Ab initio study of HVP contributions to $a_{\ell=e,\mu,\tau}$

At Physical Point Mass with Full Systematics

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Budapest-Marseille-Wuppertal Collaboration
Refs: 1711.04980 [hep-lat] and PRD (2016)

Muon Anomalous Magnetic Moment $a_{\ell=e,\mu,\tau}$

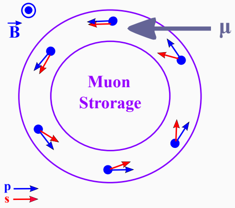
- Dirac Eq. with B:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\boldsymbol{\alpha}\cdot\left(-i\hbar c\nabla - e\mathbf{A}\right) + \beta c^2 m_\ell + eA_0\right]\psi \ ,$$

- Nonlelativistic Limit, Pauli Eq.:

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{(-i\hbar c\nabla - e\mathbf{A})^2}{2m_\ell c} - \mathbf{M}_\ell\cdot\mathbf{B} + eA_0\right]\phi \ ,$$

- Magnetic Moment: $\mathbf{M}_\ell = g_\ell \frac{e}{2m_\ell c}\frac{\hbar\boldsymbol{\sigma}}{2}$,
- In Dirac Theory:
  $g_\ell = 2 \ , \quad a_\ell \equiv (g_\ell - 2)/2 = 0 \ , \quad \omega_{\text{cyc}} = \omega_{\text{prec}}$.
- In QFT (with Loops) for Electron (M.Knecht ,NPPP2015):
  $a_e^{\text{SM}} = 1\ 159\ 652\ 180.07(6)(4)(77)\times 10^{-12} \quad (\mathcal{O}(\alpha^5))$,
  $a_e^{\text{exp}} = 1\ 159\ 652\ 180.73(0.28)\times 10^{-12} \quad [0.24 ppb]$.



$a_\mu^{exp.} = a_\mu^{\text{SM}}$?

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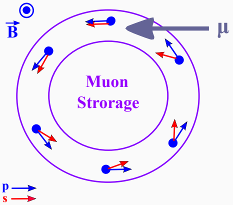


$$a_\mu^{exp.} = a_\mu^{\rm SM}?$$

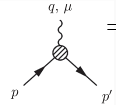
$a_\mu^{exp.}$ vs. a_μ^{SM}

SM contribution	$a_\mu^{\rm contrib.} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
HVP-LO (pheno.)	692.3 ± 4.2	[Davier et al '11]
	694.9 ± 4.3	[Hagiwara et al '11]
	681.5 ± 4.2	[Benayoun et al '16]
HVP-NLO	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
HVP-NNLO	1.24 ± 0.01	[Kurz et al '11]
HLbyL	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp − SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm

a_ℓ in QFT

- **QFT Def. for a_ℓ:**

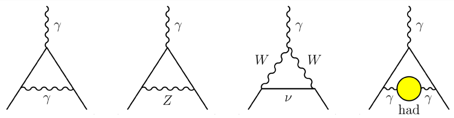


$$= \quad \langle \bar{\ell}^-(p)|\mathcal{J}^\mu|\ell^-(p')\rangle = \bar{u}(p)\Gamma^\mu(p,p')u(p') \tag{1}$$

$$\Gamma^\mu(q=p-p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2) + \cdots \ , \tag{2}$$

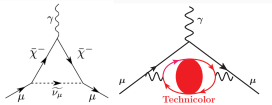
$$F_2(0) = a_\ell = (g_\ell - 2)/2 \ . \tag{3}$$

- **Standard Model, Loop Corr.:**



$$a_\ell = \alpha/(2\pi) + \cdots \ .$$

- **BSM = MSSM (Padley et.al.'15) or TC (Kurachi et.al. '13) etc.:**



$$\propto (m_\ell/\Lambda_{BSM})^2.$$

Pion Contributions to a_μ from Experimental Data

Really $a_\mu^{exp.} \neq a_\mu^{SM}$?

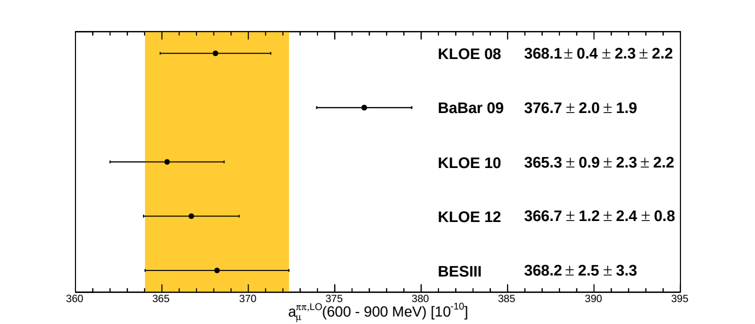


Figure: BESIII, PLB'16: Pion contributions to a_μ using $e^+e^- \to \pi^+\pi^-$ data.

Independent cross-checks by Lattice QCD must be done for Leading-Order (LO) Hadronic Vauccum Polarization (HVP) contribution to muon g-2, $a_\mu^{\text{LO-HVP}}$.

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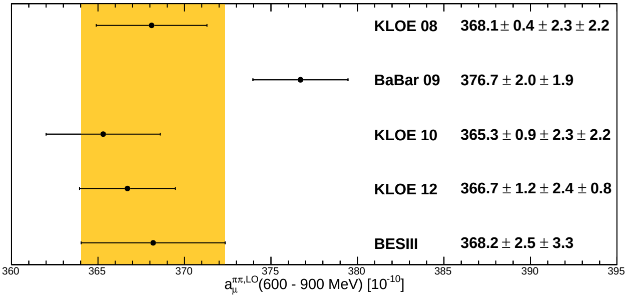

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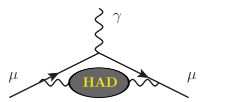
Objective in This Work

LO-HVP contribution to muon g-2 for all leptons by lattice QCD:

$$a_{\ell=e,\mu,\tau}^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\ \omega(Q^2/m_\ell^2)\hat{\Pi}^f(Q^2)\ .$$

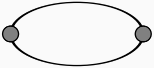


where suffix f stands for a flavor $f = l(u,d), s, c, disc$, and

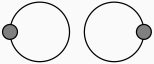
$$\hat{\Pi}^f(Q^2) = \Pi^f(Q^2) - \Pi^f(0) = \sum_t t^2 \left[1 - \left(\frac{\sin(z/2)}{z/2}\right)^2\right]_{z=Qt} \frac{1}{3}\sum_{i=1}^{3} C_{ii}^f(t)\ , \quad (4)$$

with

$$C_{\mu\nu}^{f=l,s,c}(t) = q_{f=l,s,c}^2 \sum_{\vec{x}} \langle\ j_\mu^f(x) j_\nu^f(0)\ \rangle|_{conn}\ ,$$



$$C_{\mu\nu}^{f=disc}(t) = q_{f=disc}^2 \sum_{\vec{x}} \langle(\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s)\rangle|_{disc}\ .$$

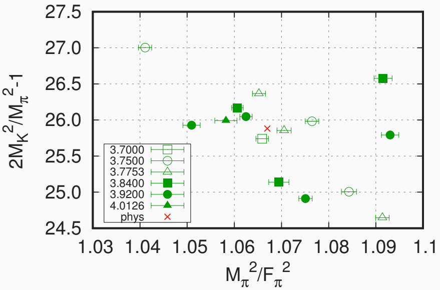


Here, charge factors are given by $(q_l^2, q_s^2, q_c^2, q_{disc}^2) = (5/9,\ -1/9,\ 4/9,\ 1/9)$.

Simulation Setup

State of The Art

- Nf=(2+1+1) simulations around Physical Mass Points.
- Large Volume: $(L, T) \sim (6, 9-12) fm$.
- Controlled Continuum Limit with 15 simulation points.



β	a[fm]	N_t	N_s	#traj.	M_π[MeV]	M_K[MeV]	#SRC (l,s,c,d)
3.7000	0.134	64	48	10000	~ 131	~ 479	$(768, 64, 64, 9000)$
3.7500	0.118	96	56	15000	~ 132	~ 483	$(768, 64, 64, 6000)$
3.7753	0.111	84	56	15000	~ 133	~ 483	$(768, 64, 64, 6144)$
3.8400	0.095	96	64	25000	~ 133	~ 488	$(768, 64, 64, 3600)$
3.9200	0.078	128	80	35000	~ 133	~ 488	$(768, 64, 64, 6144)$
4.0126	0.064	144	96	04500	~ 133	~ 490	$(768, 64, 64, -)$

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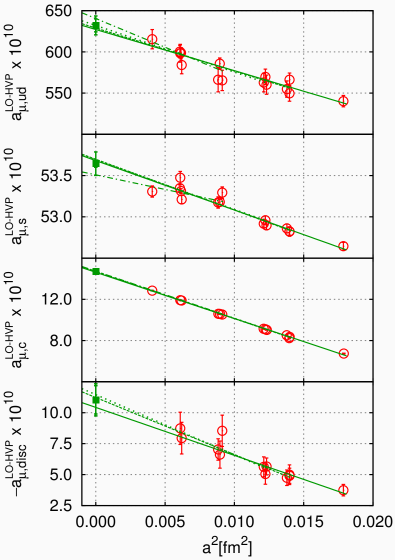
Our Challenge I: Controlled Continuum Extrapolations

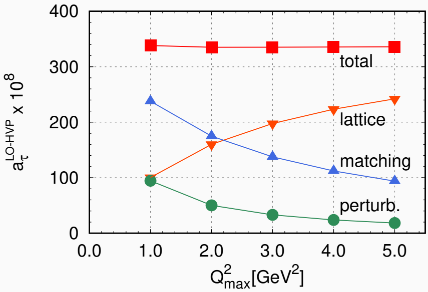


- With 6 $\beta's = 15$ $a^2[fm]$ simulations, allowing full control over continuum limit.
- Get systematic uncertainty from various cuttings: no-cut, or cutting $a \geq 0.134$, 0.111, or 0.095.
- Get good χ^2/dof with extrapolation linear in a^2, and interpolation linear in M_K^2 (strange) or M_π^2 and $M_{\eta c}$ (charm).
- Strong a^2 dependences for $a_{\mu,ud/disc}^{\text{LO-HVP}}$ due to taste violations, and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large m_c.

Our Challenge II: High Q^2 Control

$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + \gamma_\ell(Q_{max})\hat{\Pi}^f(Q_{max}) + \Delta^{pert} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ . \tag{5}$$

- For muon and electron, $Q > Q_{max}$ effects are tiny $\lesssim 0.1\%$.
- For tau, $Q > Q_{max}$ effects are large, while the total is stable.
- The lattice data have enough overlap to perturbative regime even in tau case.



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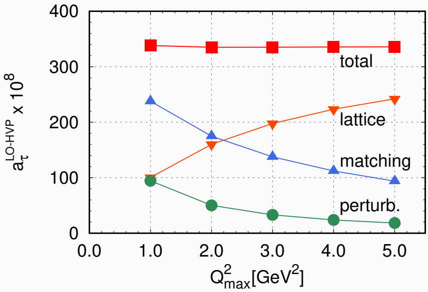


Toward Comparison to Pheno.: Isospin Breaking, QED, and FV Corr.

- **Isospin/QED Collections:**
  Thanks to F.Jegerlehner (& M. Benayoun) for correspondance and numbers:

Effect	corr. to $a_\mu^{\text{LO-HVP}} \times 10^{10}$
ρ–ω mix.	2.71 ± 1.36
FSR	4.22 ± 2.11
$M_\pi \to M_{\pi\pm}$	-4.47 ± 4.47
$\pi^0\gamma$	4.64 ± 0.04
$\eta\gamma$	0.65 ± 0.01
Total	7.8 ± 5.1

- **FV Collections:**
  Long-distance $I = 1$ ($I = 0$) contribution dominated by 2-pions (3-pions). The dominant FV in $I = 1$ channel could be estimated by XPT for $\pi^+\pi^-$ loop (Aubin et al '16):

$$\left(a_{\mu,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,I=1}^{\text{LO-HVP}}(6fm)\right)\Big|_{\mathrm{XPT}} = 13.42(13.42) \times 10^{-10}\ ,(1.9\%)\ . \quad (6)$$

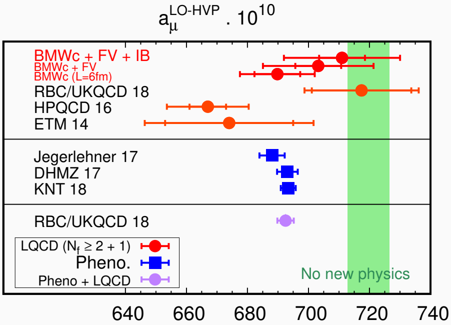
Summary on $a_\mu^{\text{LO-HVP}}$ 1711.04980

$a_\mu^{\text{LO-HVP}}$ BMWc

$I = 1$	$582.8(6.7)_{st}(7.2)_{acut}(0.1)_{tcut}(0.0)_{qcut}(4.5)_{da}(13.4)_{fv}$
$I = 0$	$120.5(3.4)_{st}(3.5)_{acut}(0.2)_{tcut}(0.0)_{qcut}(1.0)_{da}$
total	$711.0(7.5)_{st}(8.0)_{acut}(0.2)_{tcut}(0.0)_{qcut}(5.5)_{da}(13.4)_{fv}(5.1)_{iso}$

Remarks

- Our Lattice QCD results are consistent with both "No New Physics" and Dispersive Method.
- Total error of our LQCD is 2.6%, dominated FV effects.



$\hat{\Pi}^{lat}(Q^2)$ vs $\hat{\Pi}^{pheno}(Q^2)$ for Various Q^2

$$\hat{\Pi}^{lat}(Q^2) = \lim_{a\to 0} \sum_{t=0}^{T/2} \left[t^2 - \left(\frac{\sin Qt/2}{Qt/2} \right)^2 \right] \frac{C_{ii}^f(t)}{3} , \quad \hat{\Pi}^{pheno}(Q^2) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R_{had}(s)}{s(s+Q^2)} .$$

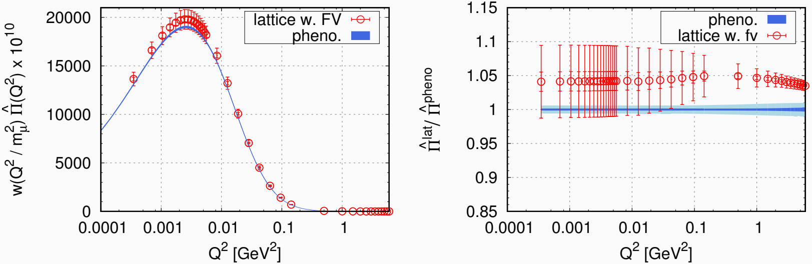


Figure: Lat (BMWc) vs Pheno (alphaQEDc17 by Jegerlehner) for $\omega(Q^2/m_\mu^2)\hat{\Pi}(Q^2)$.

- The contributions at $Q^2 \sim (m_\mu/2)^2$ are dominant, and the lattice and phemenology are consistent within the error-bars there.
- However, the lattice error gets larger at $Q^2 \sim (m_\mu/2)^2$, which is due to poor knowledge on FV effects.

Summary on $a_{e,\tau}^{\text{LO-HVP}}$ 1711.04980

$a_e^{\text{LO-HVP}}$ BMWc

$I=1$	$156.9(2.4)_{st}(2.1)_{acut}(0.0)_{tcut}(0.0)_{qcut}(1.2)_{da}(4.6)_{fv}$
$I=0$	$30.7(1.2)_{st}(1.0)_{acut}(0.1)_{tcut}(0.0)_{qcut}(0.2)_{da}$
total	$188.5(2.6)_{st}(2.3)_{acut}(0.1)_{tcut}(0.0)_{qcut}(1.5)_{da}(4.6)_{fv}(0.9)_{iso}$

$a_\tau^{\text{LO-HVP}}$ BMWc

$I=1$	$253.2(0.7)_{st}(1.4)_{acut}(0.0)_{tcut}(0.1)_{qcut}(1.2)_{da}(1.8)_{fv}$
$I=0$	$84.4(0)_{st}(0.4)_{acut}(0.7)_{tcut}(1.0)_{qcut}(3.4)_{da}$
total	$341.1(0.8)_{st}(2.0)_{acut}(0.0)_{tcut}(1.0)_{qcut}(1.5)_{da}(1.9)_{fv}(1.1)_{iso}$

Burger et.al.('15): $a_e^{\text{LO-HVP}} = 178.2(6.4)(8.6)$, $a_\tau^{\text{LO-HVP}} = 341(8)(6)$
HPQCD ('16): $a_e^{\text{LO-HVP}} = 177.9(3.9)$
Jeherlehner('16): $a_e^{\text{LO-HVP}} = 185.11(1.24)$.
Eidelman et.al.('07): $a_\tau^{\text{LO-HVP}} = 338(4)$.

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Summary and Perspective

- We have obtained $a_\mu^{\text{LO-HVP}}$ directly at physical point masses: $a_\mu^{\text{LO-HVP}} = 711.0(7.5)(17.3)$.
- Full controlled continuum extrapolation and matching to perturbation theory. Model assumptions are put on only for small corrections from FV/QED/isospin breaking. Total error is 2.6%, dominated by FV.
- Our Lattice QCD results are consistent with "No New Physics" as well as Phenomenological Dispersive Methods with a conservative systematic errors.
- Lat-Pheno. comparisons are made for HVP: consistent at small Q^2, but lattice tends to be larger, leading to larger $a_{\mu,lat}^{\text{LO-HVP}}$.
- Need $\sim 0.2\%$ precision to match Fermilab/J-PARC experiments!!
  1. lat-pheno collaborations for $a_\mu^{\text{LO-HVP}}$ and $\hat{\Pi}(Q^2)$.
  2. increase statistics by $50 - 100$ times.
  3. control FV effects directly based on the first-principle.
  4. simulations with QED and isospin breaking corrections taken account.

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Form Factors

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

$$\left. + \gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

$F_1(q^2) \quad \rightarrow \quad$ Dirac form factor: $F_1(0) = 1$

$F_2(q^2) \quad \rightarrow \quad$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \dfrac{g_\ell - \overbrace{2}^{g_\ell|_{\text{Dirac}}}}{2}$

$F_3(q^2) \quad \rightarrow \quad$ $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell / e_\ell$

$F_4(q^2) \quad \rightarrow \quad$ $\not{P}$, anapole moment: $\vec{\sigma} \cdot (\vec{\nabla} \times \vec{B})$

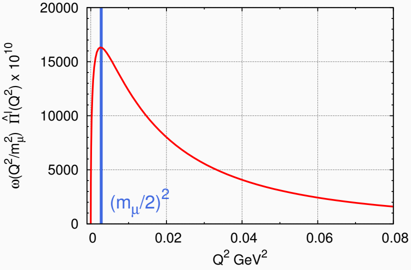
$$G_E(q^2) \equiv F_1(q^2) + \frac{q^2}{4m_\ell^2}F_2(q^2) \rightarrow \text{electric form factor: } G_E(0) = 1$$

$$G_M(q^2) \equiv F_1(q^2) + F_2(q^2) \rightarrow \text{magnetic form factor: } G_M(0) = \frac{g_\ell}{2}$$

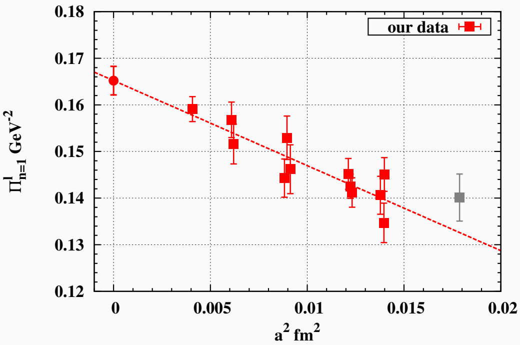
In SM, $F_2(q^2)$ & $F_{3,4}(q^2)$ are generated only by quantum fluctuations (loops)

Our Challenges

- The integrand kurnel $\omega(Q^2/m_\mu^2)$ is known and makes a peak around $Q^2 \sim (m_\mu/2)^2 \sim (0.05GeV)^2 \to 4fm$:
  Our Lattice: $(L, T) \sim (6, 9-12)fm$.
- The Pion/Kaon dynamics precisely:
  Our simulations are performed with Physical Pion/Kaon Masses.
- Large distance signal:
  10^4 Traj., 768 (9000) random sources for *ud*-conn. (*uds*-disc.) correlators.
- Need controlled continuum limit:
  15 lattice spacings ($a \sim 0.064 - 0.134$ fm).
- For a few % precision, we take account of:
  c quark w. matching onto perturb. theory.



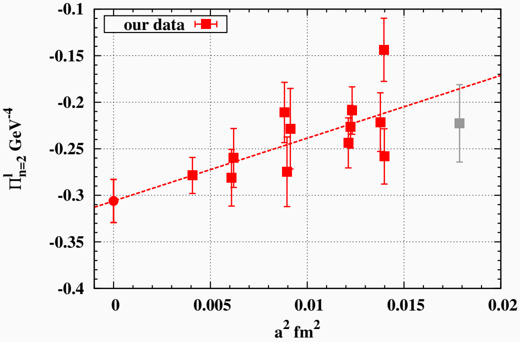
Continuum Extrapolation of $\Pi^l_{n=1}$ I



$$F(C_\Pi^{(2)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(2)}}{a^2}\ \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^l_{n=1}|_{a^2\to 0} = 0.1652(31) \ , \quad \chi^2/\mathrm{d.o.f.} = 24.3/20$$

Continuum Extrapolation of $\Pi^l_{n=2}$



$$F(C^{(4)}_\Pi, A, B, C_{M_\pi}, C_{M_K}) = \frac{C^{(4)}_\Pi}{a^4}\,\frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^l_{n=2}\big|_{a^2\to 0} = -0.306(23) .$$

Continuum Extrapolation of $\Pi^s_{n=1,2}$

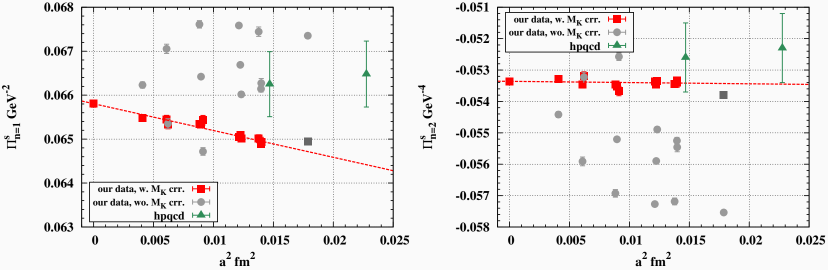


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) \,.$$

$$\Pi^s_{n=1}|_{a^2 \to 0} = 0.0658(1) \,, \quad \Pi^s_{n=2}|_{a^2 \to 0} = -0.0534(2) \,, \quad \chi^2/\text{d.o.f.} = 20.9/18$$

Continuum Extrapolation of $\Pi^c_{n=1,2}$

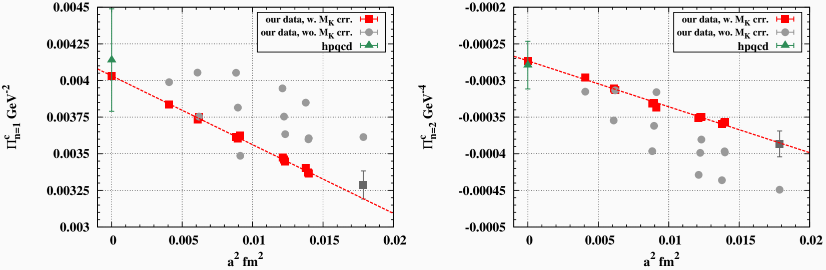


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1208.2855.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}}\;\frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^c_{n=1}|_{a^2\to 0} = 0.00403(2) , \quad \Pi^c_{n=2}|_{a^2\to 0} = -2.73(2)\times 10^{-4} .$$

Disconnected Contributions

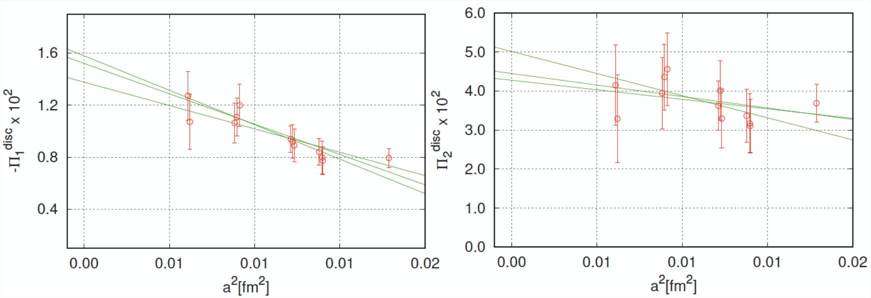


Figure: From Presentaion of T.Kawanai in Lattice 2016.

$$\Pi_1^{\rm disc} = -1.5(2)(1) \times 10^{-3}\ \mathrm{GeV}^{-2}\ , \quad (7)$$

$$\Pi_2^{\rm disc} = -4.6(1.0)(0.4) \times 10^{-3}\ \mathrm{GeV}^{-4}\ . \quad (8)$$

Summary Table of Moments (Preliminary)

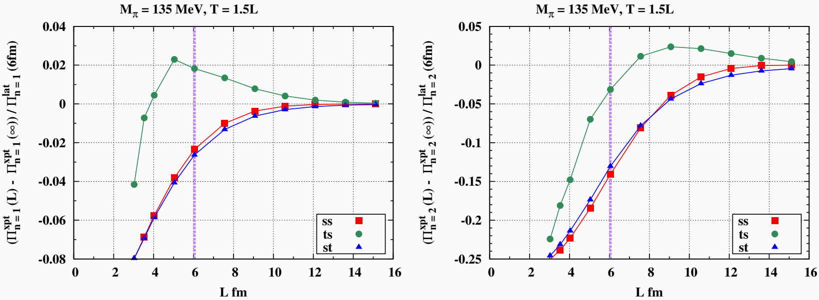
	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)

Table: Preliminary results on the first two moments of the HVP function.

TOTAL ERROR: 1.4% for Π_1, and 4.0% for Π_2 .

FV via Box Asymmetry, XPT Estimate for Various L I

c.f. Aubin et.al., PRD (2016).

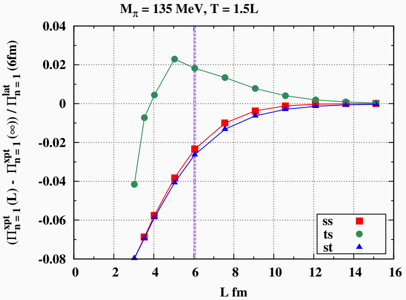


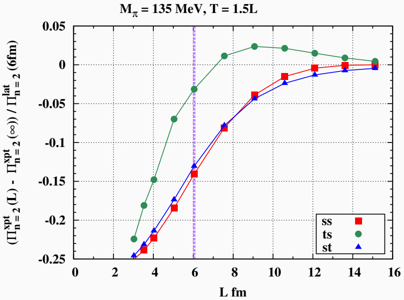
$$\Delta^i_{n=1,2}(L) \quad = \quad \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] \ , \qquad i = ss, ts, st \ ,$$

$$\frac{\Delta^i_n(L = 6\mathrm{fm})}{\Pi^{\mathrm{lat},i}_n} \quad \sim \quad \begin{cases} 2\% & \text{(for the 1st moment, } n = 1) \ , \\ 10\% & \text{(for the 2nd moment, } n = 2) \ . \end{cases} \tag{9}$$

FV via Box Asymmetry, XPT Estimate for Various L II

c.f. Aubin et.al., PRD (2016).





$$\Delta^i_{n=1,2}(L) = \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] \ , \quad i = ss, ts, st \ ,$$

$$FV.(L) \pm dFV.(L) = \left[\max\{\Delta_i\} + \min\{\Delta_i\}\right]/2 \pm \left[\max\{\Delta_i\} - \min\{\Delta_i\}\right]/2$$

$$\xrightarrow[L\to 6fm]{} \begin{cases} 0.0006(22) & (\text{for the 1st moment, } n=1) \ , \\ -0.015(19) & (\text{for the 2nd moment, } n=2) \ . \end{cases} \tag{10}$$

Summary Table of Moments with FV I (Preliminary)

	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)
$I = 1$ FV corr.	0.0006(23)	−0.015(10)
$I = 1$ + FV corr.	0.0834(8)(9)(23)	−0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	−0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. HPQCD(arXiv:1601.03071 and PRD2014):

$$\Pi_1^l = 0.1606(25)\ \mathrm{GeV}^{-2}, \quad \Pi_2^l = -0.368(16)\ \mathrm{GeV}^{-4}\ ,$$
$$\Pi_1^s = 0.06625(74)\ \mathrm{GeV}^{-2}, \quad \Pi_2^s = -0.0526(11)\ \mathrm{GeV}^{-4}\ .$$

Summary Table of Moments with FV II (Preliminary)

	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
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$I = 0$	0.0166(2)(2)	−0.017(1)(1)
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total	0.0995(9)(10)	−0.166(6)(3)
$I = 1$ FV corr.	0.0006(23)	−0.015(10)
$I = 1$ + FV corr.	0.0834(8)(9)(23)	−0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	−0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. Phenomenology(Benayoun et.al.1605.04474):
$\Pi_1 = 0.990(7)\ \mathrm{GeV}^{-2}, \quad \Pi_2 = -0.206(2)\ \mathrm{GeV}^{-4}.$

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell}\, \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{11}$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ , \tag{12}$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: computed by lattice $\hat{\Pi}^f(Q_{max})$ and perturbations .

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$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{13}$$

$$\begin{aligned}
&= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \Pi^f(Q_{max}^2)\right) + \left(\Pi^f(Q_{max}^2) - \Pi^f(0)\right) \right] \ , \\
&= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \Pi^f(Q_{max}^2)\right) \quad \left(\to \Delta a_{\ell,f}^{\text{LO-HVP}}\right) \\
&\quad + \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] \left(\Pi^f(Q_{max}^2) - \Pi^f(0)\right) \quad \left(\to \gamma_\ell(Q_{max}^2) \hat{\Pi}^f(Q_{max}^2)\right) .
\end{aligned} \tag{14}$$

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \quad (13)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) + \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right)\right] \ ,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) \quad (\to \Delta a_{\ell,f}^{\text{LO-HVP}})$$

$$+ \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right)\right] \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \quad (\to \gamma_\ell(Q_{max}^2) \hat{\Pi}^f(Q_{max}^2)) \ . \quad (14)$$

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$$+\left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)\right]\left(\mathbf{\Pi}^f(Q^2_{max}) - \Pi^f(0)\right) \quad (\to \gamma_\ell(Q^2_{max})\hat{\Pi}^f(Q^2_{max}))\ . \quad (14)$$

Perturbative Corrections

Consider separation of momentum integral range as

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \,, \qquad (15)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \,, \qquad (16)$$

where

$a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max})$: lattice simulations, investigated so far ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: $\gamma_l(Q_{max})\hat{\Pi}^f(Q_{max}) + \Delta^{pert} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$.

For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{max})$ is small. For example,

$$a_{\mu}^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=2GeV} 0.678(1)(1) \,, (0.1\%). \qquad (17)$$

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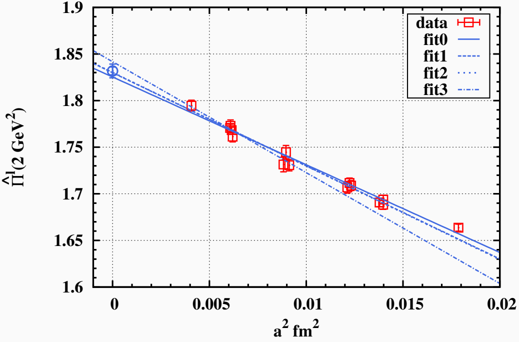
For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{max})$ is small. For example,

$$a_\mu^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=2GeV} 0.678(1)(1) \ , (0.1\%). \tag{17}$$

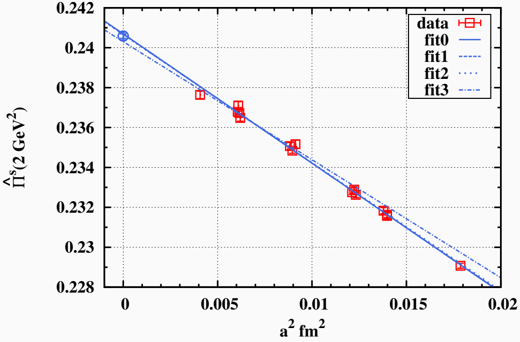
Continuum Extrap. of Light Component: $\hat{\Pi}^l$



$$F(\hat{\Pi}^l, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^l \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$\hat{\Pi}^l = 1.8318(42)(60)$, $\chi^2/\mathrm{d.o.f.} = 8.2/12$ (fit1 case).

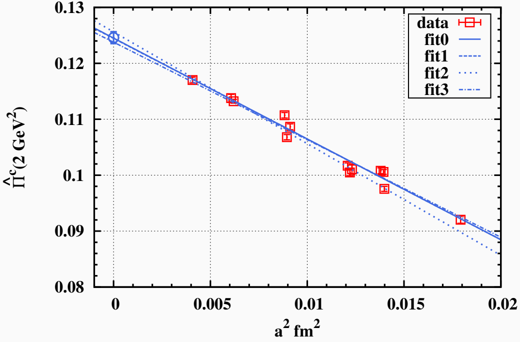
Continuum Extrap. of Strange Component: $\hat{\Pi}^s$



$$F(\hat{\Pi}^s, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^s \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$\hat{\Pi}^s = 0.2406(1)(2)$, $\chi^2/\mathrm{d.o.f.} = 13.6/11$ (fit1 case).

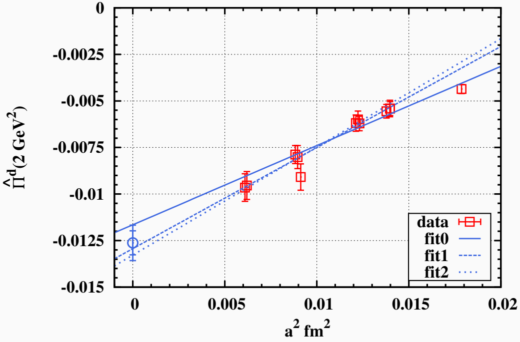
Continuum Extrap. of Charm Component: $\hat{\Pi}^c$



$$F(\hat{\Pi}^c, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^c \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) \ .$$

$\hat{\Pi}^c = 0.1246(9)(7)$, $\chi^2/\mathrm{d.o.f.} = 26.1/9$ (fit1 case).

Continuum Extrap. of Disc. Component: $\hat{\Pi}^d$



$$F(\hat{\Pi}^d, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^d \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K) \ .$$

$$\hat{\Pi}^d = -0.0126(6)(7) \ , \quad \chi^2/\mathrm{d.o.f.} = 3.5/9 \text{ (fit1 case)}.$$

Light-Conn. (and Disc.) Correlator: An Example

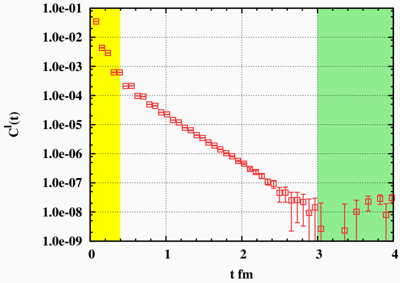
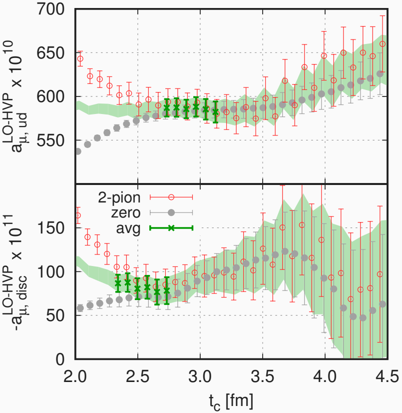


Figure: $C^{ud}(t) = \frac{5}{9}\sum_{\vec{x}}\frac{1}{3}\sum_{i=1}^{3}\langle j_i^{ud}(\vec{x},t) j_i^{ud}(0)\rangle$

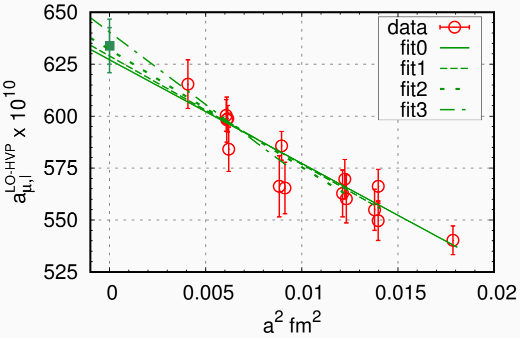
- The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3fm$. To control statistical error, consider $C^{ud}(t > t_c) \to C^{ud}_{\mathrm{up/low}}(t, t_c)$, where $C^{ud}_{\mathrm{up}}(t, t_c) = C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$, $C^{ud}_{\mathrm{low}}(t, t_c) = 0.0$, with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$, and $E_{2\pi} = 2(M_\pi^2 + (2\pi/L)^2)^{1/2}$.
- Similarly, $C^{disc}(t) \to C^{disc}_{\mathrm{up/low}}(t, t_c)$, $-C^{disc}_{\mathrm{up}}(t > t_c) = 0.1 C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$, $-C^{disc}_{\mathrm{low}}(t > t_c) = 0.0$.
- $C^{ud,disc}_{low}(t, t_c) \le C^{ud,disc}(t) \le C^{ud,disc}_{up}(t, t_c)$.

IR-CUT (t_c) Deps. of Light/Disc. Component: $a_{\ell,ud/disc}^{\text{LO-HVP}}$



- Corresponding to $C_{up/low}^{ud,disc}(t_c)$, we obtain upper/lower bounds for $g-2$: $a_{\ell,up/low}^{ud,disc}(t_c)$.
- Two bounds meet around $t_c = 3fm$. Consider the average of bounds: $\bar{a}_{\ell}^{ud,disc}(t_c) = 0.5(a_{\ell,up}^{ud,disc} + a_{\ell,low}^{ud,disc})(t_c)$, which is stable around $t_c = 3fm$.
- We pick up such averages $\bar{a}_{\ell}^{ud,disc}(t_c)$ with $4-6$ kinds of t_c around $3fm$. The average of average is adopted as $a_{\ell,\text{ud/disc}}^{\text{LO-HVP}}$ to be analysed, and a fluctuation over selected t_c is incorporated into the systematic error.

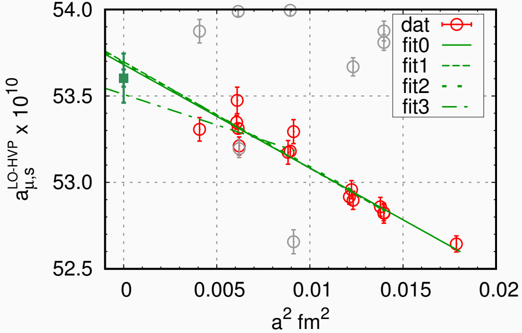
Continuum Extrap. of Light Conn. Component: $a_{\mu,ud}^{\text{LO-HVP}}$



$$F(a_{\mu,ud}^{\text{LO-HVP}}, A, C_\pi, \cdots) = a_{\mu,ud}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + \cdots .$$

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24) , \quad \chi^2/\text{dof} = 7.8/12 \text{ (fit1 case)}.$$

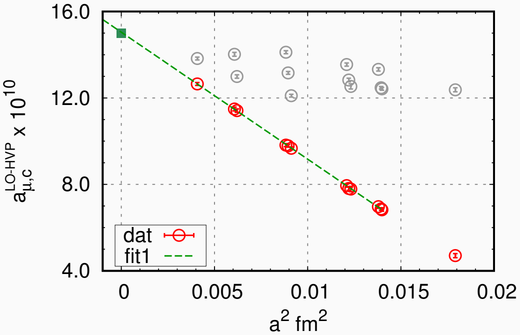
Continuum Extrap. of Strange Conn. Component: $a_{\mu,s}^{\text{LO-HVP}}$



$$F(a_{\mu,s}^{\text{LO-HVP}}, A, C_K) = a_{\mu,s}^{\text{LO-HVP}}(1 + Aa^2) + C_K \Delta M_K^2 \ .$$

$$a_{\mu,s}^{\text{LO-HVP}} = 53.60(05)(13) \ , \quad \chi^2/\text{dof} = 16.7/11 \ (\text{fit1 case}).$$

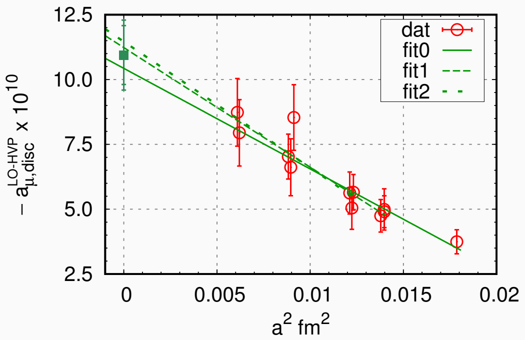
Continuum Extrap. of Charm Conn. Component: $a_{\mu,c}^{\text{LO-HVP}}$



$$F(a_{\mu,c}^{\text{LO-HVP}}, A, C_{\pi,K,\eta_c}) = a_{\mu,c}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + C_M \Delta M_K^2 + C_{\eta_c} \Delta M_{\eta_c} \ .$$

$$a_{\mu,c}^{\text{LO-HVP}} = 14.99(08)(08) \ , \quad \chi^2/\text{dof} = 1/7 \ \text{(fit1 case)}.$$

Continuum Extrap. of Disc. Component: $a_{\mu,disc}^{\text{LO-HVP}}$



$$F(a_{\mu,disc}^{\text{LO-HVP}}, A, C_\pi, \cdots) = a_{\mu,disc}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + \cdots .$$

$$a_{\mu,disc}^{\text{LO-HVP}} = -10.9(1.1)(0.7)\ , \quad \chi^2/\text{dof} = 2.4/10 \text{ (fit1 case)}.$$

Continuum Extrap. of Light Component: $a_{\mu,ud}^{\text{LO-HVP}}$

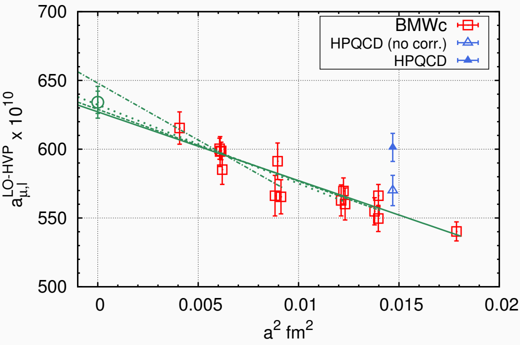
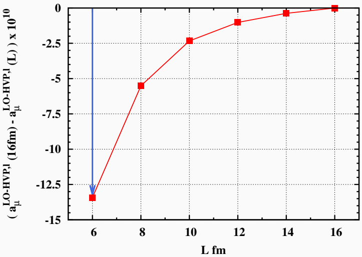


Figure: Red-squares = Our data. Blue-triangles = HPQCD, 1403.1778.

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24)\ , \quad \chi^2/\text{d.o.f.} = 7.8/12 \text{ (fit1 case)}.$$

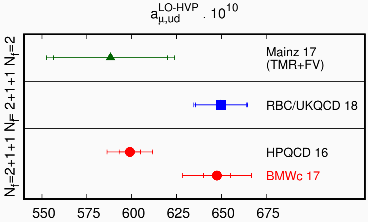
FV for $a_{\mu,ud}^{\text{LO-HVP}}$ by XPT

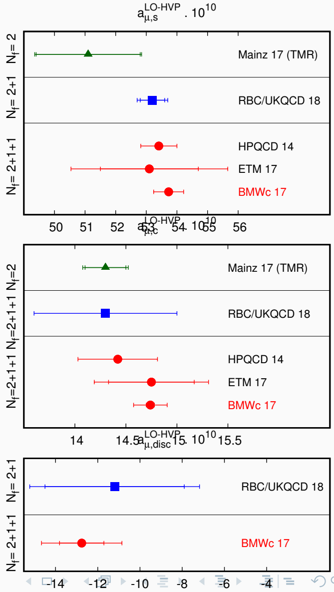
- The $a_\mu^{\text{LO-HVP}}$ comes from Euclidean mommenta; Exponentially suppressed FV with $LM_\pi \sim 4$ for $L \sim 6fm$.
- We work with $L \sim$ *fixed*, and FV effects cannot be estimated from simulations and need model.
- Long-distance $I = 1$ ($I = 0$) contribution dominated by 2-pions (3-pions). The dominant FV in $I = 1$ channel could be estimated by XPT for $\pi^+\pi^-$ loop (Aubin et al '16).



$$(a_{\mu,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,I=1}^{\text{LO-HVP}}(6fm))|_{\text{XPT}} = 13.42(13.42) \times 10^{-10}\ , (1.9\%).$$

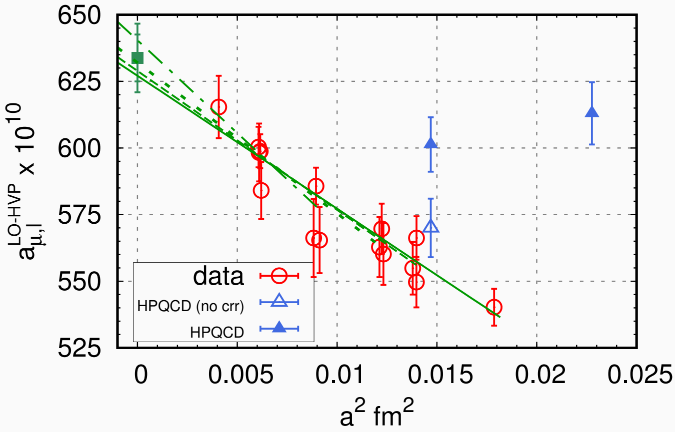
More detailed comparison





- BMWc '17 *ud* contribution is signficantly larger than other Nf=2+1+1 results
  $\rightarrow$ difference with HPQCD '14/Mainz '17 is $\sim 2.4/1.5\sigma$.
- BMWc '17 *c* contribution is slightly smaller than other Nf=2+1+1 results
- BMWc '17 is only calculation performed directly at physical quark masses with 6 β's to fully control continuum extrapolation.
- BMWc '17 $\delta a_{\mu,\,\text{disc}}^{\text{LO-HVP}} = 1.5 \times 10^{-10}$
  $\rightarrow$ contributes only 0.2% to error on $a_\mu^{\text{LO-HVP}}$.

Comparison of Light Conn. Component: $a_{\mu,ud}^{\text{LO-HVP}}$



$\hat{\Pi}^{lat}(Q^2)$ vs $\hat{\Pi}^{pheno}(Q^2)$ for Various Q^2

$$\hat{\Pi}^{lat}(Q^2) = \lim_{a\to 0}\sum_{t=0}^{T/2}\left[t^2 - \left(\frac{\sin Qt/2}{Qt/2}\right)^2\right]\frac{C_{ii}^f(t)}{3}\,, \quad \hat{\Pi}^{pheno}(Q^2) = \frac{Q^2}{12\pi^2}\int_0^\infty ds\,\frac{R_{had}(s)}{s(s+Q^2)}\,.$$

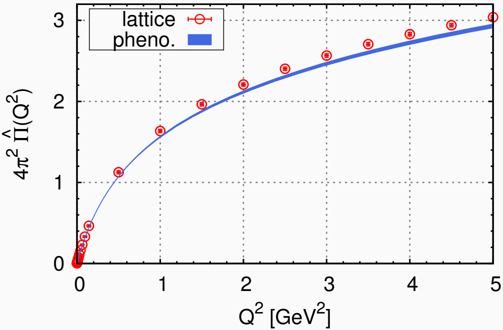
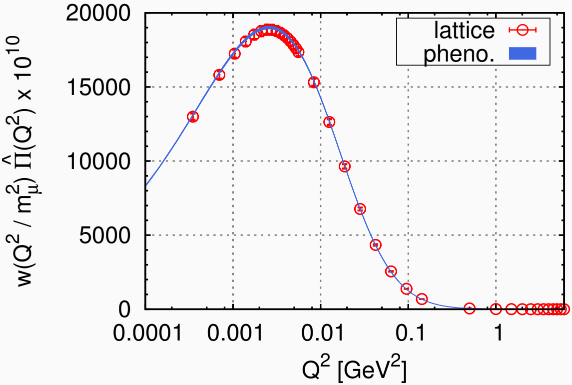


Figure: Lat. (BMW) vs Pheno. (Code alphaQCDc17 by Jegerlahner) for $\hat{\Pi}(Q^2)$.

Lat-Pheno Comp. w. Kernel: $\omega(Q^2/m_\mu^2)\hat{\Pi}(Q^2)$



The contributions at $Q^2 \sim (m_\mu/2)^2$ are dominante, and the lattice and phemenology are consistent there.

Lat-Pheno-Ratio: $\hat{\Pi}^{lat}/\hat{\Pi}^{pheno}$

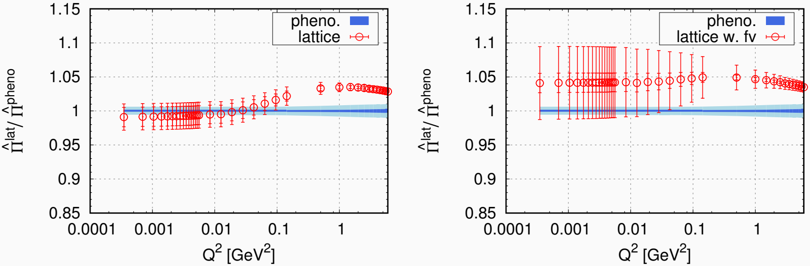


Figure: Lat-Pheno-Ratio $\hat{\Pi}^{lat}/\hat{\Pi}^{pheno}$ as a function of Q^2. Left/Right: With/Without FV.

FV is larger at smaller Q^2 as expected. As a result, the ratio becomes flat and larger than unity in whole Q^2. This results in $a_{\mu,lat}^{\text{LO-HVP}} \gtrsim a_{\mu,pheno}^{\text{LO-HVP}}$ as we have seen.