	35	30	33
k_{cc}	[%]	1.9	1.52	1.8
E_{peak}/E_{acc}	-	1.98	2.36	2.21
B_{peak}/E_{acc}	[mT/(MV/m)]	4.15	3.61	3.76
R/Q	[Ω]	113.8	133.7	126.8
G	[Ω]	271	284	277
$R/Q \cdot G$	[$\Omega \cdot \Omega$]	30840	37970	35123

Outline

1

Charged Particle Acceleration and RF Gaps

2

Rectangular & Circular Waveguides

3

Resonant RF Cavities

4

Cavity optimization

5

Disc-loaded Structure

6

Summary

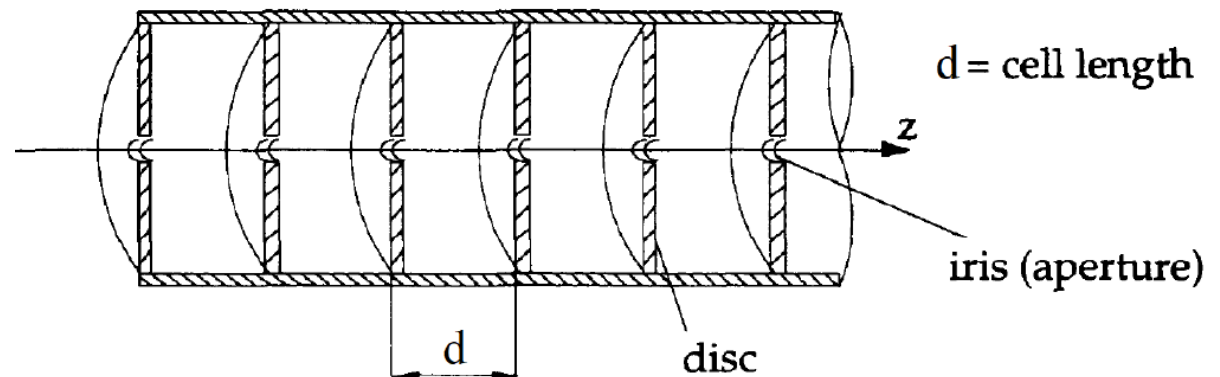
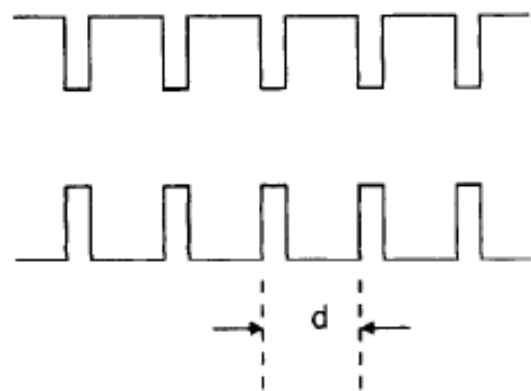
Periodically Loaded Structure

The most widely used solution for obtaining phase velocity $v_p < c$ in accelerating structure is the use of **periodic structure**.

- Periodic structure has the modes that are composed of a Fourier sum of waves, which are suitable for **synchronous particle acceleration**.

Periodically loaded structure:

- Reflections from the periodic loading elements reduce phase velocity
- **Iris-loaded structure (disc-loaded structure)** may be viewed as a uniform waveguide with periodic loading elements.



In uniform waveguide, the longitudinal E-field of TM₀₁ mode is

$$E_z(r, z, t) = EJ_0(Kr)e^{j(\omega t - k_0 z)}$$

Dispersion relation of uniform waveguide:

$$\omega^2 = \omega_c^2 + (k_0 c)^2 = (Kc)^2 + (k_0 c)^2$$

K is the cutoff wavenumber for TM₀₁ mode

Phase velocity:

$$v_p = \frac{\omega}{k_0} = \frac{c}{\sqrt{1 - (Kc)^2/\omega^2}} > c$$

**Cannot be used
for synchronous
acceleration !**

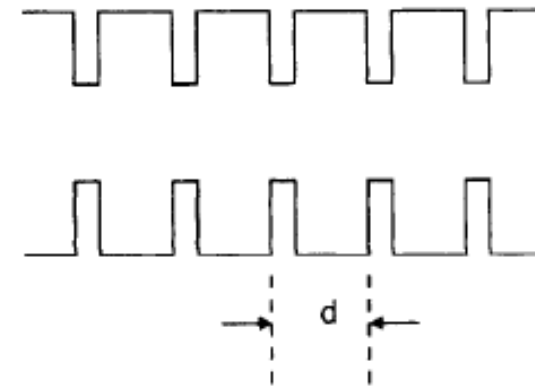
Solving this problem is based on two fundamental principles:

- **Floquet's theorem**
- **Space harmonics**

Floquet Theorem and Space Harmonics

Floquet's theorem: “For a given mode and frequency, the electric field in adjacent cells differs by a constant phase factor.”

$$\mathbf{E}(r, z, t) = \mathbf{E}_d(r, z)e^{j(\omega t - k_0 z)}$$



- A solution $E_d(r, z)$ in one cell applies to all identical cells.
- The exponential factor describes the phase shift of the E-field between adjacent cells.

Since $E(r, z)$ is periodic with period d , such that $E(r, z + d) = E(r, z)$, it can be expressed as a Fourier series:

$$E_d(r, z) = \sum_{n=-\infty}^{\infty} a_n(r) e^{-j2\pi n z/d}$$

Replace $\mathbf{E}(r, z, t)$ and $E_d(r, z)$ in wave equation:

$$e^{j\omega t} \sum_n e^{-j(k_0 + 2\pi n/d)z} \left[\frac{d^2 a_n(r)}{dr^2} + \frac{1}{r} \frac{da_n(r)}{dr} + K_n^2 a_n(r) \right] = 0$$

$$a_n''(r) + \frac{a_n'(r)}{r} - K_n^2 a_n(r) = 0$$

$$K_n^2 = (\omega/c)^2 - (k_0 + 2\pi n/d)^2 \rightarrow \text{Dispersion relation of periodic structure}$$

$$k_n = k_0 + \frac{2\pi n}{d}$$

For $K_n^2 > 0$, field components in TM₀₁-mode are

$$E_z(r, z, t) = \sum_{n=-\infty}^{\infty} E_n J_0(K_n r) e^{j(\omega t - k_n z)}$$

$$E_r(r, z, t) = j \sum_{n=-\infty}^{\infty} E_n \frac{k_n}{K_n} J_1(K_n r) e^{j(\omega t - k_n z)}$$

$$B_\theta(r, z, t) = j \sum_{n=-\infty}^{\infty} E_n \frac{\omega}{K_n c^2} J_1(K_n r) e^{j(\omega t - k_n z)}$$

$$K_n^2 = (\omega/c)^2 - (k_0 + 2\pi n/d)^2$$

$$k_n = k_0 + \frac{2\pi n}{d}$$

J_0 and J_1 are the Bessel functions

Dispersion relation of periodic structure:

$$K_n^2 = (\omega/c)^2 - (k_0 + 2\pi n/d)^2$$

Phase velocity for n^{th} space harmonic:

$$v_{p,n} = \frac{\omega}{k_n} = \frac{v_{p,0}}{1 + (2\pi n/k_0 d)}$$

$$\beta_n = \frac{\omega}{k_n c} = \frac{\beta_0}{1 + (n\beta_0 \lambda)/d}$$

- A periodic structure generates space harmonics with the same frequency but different wavenumbers K .
- By selecting a suitable harmonic order n , the phase velocity can be reduced to match the particle velocity, enabling synchronous acceleration.

When consider $v_{ph} < c$, $K_n^2 < 0 = -k_r^2$, or $K_n = jk_r$

$$k_r = (k_0 + 2\pi n/d)$$

Field components in TM₀₁-mode are

$$\begin{aligned} E_z(r, z, t) &= \sum_{n=-\infty}^{\infty} E_n I_0(K_n r) e^{j(\omega t - k_n z)} \\ E_r(r, z, t) &= j \sum_{n=-\infty}^{\infty} E_n \frac{k_n}{K_n} I_1(K_n r) e^{j(\omega t - k_n z)} \\ B_\theta(r, z, t) &= j \sum_{n=-\infty}^{\infty} E_n \frac{\omega}{K_n c^2} I_1(K_n r) e^{j(\omega t - k_n z)} \end{aligned}$$

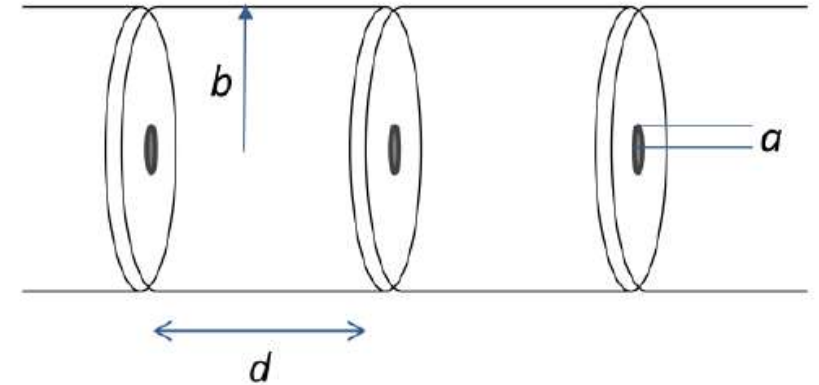
I_0 and I_1 are the modified Bessel functions.

Field components in TM₀₁-mode for n = 0:

$$E_z = E_0 I_0(K_0 r) \cos(\phi)$$

$$E_r = -\gamma_s E_0 I_1(K_0 r) \sin(\phi)$$

$$B_\theta = -\frac{\gamma_s \beta_s}{c} E_0 I_1(K_0 r) \sin(\phi)$$



when $K_0 = \sqrt{|K_0^2|}$ and $\phi = \omega t - \frac{2\pi}{\lambda} \int \frac{dz}{\beta_s(z)}$

Energy gain of synchronous particle after acceleration in length d :

$$\Delta W = q \int_0^d dz \operatorname{Re} \left[E_z(r, z, t = \frac{z}{\beta_0 c} + t_0) \right] = q E_0 I_0(K_0 r) d \cos \phi_0$$

Phase velocity for n^{th} space harmonic:

$$\beta_n = \frac{\omega}{k_n c} = \frac{\beta_0}{1 + (n\beta_0 \lambda / d)}$$

Dispersion relation of periodic structure:

$$K_n^2 = (\omega / c)^2 - (k_0 + 2\pi n / d)^2$$

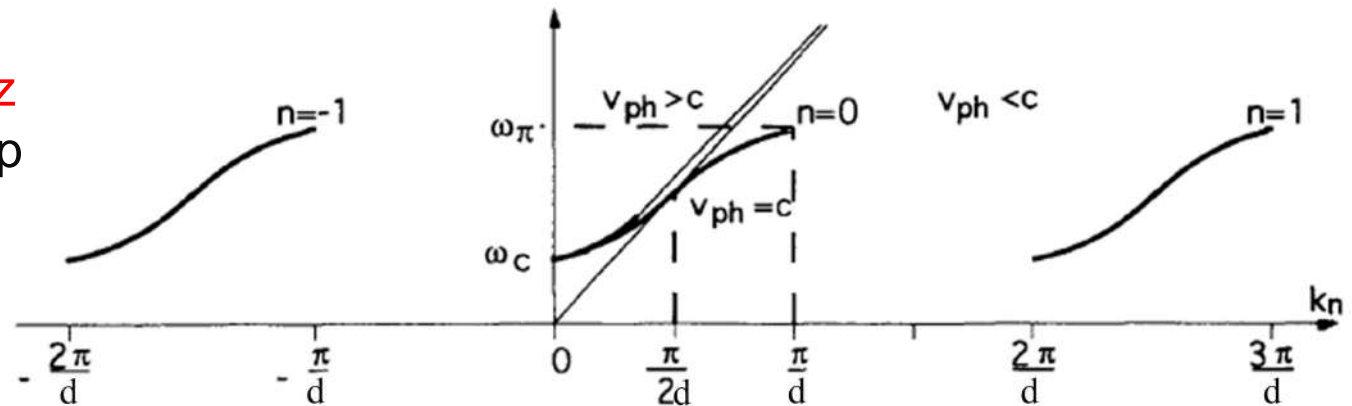
➤ Consider $n = 0$: $k_0 = \omega / \beta_0 c$

$$K_0^2 = (\omega / c)^2 - k_0^2 = -\omega^2 / \gamma_s^2 \beta_s^2 c^2$$

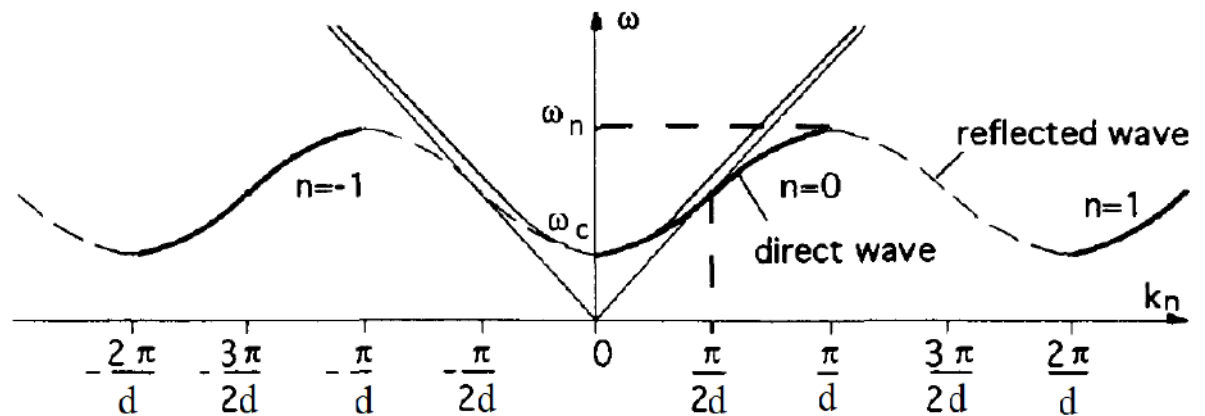
$$\gamma_s = [1 - (v_s / c)^2]^{-1/2} = (1 - \beta_s^2)^{-1/2}$$

RF power and energy propagate along the accelerating structure in the direction of beam acceleration.

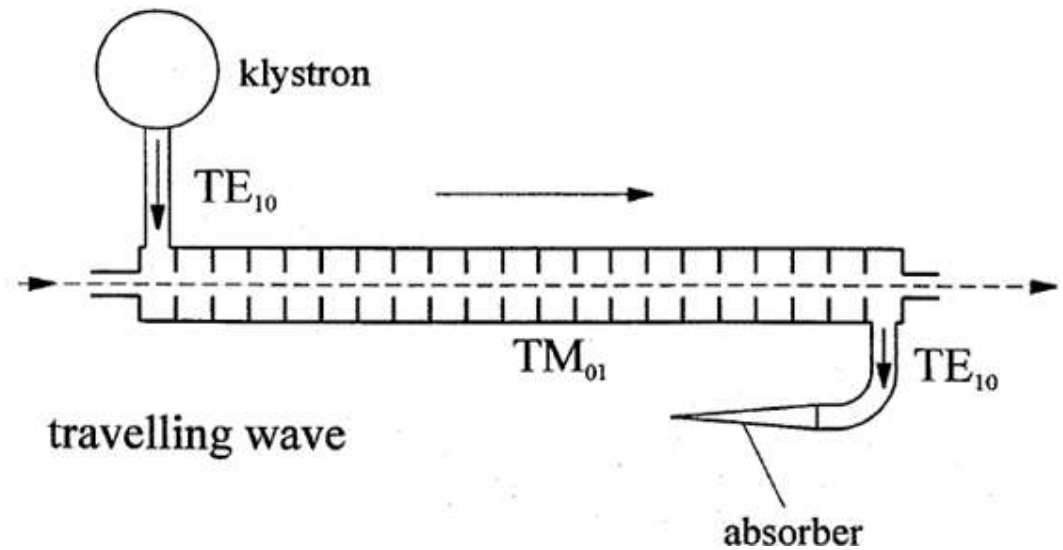
- **Travelling-wave structure:**
RF wave propagates in the **+z direction**, with a positive group velocity ($v_g > 0$)



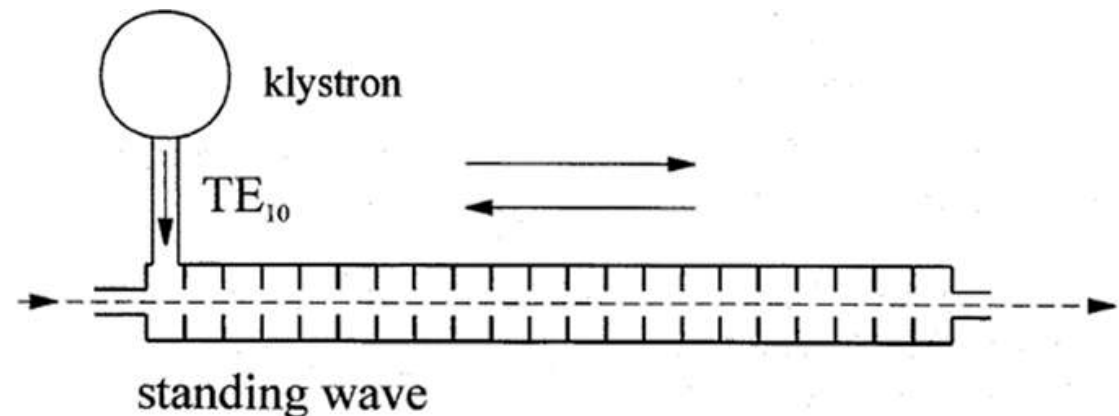
- **Standing-wave structure:**
RF waves reflect between walls, forming standing waves.
 - A negative group velocity ($v_g < 0$) indicates a backward-wave mode.
 - A standing wave has zero net energy flow ($v_g = 0$) for the ideal standing-wave field.



- **Travelling wave structures:**
Disk loaded waveguides (TM_{01})



- **Standing wave structures:**
Resonant cavities (TM_{010})



Wille, Klaus. The Physics of Particle Accelerators: An Introduction. Oxford University Press, 2000.

Conclusion

1. RF Acceleration

Longitudinal RF electric fields accelerate particles; synchronous phase controls energy gain.

2. Waveguide Modes

TE/TM modes follow Maxwell's equations and boundary conditions; uniform guides cannot provide synchronous acceleration.

3. Resonant Cavities

Standing-wave cavities store RF energy; TM_{010} mode provides a longitudinal accelerating field.

4. Transit-Time Factor

Finite transit time reduces the effective voltage:

$$\Delta W = qV_0 T \cos \phi$$

5. Cavity Optimization

Beam apertures and nose cones affect fields, frequency, Q, shunt impedance, and energy gain.

6. Periodic Structures

Space harmonics enable velocity matching; travelling-wave and standing-wave structures use different RF field patterns.

KEY MESSAGE • Synchronization + accelerating field + cavity geometry → efficient energy gain