Flavor's Delight

対称性をツールとして
フレーバー模型を探る

谷本盛光（新潟大学）

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Flavor Frontier 2026

目次

1　フレーバー問題

20世紀の初めの素粒子　　electron, proton, photon

中性子の発見　1932　　チャドウィック

μ粒子の発見 ② 1937年 PR52(1937)1003 霧箱

Carl David Anderson, Seth Henry Neddermeyer

電子と同じ電荷、 電子より重い新粒子

Who ordered that ? I.I. Rabi

Isidor Issac Rabi

π中間子の発見　1947年

ニュートリノ(電子型)の発見　　1953年　原子炉

ストレンジネス(K, Λ)の発見 ②　1953年　泡箱

中野-西島-ゲルマンの法則

1955年　坂田モデル　基本粒子は(P, N, Λ)

1956年　Lee-Yang　Parityの破れの予想

1957年　Wuの実験　Parityの破れの実証

1959年　小川修三、山口嘉夫
3個の基本粒子 の入れ替えで強い相互作用は変わらない
池田、大貫、小川　IOO理論　U(3)対称性

1960年　**名古屋モデル** **三つのバリオンとレプトンの対称性**
（P N Λ）（ν e μ）

1962年　**Neutrino（muon type）の発見**　**（p n p'λ）（ν_e e ν_μ μ）**
新名古屋モデル（MNS混合)

しかし　Cabibbo Angleには到達できなかった

Octetway (八道説)

1964年　**クォークモデル (u, d), s**　**Ω粒子の発見**

CP対称性の破れの発見　**Higgs粒子の予言**　....

1968年ころ　“フレーバー”　**“カラー”1973年**

1972年　小林・益川理論

1974年　チャームクォークの発見

1977年　ボトムクォークの発見

1987年　ARGUS　B_d–$\bar{B}_d$　混合　$P(B_d \to \bar{B}_d) / P(B_d \to B_d) \sim 20\%$　予想よりかなり大きい

当時トップクォークの質量は30GeV程度と想定

1995年　トップクォークの発見

1998年　ニュートリノ振動の発見

ゲージ対称性による電磁気力と弱い相互作用の統一

電磁気と弱い相互作用はあまりにも違いすぎる

パリティの破れ、　CP対称性の破れ

Partial-symmetries of weak interactions

Glashow 1961年

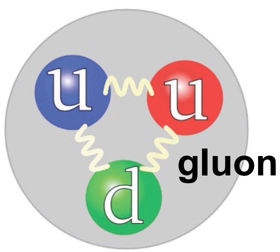
A Model of Leptons

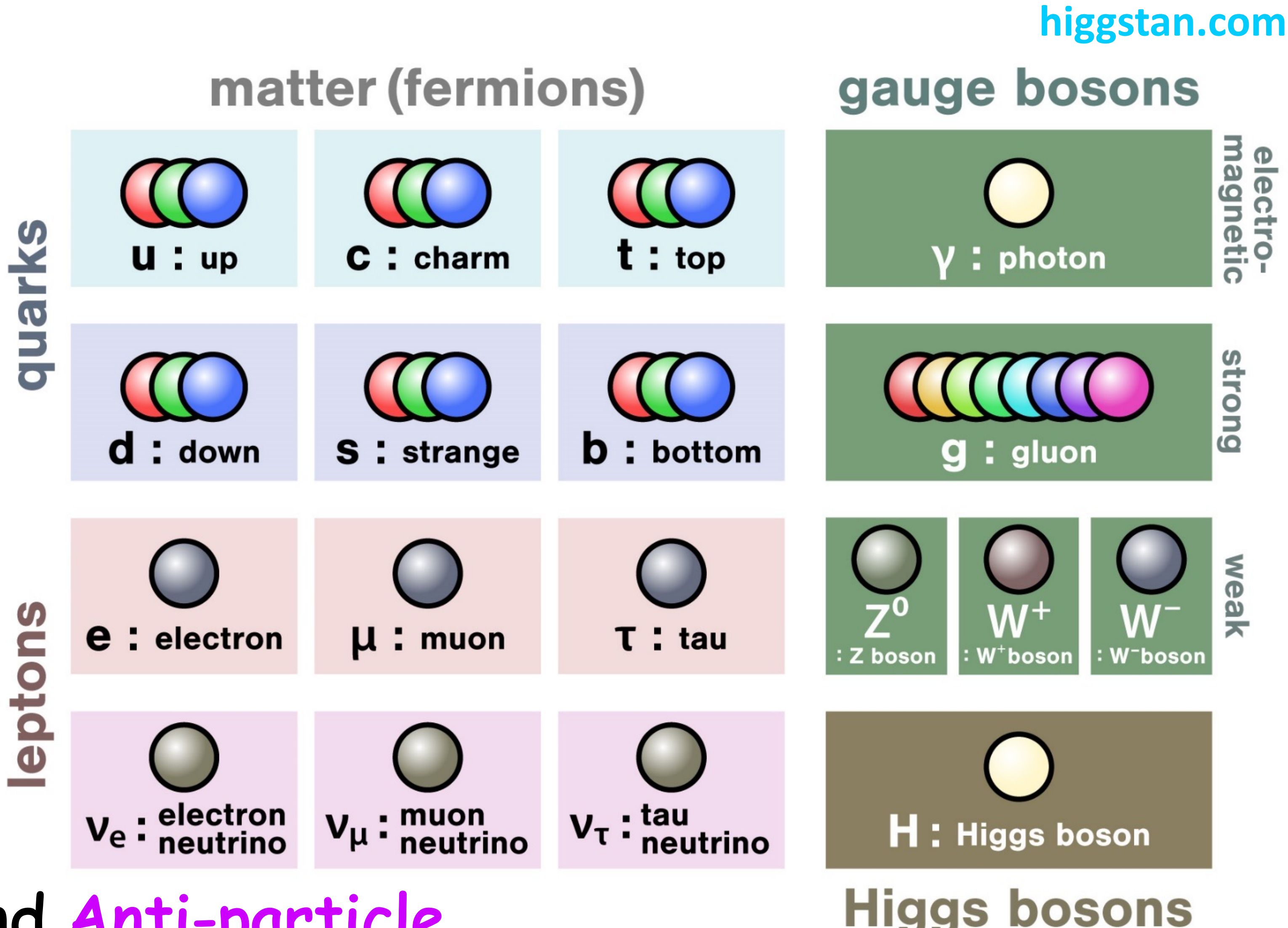
S.Weinberg　1967年

A.Salam　1968年

Particles in the Standard Model

Proton





and Anti-particle

何を理解したいか

世代の起源が知りたい
そうするとフレーバーを支配する法則がわかる

"Although nature commences with reason and ends in experience, it is necessary for us to do the opposite, that is to commence with experience and from this to proceed to investigate the reason."

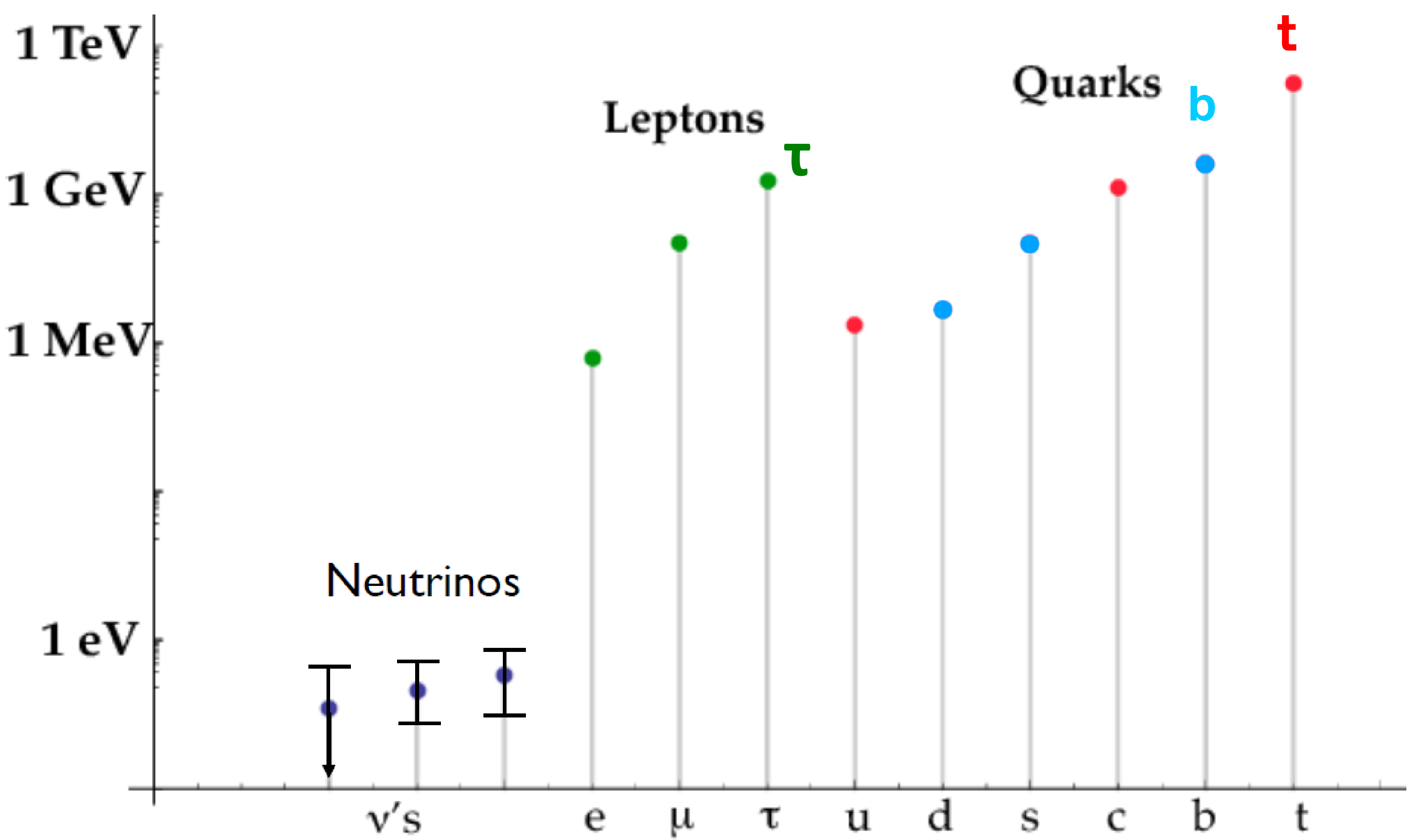
自然は理に始まり経験に終わるが、私たちはその逆を行う必要がある。
つまり、経験から始め、そこから理を探究しなければならない。

Leonardo da Vinci

そのために, これまでの実験事実からモデルを作り考察する。

質量スペクトル
フレーバー混合
CPの破れ

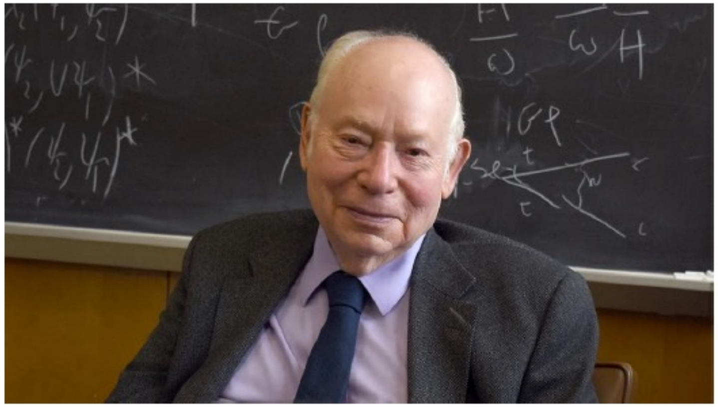
Masses



S. King 2022

CERN Courier 13 Oct. 2017

Steven Weinberg: ``Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesn't have to think for long: he wants to be able to explain **the observed pattern of quark and lepton masses**. …''



PHYSICAL REVIEW D **101,** 035020 (2020)

Models of lepton and quark masses

Steven Weinberg*

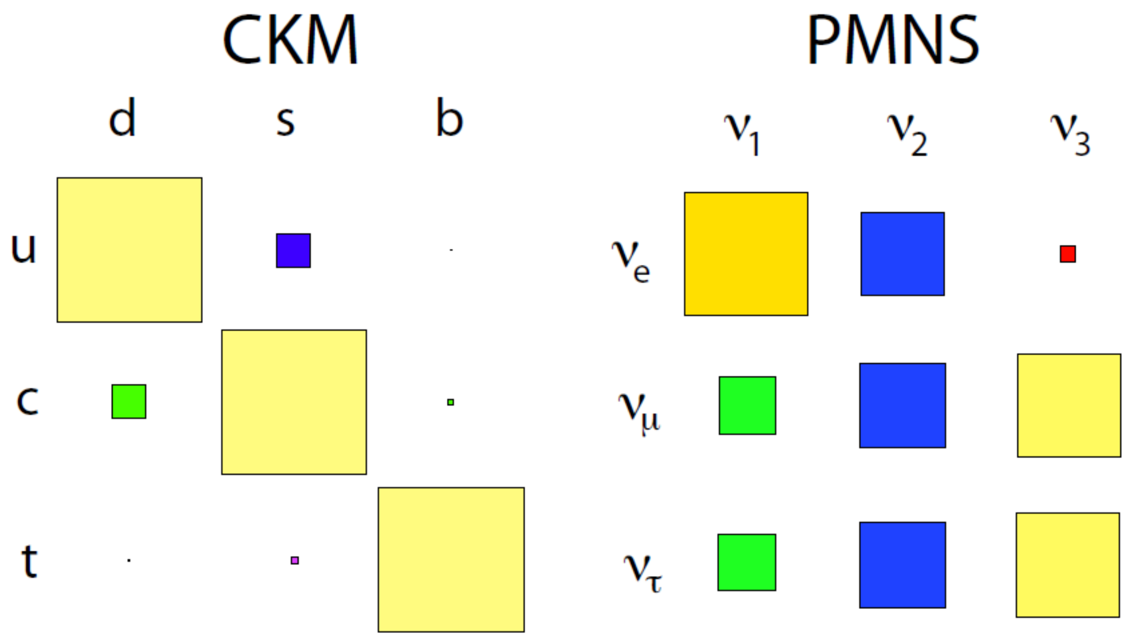
Theory Group, Department of Physics, University of Texas, Austin, Texas 78712, USA

(Received 15 December 2019; accepted 27 January 2020; published 19 February 2020)

A class of models is considered in which the masses only of the third generation of quarks and leptons arise in the tree approximation, while masses for the second and first generations are produced respectively by one-loop and two-loop radiative corrections. So far, for various reasons, these models are not realistic.

Can we understand Flavor mixing?

Mixing



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2　クォークの質量とカビボ角の理解に向けて

任意の2 x 2 Mass Matrix

$$M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

parameter数を減らす　A=0 by symmetry?　**U(1)ではできない**

Brane in extra dim.

mixing angleを予言できる　non-invertible selection rule

Modular zeros

parameterの大きさを制限する　B << C by symmetry?

Frogatt-Niesen

mass hierarchyを予言できる　Modular symmetry

Texture zeros approach

Two generations of quarks

- In the basis in which the up-type quark mass matrix is diagonal,

S. Weinberg, Trans. New York Acad. Sci. 38 (1977) 185

$$m_{\text{down}} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix}$$

2 parameters

$$\xrightarrow{M \gg m} \begin{pmatrix} m^2/M & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} m_d & 0 \\ 0 & m_s \end{pmatrix}$$

one prediction

The Cabibbo angle is successfully predicted to be

$$\sin\theta_c \sim \tan\theta_c = m/M = \sqrt{m_d/m_s}$$

GST relation (1968)

S. Weinberg (1977)

R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B 28 (1968) 128.

- How do we derive Yukawa interaction $Y_{ij}\psi_i^L\psi_j^R H$ with

$$Y_{ij} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$$

- Ex., let us consider $U(1)$ or $\mathbb{Z}_N$ symmetry
- U(1) or $\mathbb{Z}_N$ charges for fermions and Higgs should satisfy

5 charges

Q_1^L Q_2^L Q_1^R Q_2^R Q_H

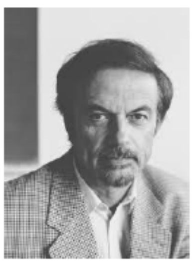
$$\boxed{\begin{aligned} Q_1^L + Q_1^R + Q_H &\neq 0 \\ Q_1^L + Q_2^R + Q_H &= 0 \\ Q_2^L + Q_1^R + Q_H &= 0 \\ Q_2^L + Q_2^R + Q_H &= 0 \end{aligned}}$$

No solutions!

Texture zeros approach

Three generations of quarks

- Fritzsch extended this approach to the three family case (1978)



$$m_{u,d}^{(\mathrm{Fritzsch})} = \begin{pmatrix} 0 & a & 0 \\ a^* & 0 & b \\ 0 & b^* & c \end{pmatrix}$$

leading to various relations between masses and mixing angles

See for systematic approach with four or five zeros for symmetric
or hermitian quark mass matrices by Ramond-Roberts-Ross (1993))

 These textures are not viable today under the precise experimental data

"Occam's Razor" approach for quark mass matrices

For quark masses and CKM, Tanimoto and Yanagida arXiv:1601.04459

Three zeros of down-type quark mass matrix

Diagonal basis of up-type quarks Mu

$$M_D = \begin{pmatrix} 0 & a_D & 0 \\ a'_D & b_D e^{-i\phi} & c_D \\ 0 & c'_D & d_D \end{pmatrix}, \qquad M_U = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

9 real parameters and one phase = CKM phase

Lepton sector

Successful neutrino mass matrix with 2 zeros

P.H. Frampton, S.L. Glashow and D. Marfatia, Phys. Lett. B536 (2002) 79.

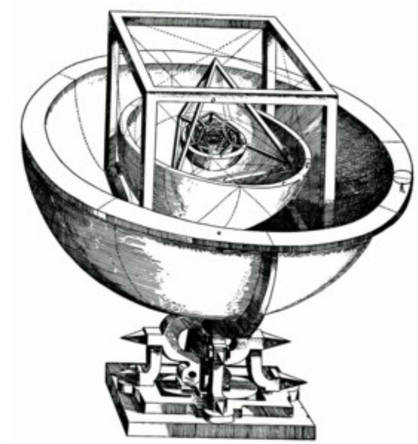
ダイナミックスがわからないとき
対称性によるアプローチは有効である。

対称性は美しい

対称性は窮屈である

対称性は破れる

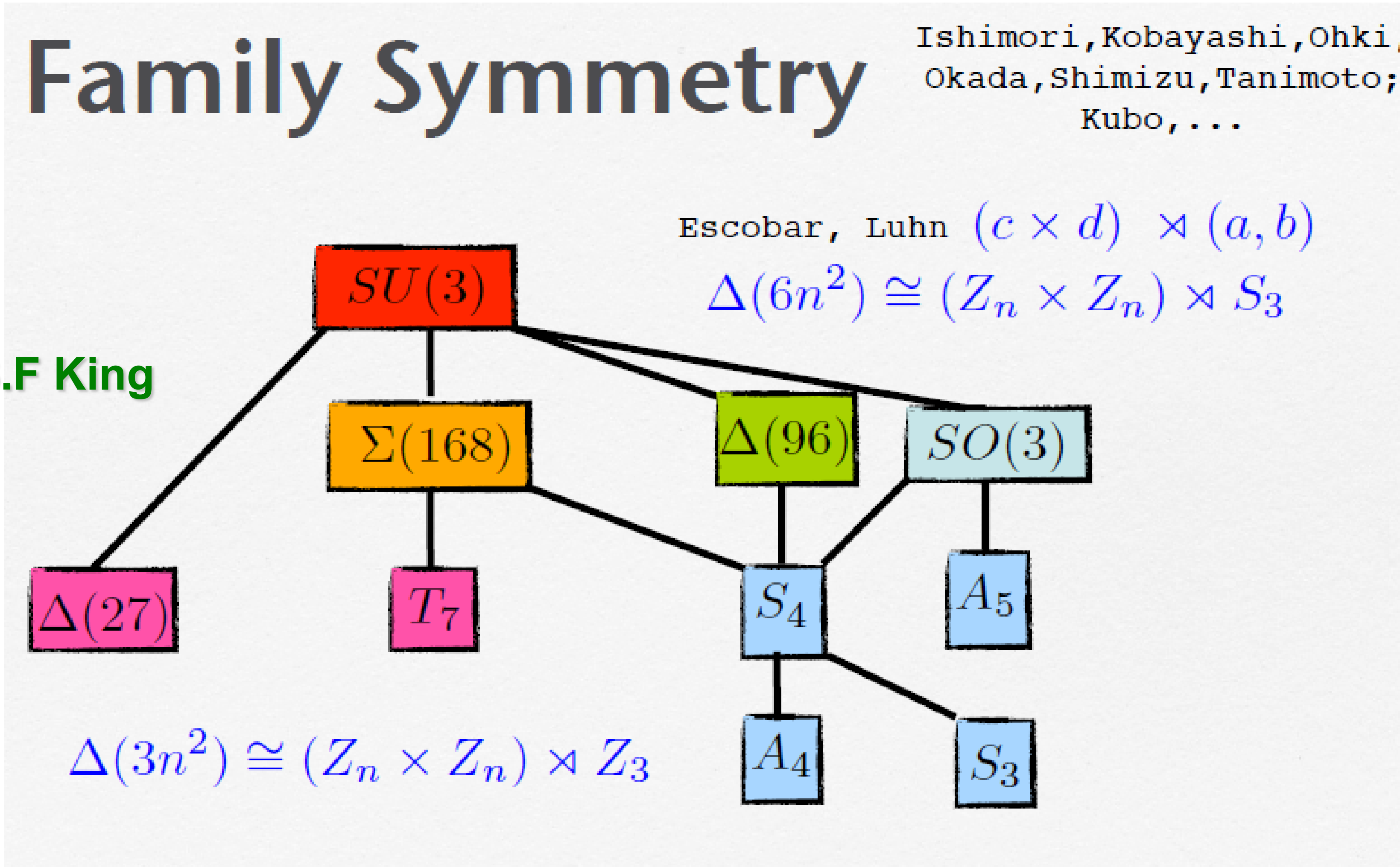
あらわ破れか自発的破れか



ケプラーの惑星の軌道を支える正多面体

Subgroups of SU(3)

by S.F King



C.Luhn, JHEP 1103, 108(2011) arXiv:1101.2417
A.Merle, R. Zwicky, JHEP, 1202 128(2012) 1110.4891

フレーバーに対称性を直接適用する

Discrete Symmetry and Cabibbo Angle, 2 generations (Doublet)

Phys. Lett. 73B (1978) 61, S.Pakvasa and H.Sugawara

S_3 symmetry is assumed for the Higgs interaction with the quarks and the leptons, and for the self-coupling of the Higgs bosons.

S_3: 1, 1', 2

$$\overset{\mathbf{2}}{\left\{\begin{pmatrix} p_1 \\ n_1 \end{pmatrix}_L, \begin{pmatrix} p_2 \\ n_2 \end{pmatrix}_L\right\}} \Bigg| \; \{p_{1R}\}, \{p_{2R}\}, \overset{\mathbf{2}}{\{n_{1R}, n_{2R}\}}$$

Higgs bosons, we introduce one S_3 singlet $\{\phi_0\}$ and one S_3 doublet $\{\phi_1, \phi_2\}$ bosons.

$$\langle\phi_1\rangle_0 = \xi_1 \cos\alpha\, e^{i\psi}, \quad \langle\phi_2\rangle_0 = \xi_1 \sin\alpha\, e^{i\phi}, \quad \langle\phi_0\rangle_0 = \xi_0, \qquad \alpha = \pi/4, \quad \phi + \psi = 0,$$

$$\text{Md} = \frac{g_2\xi_1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & e^{-i\phi} \\ e^{-i\phi} & -e^{i\phi} \end{pmatrix} \qquad \text{Mu} = \frac{f_2\xi_1}{\sqrt{2}} \begin{pmatrix} 0 & e^{i\phi} \\ 0 & e^{-i\phi} \end{pmatrix}$$

$$\boxed{\tan\theta_C = \tan(\phi - \pi/4) \simeq m_d/m_s}$$

SU(2)$_F$

Progress of Theoretical Physics, Vol. 61, No. 5, May 1979

Gauge Symmetry of Horizontal Flavour

Toshinobu MAEHARA and Tsutomu YANAGIDA*)

Department of Physics, Hiroshima University, Hiroshima 730

(Received January 19, 1979)

A detailed discussion on CP violation, the electric-dipole moment of the neutron and several muon-number changing processes is given based on the $\boxed{SU(2)\times U(1)\times SU_F(2)}$ gauge model, where $SU_F(2)$ corresponds to the symmetry of horizontal flavour. The model contains the neutral gauge bosons S_μ^a of a new type which carry the electronic and muonic numbers. We have obtained, from the present experimental data, the following bounds on the effective coupling constant G_S of S_μ^a with leptons and quarks: $3\times10^{-11}<G_S/G_F<5\times10^{-5}$ and, if $U_L^f=U_R^f$, $4\times10^{-6}<G_S/G_F<5\times10^{-5}$.

on the CP violation in $K_L{}^0\to2\pi$ decay, as $G_S/G_F>3\times10^{-11}$

大きなニュートリノ混合角の発見

Tri-Bi-Maximal

Harrison, Perkins, Scott (2002)

$\sin^2\theta_{12} = 1/3$, $\sin^2\theta_{23} = 1/2$

$$U \sim \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{aligned} \nu_2 &= \text{tri-maximal mixture of } \nu_e, \nu_\mu, \nu_\tau \\ \nu_3 &= \text{bi-maximal mixture of } \nu_\mu, \nu_\tau \end{aligned}$$

$$M_\nu^{\exp} \simeq V_{\rm tri-bi}^{*} \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} V_{\rm tri-bi}^{\dagger}$$

$$= \boxed{\frac{m_1+m_3}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}}$$

A_4 symmetric

- integer (inter−family related) matrix elements

$\Longleftrightarrow$ non−abelian discrete flavor sym

Mixing angles are independent of mass eigenvalues

A_4 symmetry considered by E. Ma and G. Rajasekaran 2001

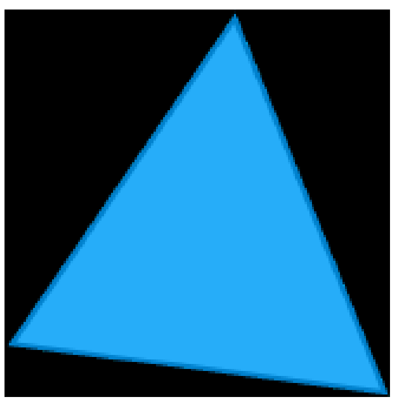
$$\left(\theta_{ij} \cancel{\times} \sqrt{\frac{m_i}{m_j}}\right)$$

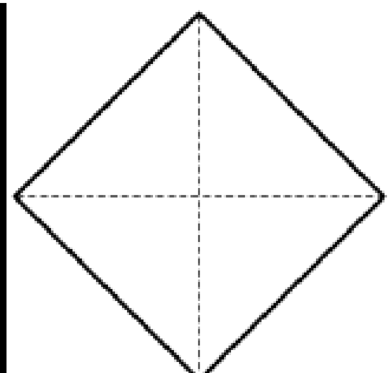
Different from quark mixing angles

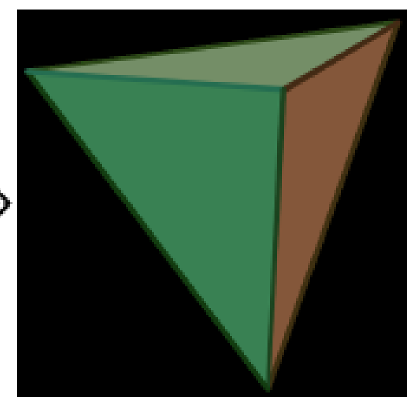
素粒子のフレーバーを支配する対称性があるのではないか？

どのような対称性？

離散群に注目が集まるが, それが選ばれる理由はわからない







S_3 **(1,1',2)**

A_4 **(1,1',1'',3)**

3　フレーバーがモジュラー形式と出会う

Are Yukawa couplings (masses) modular forms ?

F. Feruglio, arXiv:1706.08749

Many challenges began in flavor physics after 2018

Large lepton mixing angles
Quark mixing angles
Lepton CP violation
Quark / Lepton mass hierarchy
…………

「フェルマーの最終定理」の証明にはモジュラー形式が重要な役割をもつ。

Michael Ratz
@FLASY2026

what does Fermat's Last Theorem say?
$a^n + b^n = c^n$ has no integer solutions for $n > 2$

Fermat (1637)

how was this theorem proven?

- if there were solutions for $n > 2$ this would imply the existence of certain "weird" modular forms
- Taniyama–Shimura conjecture: these "weird" modular forms cannot exist

proved by Wiles (1994)

Modular zeros   Michael Ratz (FLASY2026)

Modular Zeros

Gaps in Modular Forms for $\Gamma'_3 \cong T'$

$$T' = \langle \mathsf{S}, \mathsf{T} \mid \mathsf{S}^4 = (\mathsf{S}\,\mathsf{T})^3 = \mathsf{T}^3 = \mathsf{S}^2\,\mathsf{T}\,\mathsf{S}^{-2}\,\mathsf{T}^{-1} = \mathbb{1} \rangle$$

☞ irreps s: 3-plet $\mathbf{3}$, 3 2-plets $\mathbf{2}$, $\mathbf{2'}$ & $\mathbf{2''}$, 3 1-plets $\mathbf{1}$, $\mathbf{1'}$ & $\mathbf{1''}$

$k_{\mathcal{Y}}$	VVMFs $\mathcal{Y}_s^{(k_{\mathcal{Y}})}$
1	$\mathcal{Y}^{(1)}_{\mathbf{2''}}$
2	$\mathcal{Y}^{(2)}_{\mathbf{3}}$
3	$\mathcal{Y}^{(3)}_{\mathbf{2'}}$, $\mathcal{Y}^{(3)}_{\mathbf{2''}}$
4	$\mathcal{Y}^{(4)}_{\mathbf{1}}$, $\mathcal{Y}^{(4)}_{\mathbf{1'}}$, $\mathcal{Y}^{(4)}_{\mathbf{3}}$
5	$\mathcal{Y}^{(5)}_{\mathbf{2}}$, $\mathcal{Y}^{(5)}_{\mathbf{2'}}$, $\mathcal{Y}^{(5)}_{\mathbf{2''}}$
6	$\mathcal{Y}^{(6)}_{\mathbf{1}}$, $\mathcal{Y}^{(6)}_{\mathbf{3}\mathrm{I}}$, $\mathcal{Y}^{(6)}_{\mathbf{3}\mathrm{II}}$
$\vdots$	$\vdots$

$k \backslash s$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{1''}$	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2''}$	$\mathbf{3}$
0	✓	∅	∅	∅	∅	∅	∅
1	∅	∅	∅	∅	∅	✓	∅
2	∅	∅	∅	∅	∅	∅	✓
3	∅	∅	∅	∅	✓	✓	∅
4	✓	✓	∅	∅	∅	∅	✓
5	∅	∅	∅	✓	✓	✓	∅
6	✓	∅	∅	∅	∅	∅	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Modular Zeros

Weinberg's Mass Texture with a Modular Zero

- assign both the quark doublet & d-type quarks the representations $(1'', 1)$ under a modular T' & weight $-5/2$

modular zero

- Weinberg's mass texture can be reproduced as [Liu & MR (2025)]

$$M_{\text{modular zero}} = \begin{pmatrix} 0 & \mathcal{Y}^{(4)}_{1'} \\ \mathcal{Y}^{(4)}_{1'} & \mathcal{Y}^{(4)}_{1} \end{pmatrix}$$

- mass hierarchies

$$M_{\text{modular zero}} \xrightarrow{\operatorname{Im}\varrho>1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

$$\varepsilon := \mathrm{e}^{2\pi\,\mathrm{i}\,\varrho/3}$$

What is Modular form ?

楕円函数の拡張

$$\boxed{f(x) = \sin 2\pi x\,, \qquad \mathrm{T}: x \to x+1 \Rightarrow f(x+1) = f(x)}$$

shift-symmetry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a, b, c, d) \text{ are integer} \quad \text{and } ad - bc = 1$$

$$\boxed{\gamma: \; z \to \frac{az+b}{cz+d}}$$

z is complex

(Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{T}: z \to z+1$$

shift-symmetry

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathrm{S}: z \to -\frac{1}{z}$$

Modular form f(z) is defined by imposing three conditions

$$\gamma: \; z \to \frac{az+b}{cz+d}$$ Modular transformation

① $f(z)$ **is holomorphic** @ Im z> 0

② $f(z)$ **is holomorphic** @ $z \to i\infty$

k: weight

③ $\cancel{f\left(\frac{az+b}{cz+d}\right) = f(z)}$

Modular function only constant

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

保型因子

Automorphy factor

Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$f(z+1) = f(z)\,, \qquad f\left(-\frac{1}{z}\right) = z^k f(z)$$

a=1, b=1, c=0, d=1 a=0, b=1, c=-1, d=0

theta-function

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

Modular group

Three matrices construct ɣ

$$\gamma: \ z \to \frac{az+b}{cz+d}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}: \ f(z+1) = f(z)$$

z → z + 1

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}: \ f\left(\frac{1}{-z}\right) = (-z)^k f(z)$$

z → -1/z

$$I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}: \ f\left(\frac{-z}{-1}\right) = (-1)^k f(z)$$

⇒ k=even

$$S: \tau \longrightarrow -\frac{1}{\tau},$$
$$T: \tau \longrightarrow \tau + 1.$$

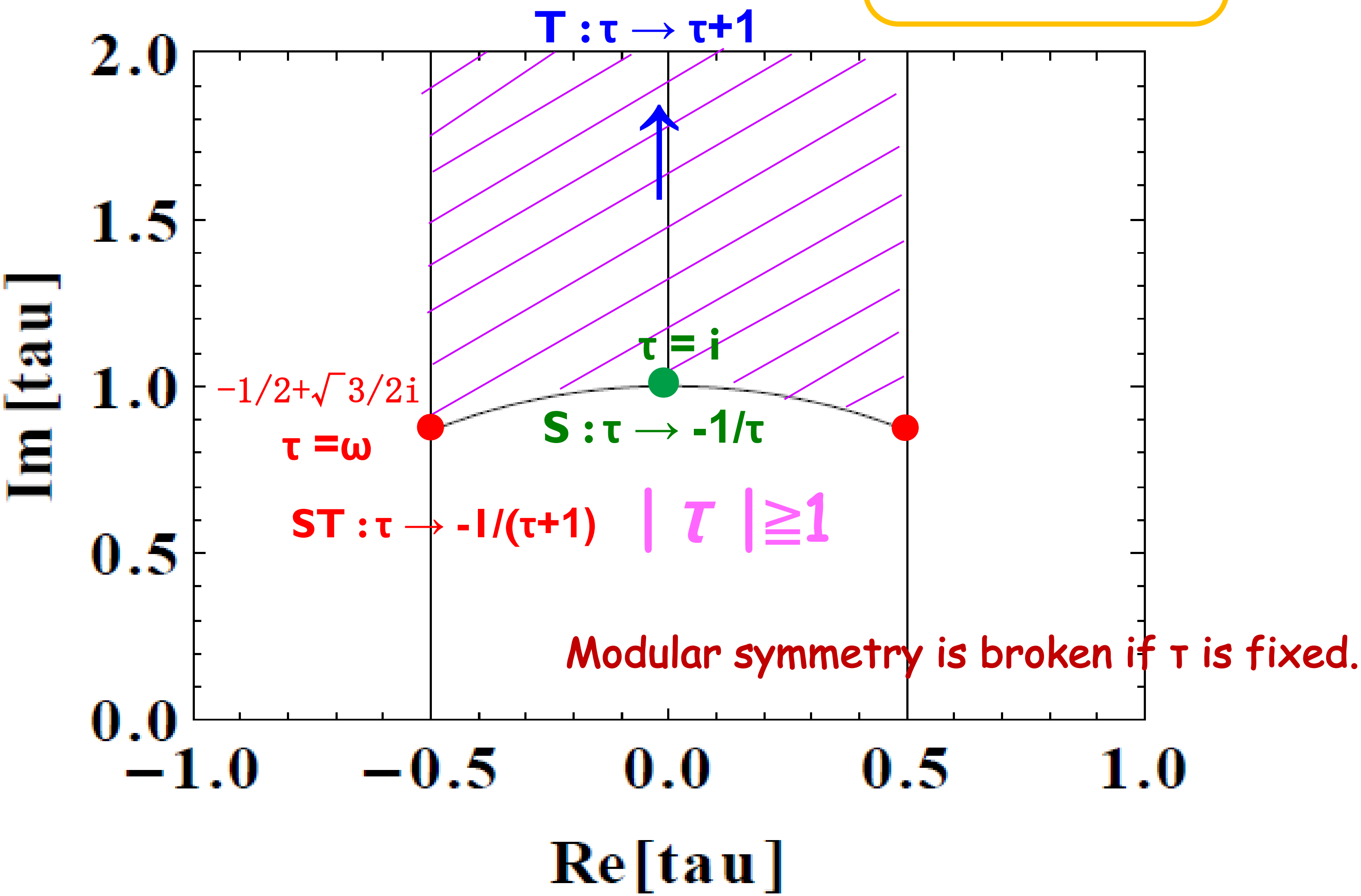
τ : modulus

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group
PSL(2,Z)

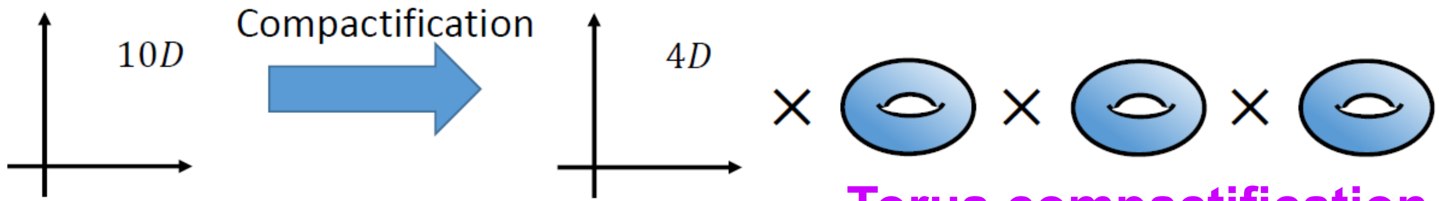
Fundamental Domain of τ

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$



● ● **Symmetric point of τ** **(Residual symmetry)**

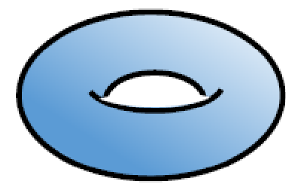
Modular forms appear naturally in top-down scenarios based on a class of string compactifications



Torus compactification

We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

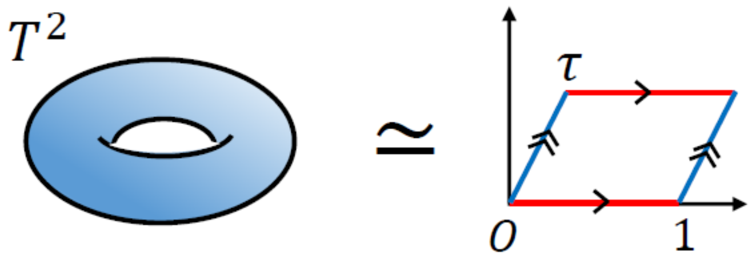
$\mathcal{L}_{\mathrm{eff}}$ depends on the structure of 

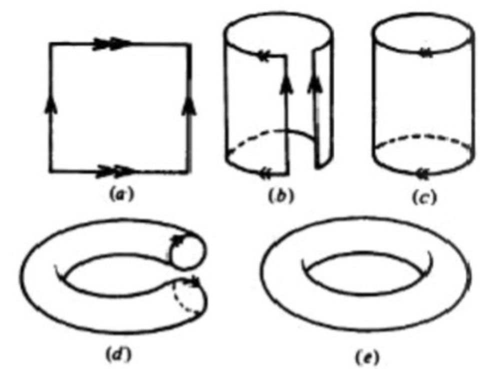
- $4D$ effective theory depends on internal space

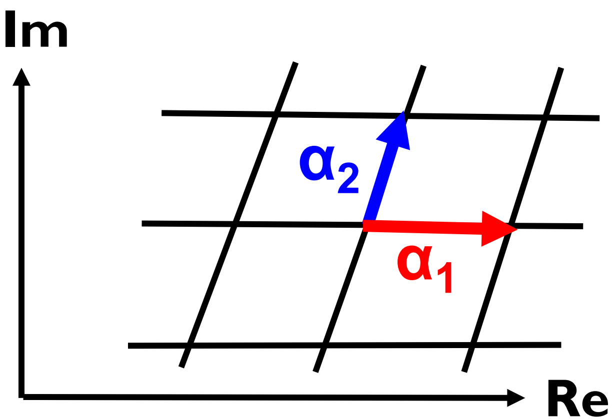
2 dim. torus has Modular symmetry

$2D$ torus (T^2) is equivalent to
parallelogram with identification of confronted sides.

by Feruglio







$(x,y) \sim (x,y)+n_1\alpha_1+n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice,
which is spanned by two lattice vectors

$\alpha_1 = 2\pi R$ **and** $\alpha_2 = 2\pi R\tau$

$\tau = \alpha_2 / \alpha_1$ **is a modulus parameter (complex).**

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1
a,b,c,d are integer SL(2,Z)

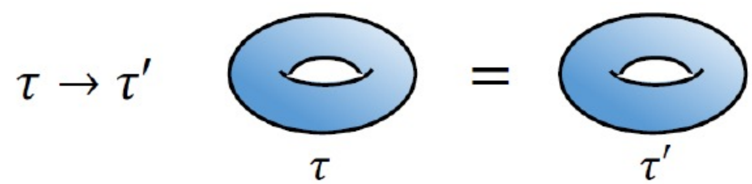
$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

ad-bc=1
a,b,c,d are integer

$\tau = \alpha_2 / \alpha_1$

$$\boxed{\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}}$$

Modular transformation



Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij} \phi \overline{\psi_i} \psi_j$

4D effective theory

Coupling is not constant !

- **depends on a modulus τ**
- **is independent under modular transformation**

An example

$$\mathcal{L}_1 = f(\tau)\phi_1\phi_2\cdots\phi_n$$

$f(\tau)$: coupling constant
ϕ_i: scalar fields

$$\boxed{\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}}$$

$$f(\tau) \xrightarrow{\gamma} \boxed{(c\tau + d)^k} f(\tau)$$

Modular form with $\boxed{\text{weight } k}$

$$\phi_i \rightarrow (c\tau + d)^{-k_i}\phi_i$$

Automorphy factor (保型因子)

When $k = \sum_i k_i$, $\mathcal{L}_1$ is modular invariant.

Generate finite modular group

Modular group
$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$ infinite discrete group

Modular group has subgroups

Impose
congruence condition $\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$

called principal congruence subgroups (normal subgroup)

$\Gamma_N \equiv \Gamma / \Gamma(N)$ quotient group finite group of level N

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_2 \simeq S_3$ $\Gamma_3 \simeq A_4$ $\Gamma_4 \simeq S_4$ $\Gamma_5 \simeq A_5$

isomorphic

例） レベル N=3 のモジュラー形式

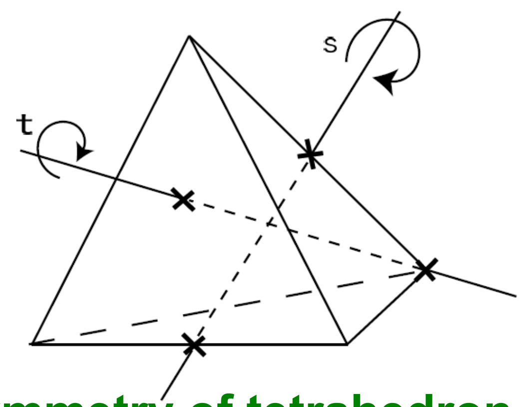
Consider simple case of mass matrix in A_4 symmetry N=3

A_4 assignments: left-handed doublet 3 right-handed singlets 1, 1”, 1’

A_4 group

Even permutation group of four objects (1234)
12 elements (order 12) are generated by
S and T: $S^2=T^3=(ST)^3=1$: S=(14)(23), T=(123)

Irreducible representations: 1, 1′, 1″, 3



Symmetry of tetrahedron

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular forms with weight 2

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right)
\end{aligned}$$

$$Y_2^2 + 2Y_1Y_3 = 0$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Dedekind eta-function

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$\boxed{q = e^{2\pi i\tau}}$$

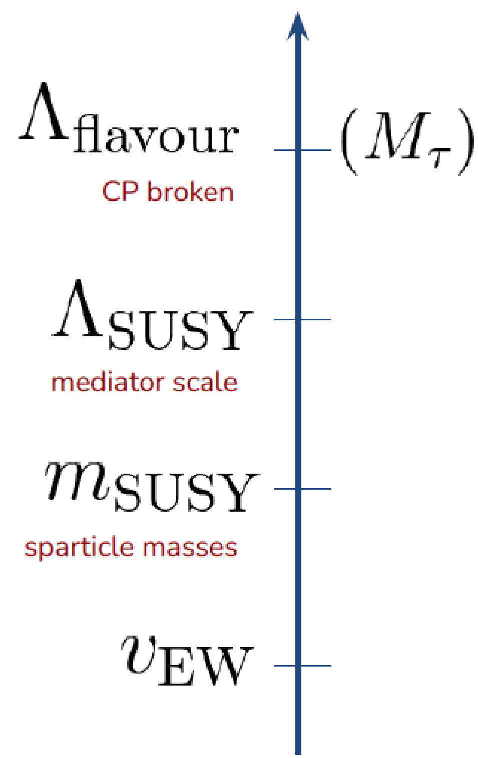
$$\rho(\mathrm{S}) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(i\frac{2}{3}\pi)$$

$$q_{\mathbf{1}} = u^c\,,\quad q_{\mathbf{1}"} = c^c\,,\quad q_{\mathbf{1}'} = t^c\,,\quad Q_{\mathbf{3}} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L\,,\quad Y_{\mathbf{3}}^{(k)}(\tau) = \begin{pmatrix} Y_1^{(k)}(\tau) \\ Y_2^{(k)}(\tau) \\ Y_3^{(k)}(\tau) \end{pmatrix}$$

$$\boxed{\alpha_u\, u^c_{\mathbf{1}}\, H\, Y_{\mathbf{3}}^{(k)}(\tau)\, Q_{\mathbf{3}} + \beta_u\, c^c_{\mathbf{1}"}\, H\, Y_{\mathbf{3}}^{(k)}(\tau)\, Q_{\mathbf{3}} + \gamma_u\, t^c_{\mathbf{1}'}\, H\, Y_{\mathbf{3}}^{(k)}(\tau)\, Q_{\mathbf{3}}}$$

Quark mass matrix

$$\boxed{M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}}$$



The modulus τ の関数

Modulus τ によりCP 対称性は破れる

CP transformation in modular invariant theory

P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov, JHEP 07(2019)165 [arXiv:1905.11970].

$$\tau \xrightarrow{\mathrm{CP}} -\tau^*\,, \qquad \psi(x) \xrightarrow{\mathrm{CP}} \overline{\psi}(x_P)\,, \qquad \mathbf{Y}^{(\mathrm{k})}_{\mathrm{r}}(\tau) \xrightarrow{\mathrm{CP}} \mathbf{Y}^{(\mathrm{k})}_{\mathrm{r}}(-\tau^*) = \mathbf{Y}^{(\mathrm{k})*}_{\mathrm{r}}(\tau)$$

bar denotes hermitian conjugation

We can construct CP invariant mass matrices in modular invariant flavor theory.

example

$$M_E(-\tau^*) = M_E(\tau)^*\,, \qquad M_\nu(-\tau^*) = M_\nu(\tau)^*$$

CP violation could be realized by fixing τ.

CP invariance in Lepton model

H.Okada, M.Tanimoto, JHEP 03(2021),010 [arXiv:2012.01688 [hep-ph]]

	L	(e^c,μ^c,τ^c)	H_u	H_d	$\mathrm{Y}_{\mathrm{r}}^{(2)}$, $\mathrm{Y}_{\mathrm{r}}^{(4)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1}'',\ \mathbf{1}')$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$, $\{\mathbf{3},\mathbf{1},\mathbf{1}'\}$
$-k_I$	-2	$(0,\ 0,\ 0)$	0	0	2, 4

Weight 4
Modular forms
3 + 1 + 1'

$$M_E = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

$Y^{(k)}(\tau)\ E^c\ L\ H_d$

$3\ \times 1^{('')}\times 3\ \times 1$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L \mathbf{Y}_{\mathrm{r}}^{(\mathrm{k})})_{\mathbf{1}}$$

Weinberg operator by using weight 4 modular forms

3 real parameters + τ

$$M_\nu = \frac{v_u^2}{\Lambda}\left[\begin{pmatrix} 2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\ -Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\ -Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} \end{pmatrix} + g_{\nu 1} \mathrm{Y}_{\mathbf{1}}^{(4)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\nu 2} \mathrm{Y}_{\mathbf{1}'}^{(4)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]_{LL}$$

3 x 3 x 3 (1,1')
L L Y

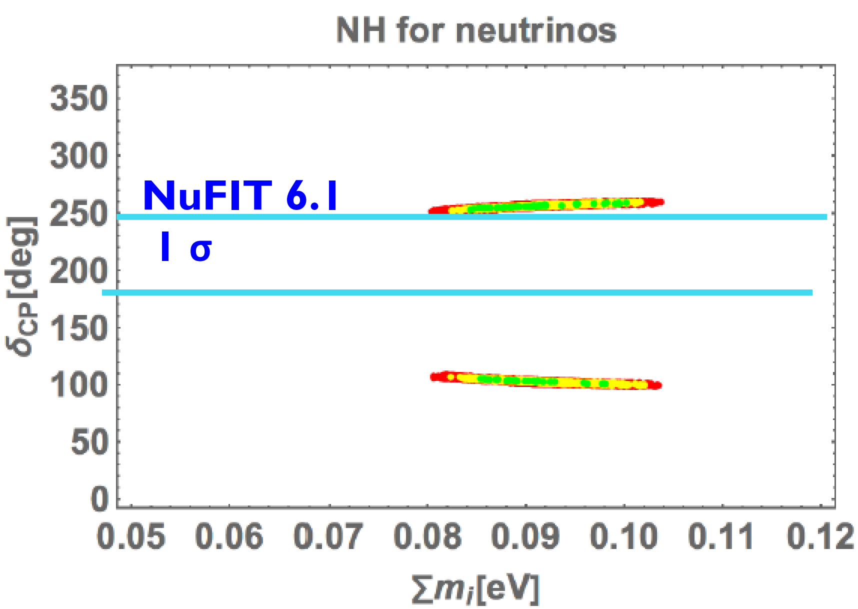
$$M_E(-\tau^*) = M_E(\tau)^*\,, \qquad M_\nu(-\tau^*) = M_\nu(\tau)^*$$

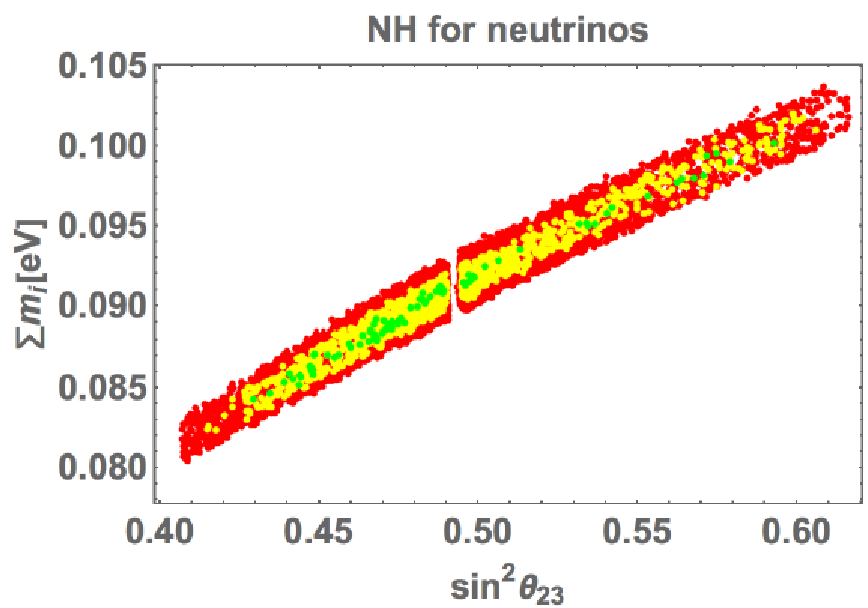
Parametes are 8; Input of *observed* values:

m_e , m_μ , m_τ , θ_{12} , θ_{23} , θ_{13} , Δm^2_{sol} , Δm^2_{atm}

output : δ_{CP} , $\langle m_{ee} \rangle$, $\sum m_i$

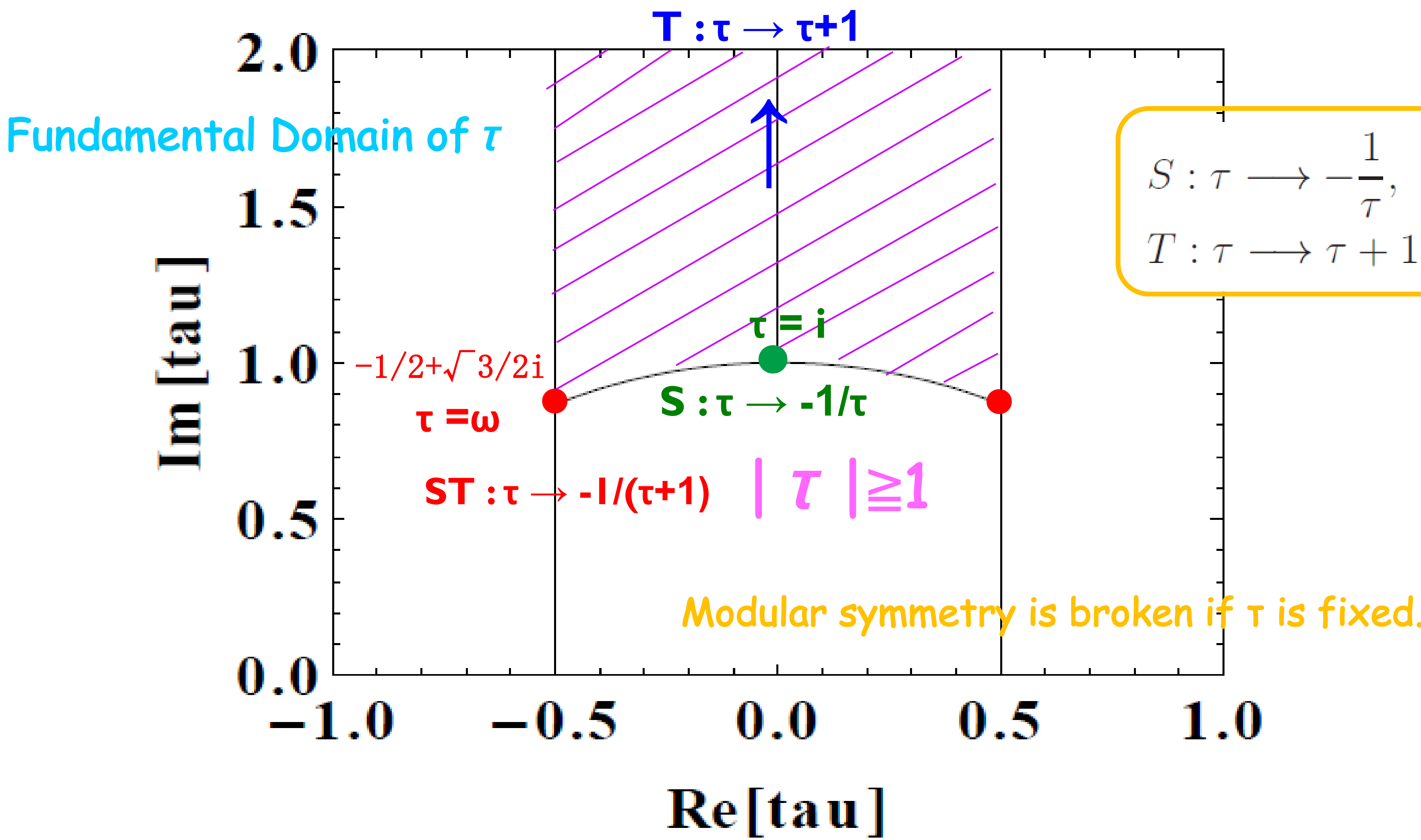
δ_{CP} is predicted clearly in $[98°, 110°]$ and $[250°, 262°]$ at 3σ confidence level.





τ は i 付近

Symmetric point 付近でのモジュラー形式からmass hierarchyが実現する



$$S: \tau \longrightarrow -\frac{1}{\tau},$$
$$T: \tau \longrightarrow \tau + 1.$$

● ● **Symmetric point of τ** **(Residual symmetry)**

Consider A_4 triplet modular forms with weigh k=2

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots ,\\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots),\\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$\varepsilon = 6\,|q|^{1/3}$

$\tau \rightarrow \infty i$ $\quad (Y_1, Y_2, Y_3)^T \longrightarrow (1, -\varepsilon, -1/2\,\varepsilon^2)^T \longrightarrow (1, 0, 0)^T$

A_4 triplet $\qquad |\varepsilon| << 1$

k=4 $\quad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{Y}_{\mathbf{1}}^{(4)} = Y_0^2, \quad \mathbf{Y}_{\mathbf{1'}}^{(4)} = 0,$

$$\rho(T) = \begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix}$$

Z_3 symmetry

k=6 $\quad \mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(6)} = Y_0^3,$

k=8 $\quad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(8)} = Y_0^4, \quad \mathbf{Y}_{\mathbf{1'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1''}}^{(8)} = 0$

We can construct the mass matrix with hierarchical masses by using the hierarchical modular forms towards $\tau = \infty i$ and ω

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}_{RL}$$

$\varepsilon = 6\,|q|^{1/3}$

This hierarchical structure is not accidental.

Thanks to Residual symmetry Z_3 (N=3)

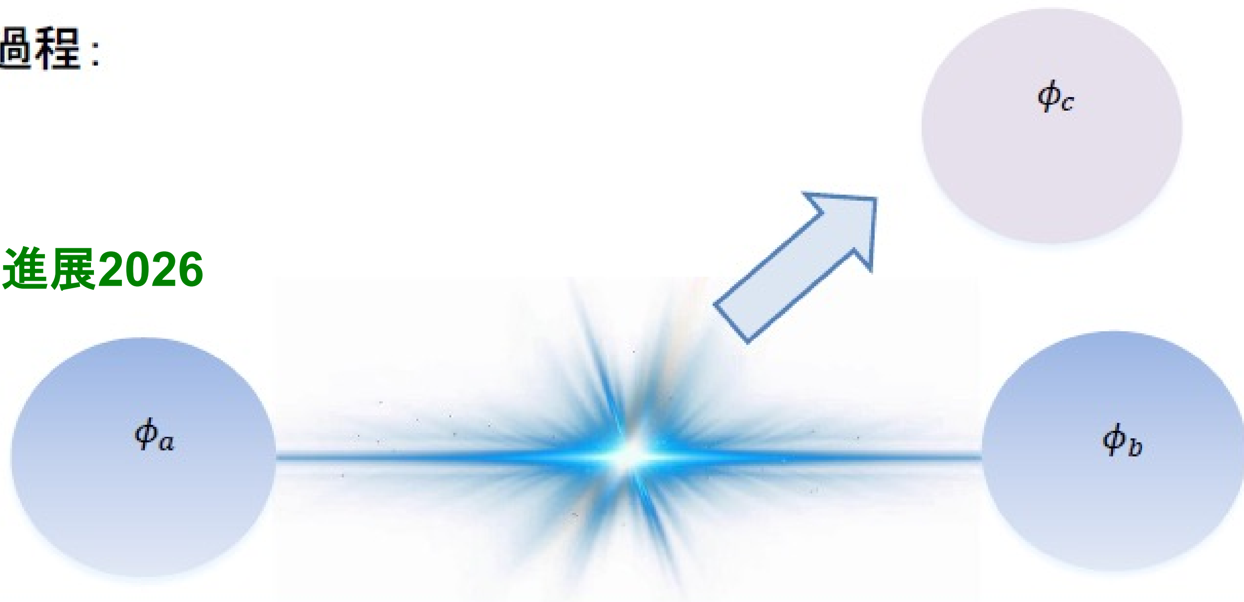
Challenge

Mass hierarchy を実現しながら τ によってCPをうまく破れるか

4　フレーバーがNon-invertible selection rules と出会う

素粒子の選択則

散乱過程：



H. Otsuka
素粒子物理学の進展2026

対称性に基づく「選択則」により、相互作用の有無が決定される。
（例えば、電荷の保存則：$\phi_a + \phi_b \to \phi_c$　$(q_a + q_b = q_c)$）

群論的対称性に由来する選択則（保存則）に紐づけられていたが、
近年対称性の概念が一般化され、非可逆対称性に基づく選択則が進展している

Non-invertible selection rules とは

復習　Group theory

Closure

If a and b are elements of group G, $c=a \circ b$ is also its element.

c is unique

each field corresponds to a (representation of) element of G

$$\phi_a + \phi_b \to \phi_c$$

Inverse process is realized

a b=1

Fusion Rule

Consider set of elements $\{U_i\}$

$$U_i U_j = \sum_k c_{ijk} U_k$$

Let us consider an example, $U_1 U_2 = U_3 + U_5$

$$\phi_1 + \phi_2 \to \phi_3 \qquad \phi_1 + \phi_2 \to \phi_5$$

Right-handed side is not unique.

Inverse operation cannot be defined !

These fusion rules may be originated from string compactification.

Ex: Type IIB magnetized D-brane models on toroidal orbifolds T.Kobayashi, H.Otsuka

Many examples in 2D CFT
E.P.Verlinde (1988),
G.W.Moore and N. Seiberg (1989),

- **Ordinary symmetry :**

  - Fusion rule of symmetry operator :

$$U_{g_1}U_{g_2} = U_{g_1 g_2} \qquad g_1, g_2 \in G$$

Ex., $U(1) : e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$$\boxed{U_\theta U_{-\theta} = 1}$$

- **Non-invertible symmetry :**

  - Fusion rule of symmetry operator :

Ex., $U_{\theta_1}U_{\theta_2} = U_{\theta_1+\theta_2} + U_{\theta_1-\theta_2}$

$U_\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ except for $\theta = 0, \pi$

For $\theta_2 = -\theta_1 = \theta$ $\quad \boxed{U_\theta U_{-\theta} = U_0 + U_{2\theta} = 1 + U_{2\theta}}$

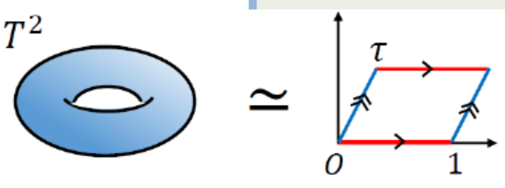
Non-invertible symmetries in type IIB string theory

T^2	$T^2/\mathbb{Z}_2$

- Operators: $e^{\frac{i}{M}\hat{D}_1}$

Z_M $\quad e^{\frac{i}{M}\hat{D}_2}$

H.Otuka



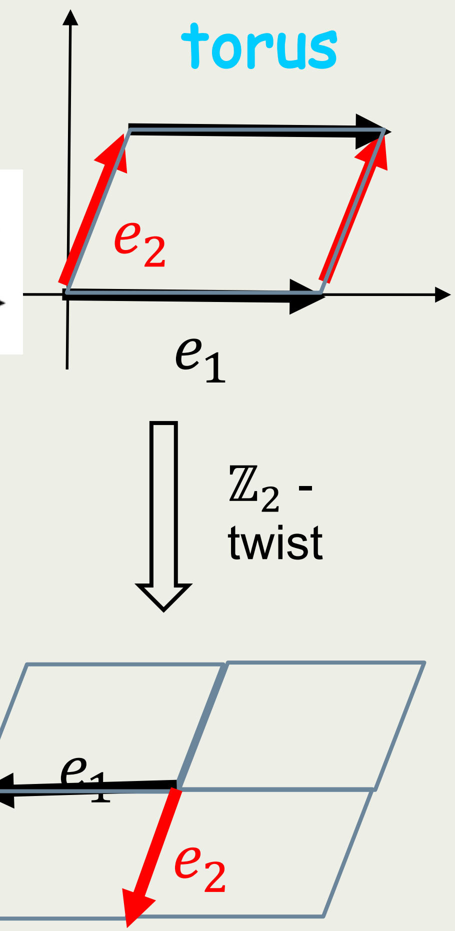
- Chiral matters:

$$\left|\psi^{j,M}\right\rangle$$

- Representations:

$$\mathbb{Z}_M^{(D_1)} : e^{\frac{i}{M}\hat{D}_1}\left|\psi^{j,M}\right\rangle = \left|\psi^{j+1,M}\right\rangle$$

$$\mathbb{Z}_M^{(D_2)} : e^{\frac{i}{M}\hat{D}_2}\left|\psi^{j,M}\right\rangle = e^{2\pi i\frac{j}{M}}\left|\psi^{j,M}\right\rangle$$



Non-invertible symmetries in type IIB string theory

	T^2	$T^2/\mathbb{Z}_2$
- Operators :	$e^{\frac{i}{M}\hat{D}_1}$ $e^{\frac{i}{M}\hat{D}_2}$	$\hat{U}_1 \equiv e^{\frac{i}{M}\hat{D}_1} + e^{-\frac{i}{M}\hat{D}_1}$ $\hat{U}_2 \equiv e^{\frac{i}{M}\hat{D}_2} + e^{-\frac{i}{M}\hat{D}_2}$
- Chiral matters:	$\lvert\psi^{j,M}\rangle$	$\mathbb{Z}_2$ -even : $\lvert\psi^{j,M}\rangle_+ = \frac{1}{2}\left(\lvert\psi^{j,M}\rangle + \lvert\psi^{M-j,M}\rangle\right.$
- Representations:	$\mathbb{Z}_M^{(D_1)} : e^{\frac{i}{M}\hat{D}_1}\lvert\psi^{j,M}\rangle = \lvert\psi^{j+1,M}\rangle$ $\mathbb{Z}_M^{(D_2)} : e^{\frac{i}{M}\hat{D}_2}\lvert\psi^{j,M}\rangle = e^{2\pi i\frac{j}{M}}\lvert\psi^{j,M}\rangle$	$\hat{U}_1 \lvert\psi^{j,M}\rangle_+ = \lvert\psi^{j+1,M}\rangle_+ + \lvert\psi^{j-1,M}\rangle_+$ $\hat{U}_2 \lvert\psi^{j,M}\rangle_+ = 2\cos\left(\frac{2\pi j}{M}\right)\lvert\psi^{j,M}\rangle_+$

Corresponding to the gauging the (outer) automorphism of each $\mathbb{Z}_M^{(D_1)}$ and $\mathbb{Z}_M^{(D_2)}$

$$j \quad \rightarrow \quad [j] \; : \; j \text{ and } M-j$$

例） Fibonacci fusion rule

$\{\ 1\ ,\ \ \tau\ \}$ Fibonacci category

anyon

フィボナッチ融合則（エニオンを記述）：

$1 \longrightarrow$

$\tau \longrightarrow$

$$\boxed{\begin{aligned} [g_0]\cdot[g_0] &= [g_0] \\ [g_0]\cdot[g_1] &= [g_1] \\ [g_1]\cdot[g_1] &= [g_0]\oplus[g_1] \end{aligned}}$$

- $[g_1]$ は逆元をもたない

一般化された選択則

T. Kobayashi, H. Otuka, M. Tanimoto, 2409.05270

- Let us start with $\mathbb{Z}_M$ *symmetry (generators : $g = e^{2\pi i/M}$)*

  Transformation of fields ϕ_j in 4D QFT : $\phi_j \to g^{k_j}\phi_j$

  Charge k_j under $\mathbb{Z}_M$

- Let us consider the automorphism of $\mathbb{Z}_M$, *i.e.*, $\boxed{D_M \simeq \mathbb{Z}_M \rtimes \mathbb{Z}_2}$

Group Gの自己同型群とは

G→G　f(g_1)・f(g_2)=f(g_1g_2)　group G'=Aut(G)≠G　を形成する。

f = h g h^{-1} の場合　内部自己同型という。それ以外は外部自己同型という。

D_M group

$$D_M \cong \mathbb{Z}_M \rtimes \mathbb{Z}_2$$

$g^M=e$ $r^2=e$ $rgr^{-1}=g^{-1}$

"conjugacy class": set of all elements $g^{-1}\, a\, g$ for $g \in G$

Z_2 invariant conjugacy class $ege^{-1} = g \quad rgr^{-1} = g^{-1}$

$$[g^k] = \boxed{\{hg^k h^{-1} \mid h = e, r\} = \{g^k, g^{M-k}\}} \qquad g^{-k} = g^{M-k}$$

Example: D_5 case 10 elements are classified by 4 classes

Z_2 invariant

$C_1=\{e\}$ $C_2^{(1)}=\{g, g^4\}$ $C_2^{(2)}=\{g^2, g^3\}$ $C_5=\{r, gr, g^2r, g^3r\ g^4r\}$

Labelling fields by class

$$\boldsymbol{\Phi_j} = \phi_j + \phi_{M-j}$$

In $\mathbb{Z}_2$ basis, they have both j and M-j $\mathbb{Z}_\mathbf{M}$ charges.

corresponding to $\mathbb{Z}_2$ invariant modes : $\boldsymbol{\Phi_j} = \phi_j + \phi_{M-j}$

labelling fields by class

No definite Z_M charge

★ ϕ_j **is not invariant under Z2 orbifolding** $\mathbf{T^{2n}/\mathbb{Z}_2}$

★ $\mathbf{\Phi_j}$ **has both charge j and M-j**

That is $\mathbb{Z}_2$ gauging of $\mathbb{Z}_M$ symmetry

一般化された選択則

H. Otsuka
素粒子物理学の進展2026

- Suppose that

1. 場 ϕ_i が $\boxed{hypergroup}$ の要素でラベルされる

> A *hypergroup* is an algebra $[g_i] \cdot [g_j] = \sum_k d_{ij}^k [g_k]$

- Ex., フュージョン代数 ($d_{ij}^k \in \mathbb{Z}$)
  群 G の共役類など

→ 場の相互作用に制限

Hamidi-Vafa (1987), Dijkgraaf-Vafa-Verlinde-Verlinde (1989),
Heckman-McNamara-Montero-Sharon-Vafa-Valenzuela, Kaidi-Tachikawa-Zhang (2024),...

2. 古典的なラグランジアンの相互作用は
場のラベルのフュージョン積にIdentityが現れると許される

一般化された選択則

Instead of labelling fields by **representations**, we label them by **class**:

$$[g^k] = \{\tilde{g}^k,\ \tilde{g}^{M-k}\}$$

$$\tilde{g}^{k_1}\tilde{g}^{k_2} = \tilde{g}^{k_1+k_2}, \quad \tilde{g}^{k_1}\tilde{g}^{-k_2} = \tilde{g}^{k_1-k_2},$$
$$\tilde{g}^{-k_1}\tilde{g}^{k_2} = \tilde{g}^{-k_1+k_2}, \quad \tilde{g}^{-k_1}\tilde{g}^{-k_2} = \tilde{g}^{-k_1-k_2}.$$

$$\boxed{[g^{k_1}][g^{k_2}] = [g^{k_1+k_2}] + [g^{M-k_1+k_2}]}$$

Suppose that Φ_j is labeld by a class $[g^{k_j}]$.
An interaction in the classical Lagrangian

$$\mathcal{L} \supset \Phi_1 \cdots \Phi_n$$

is allowed when $\tilde{g}^{k_1} \cdots \tilde{g}^{k_n} = e$ (identity) $\quad \tilde{g}^{k_j} \in [g^{k_j}]$

How do we understand the Yukawa texture without group-like structure?

$$Y_{ij} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$$

$M = 3$ ($\mathbb{Z}_2$ *gauging of* $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$)

T. Kobayashi, H.Otsuka, M. Tanimoto, 2409.05270

- Two different classes :

$$[g^0], \qquad [g^1] = \{\tilde{g}^1, \tilde{g}^2\}$$

$$[\boldsymbol{g}^{k}] \supset \tilde{\boldsymbol{g}}^{k}, \tilde{\boldsymbol{g}}^{3-k}$$

- *Fusion rule :*

$$[\boldsymbol{g}^{k_1}][\boldsymbol{g}^{k_2}] = [\boldsymbol{g}^{k_1+k_2}] + [\boldsymbol{g}^{M-k_1+k_2}]$$

$$[g^0][g^0] = [g^0]$$
$$[g^0][g^1] = [g^1]$$
$$[g^1][g^1] = [g^0] + [g^1]$$

- *Previous example for two generations of fermions:*

Left : $([g^0], [g^1])$
Right : $([g^0], [g^1])$
Higgs : $[g^1]$

$$Y = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$$

M = 5 ($\mathbb{Z}_2$ *gauging of* $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$)

- Three different classes :

$$[g^0], \qquad [g^1], \qquad [g^2]$$

$$[\boldsymbol{g}^{\boldsymbol{k}}] \supset \tilde{\boldsymbol{g}}^{\boldsymbol{k}}, \tilde{\boldsymbol{g}}^{\boldsymbol{5-k}}$$

- *Yukawa couplings for 3 generations of fermions :*

$$[\boldsymbol{g}^{\boldsymbol{k_1}}][\boldsymbol{g}^{\boldsymbol{k_2}}] = [\boldsymbol{g}^{\boldsymbol{k_1+k_2}}] + [\boldsymbol{g}^{\boldsymbol{M-k_1+k_2}}]$$

Fusion rules :

$[g^0][g^0] = [g^0]$

$[g^1][g^1] = [g^0] + [g^2]$

$[g^2][g^2] = [g^0] + [g^1]$

$[g^1][g^2] = [g^1] + [g^2]$

Left : $([\boldsymbol{g}^{\boldsymbol{0}}], [\boldsymbol{g}^{\boldsymbol{1}}], [\boldsymbol{g}^{\boldsymbol{2}}])$

Right : $([\boldsymbol{g}^{\boldsymbol{0}}], [\boldsymbol{g}^{\boldsymbol{1}}], [\boldsymbol{g}^{\boldsymbol{2}}])$

Higgs : $[\boldsymbol{g}^{\boldsymbol{1}}]$

$$Y_{[g^1]} = \begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & e \end{pmatrix}$$

Consider Direct product group theory

$\mathbb{Z}_2$ gaugings of $\mathbb{Z}_{\mathbf{M}} \times \mathbb{Z}_{\mathbf{M'}}$ symmetries

$$\mathbb{Z}_{\mathbf{M}} \times \mathbb{Z}_{\mathbf{M'}} \rtimes \mathbb{Z}_{\mathbf{2}}$$

Two multiplication rules

$$U_i U_j = \sum_k c_{ijk} U_k , \qquad U'_{i'} U'_{j'} = \sum_{k'} c'_{i'j'k'} U'_{k'} .$$

$$(U_i, U'_{i'})(U_j, U'_{j'}) = \sum_{k,k'} c_{ijk} c'_{i',j',k'} (U_k, U'_{k'}).$$

$\mathbb{Z}_5 \times \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ **3 classes** $\boxed{\{[g^0], [g^1], [g^2]\}}$

$$\begin{array}{lcl} Q_L & : & ([g_1^0][g_2^0],\ [g_1^1][g_2^1],\ [g_1^0][g_2^1]) \\ U_R & : & ([g_1^0][g_2^0],\ [g_1^1][g_2^1],\ [g_1^0][g_2^1]) \\ D_R & : & ([g_1^1][g_2^1],\ [g_1^2][g_2^2],\ [g_1^2][g_2^1]) \end{array}$$

Hu : $[g^0]\ [g^0]$, Hd : $[g^2]\,[g^2]$

Fusion rules :

$[g^0][g^0] = [g^0]$

$[g^1][g^1] = [g^0] + [g^2]$

$[g^2][g^2] = [g^0] + [g^1]$

$[g^1][g^2] = [g^1] + [g^2]$

$$M_D = \begin{pmatrix} 0 & a_D & 0 \\ a'_D & b_D e^{-i\phi} & c_D \\ 0 & c'_D & d_D \end{pmatrix}, \qquad M_U = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

which are obtained in “Occam’s Razor” approach

Textures of dim.6 operators in SMEFT

T. Kobayashi, H. Otuka, M. Tanimoto, K. Yamamoto, arXiv 2608.21707

$\mathbb{Z}_5 \times \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ 3 classes $\boxed{\{[g^0], [g^1], [g^2]\}}$

$$[g^0] \otimes [g^0] = [g^0], \qquad [g^0] \otimes [g^1] = [g^1] \otimes [g^0] = [g^1],$$
$$[g^0] \otimes [g^2] = [g^2] \otimes [g^0] = [g^2], \qquad [g^1] \otimes [g^1] = [g^0] + [g^2],$$
$$[g^1] \otimes [g^2] = [g^2] \otimes [g^1] = [g^1] + [g^2], \qquad [g^2] \otimes [g^2] = [g^0] + [g^1].$$

Let us take a simple set of quark and lepton mass matrices at cut off scale (compactification scale).

$$-\mathcal{L}_{\rm Yukawa} = Y_u \bar{q}_i \widetilde{\Phi}_u u_j + Y_d \bar{q}_i \Phi_d d_j + Y_e \bar{\ell}_i \Phi_d e_j + Y_{\mathcal{W}} \frac{\ell \widetilde{\Phi}_u \ell \widetilde{\Phi}_u}{\Lambda}$$

$$Y_u = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad Y_d = \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \quad Y_e = \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}, \quad Y_{\mathcal{W}} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Impose $[q_i] = [\ell_i], \quad [d_i] = [e_i]$ assingnment of Class

$\mathbb{Z}_5 \times \mathbb{Z}'_5 \rtimes \mathbb{Z}_2$

$$\begin{array}{lcl}
Q_L = L_L & : & ([g_1^0][g_2^0],\ [g_1^1][g_2^1],\ [g_1^0][g_2^1]) \\
U_R & : & ([g_1^0][g_2^0],\ [g_1^1][g_2^1],\ [g_1^0][g_2^1]) \\
D_R = e_R & : & ([g_1^1][g_2^1],\ [g_1^2][g_2^2],\ [g_1^2][g_2^1]) \\
H_u & : & [g_1^0][g_2^0]\,, \qquad\qquad H_d : [g_1^2][g_2^2]
\end{array}$$

Case (a)

There are other 4 independent assignments.

4 fermi Semi-leptonic operator $(L\bar{L})\,(L\bar{L})$

$$\boxed{\begin{aligned} C_{ijkl}Q^{(1)}_{\ell q} &= C_{ijkl}(\bar{\ell}^i_L\gamma_\mu \ell^j_L)(\bar{q}^k_L\gamma^\mu q^l_L), \\ C_{ijkl}Q^{(3)}_{\ell q} &= C_{ijkl}(\bar{\ell}^i_L\gamma_\mu \tau^I \ell^j_L)(\bar{q}^k_L\gamma^\mu \tau^I q^l_L). \end{aligned}}$$

Flavor basis

$$C^{(a)}_{ijkl} = \begin{pmatrix} [0]_1^2[0]_2^2 & [0]_1[1]_1[0]_2[1]_2 & [0]_1^2[0]_2[1]_2 \\ [0]_1[1]_1[0]_2[1]_2 & [1]_1^2[1]_2^2 & [0]_1[1]_1[1]_2^2 \\ [0]_1^2[0]_2[1]_2 & [0]_1[1]_1[1]_2^2 & [0]_1^2[1]_2^2 \end{pmatrix}_{\ell\ell} \begin{pmatrix} [0]_1^2[0]_2^2 & [0]_1[1]_1[0]_2[1]_2 & [0]_1^2[0]_2[1]_2 \\ [0]_1[1]_1[0]_2[1]_2 & [1]_1^2[1]_2^2 & [0]_1[1]_1[1]_2^2 \\ [0]_1^2[0]_2[1]_2 & [0]_1[1]_1[1]_2^2 & [0]_1^2[1]_2^2 \end{pmatrix}_{qq}$$

$$= \begin{array}{c c} & \begin{array}{ccc} l=1 & l=2 & l=3 \end{array} \\ \begin{array}{c} k=1 \\ \\ \\ k=2 \\ \\ \\ k=3 \end{array} & \left(\begin{array}{ccc|ccc|ccc} \checkmark & 0 & 0 & 0 & \checkmark & 0 & 0 & 0 & \checkmark \\ 0 & \checkmark & 0 & \checkmark & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \checkmark & 0 & 0 & 0 & \checkmark & 0 & 0 \\ \hline 0 & \checkmark & 0 & \checkmark & 0 & 0 & 0 & 0 & 0 \\ \checkmark & 0 & 0 & 0 & \checkmark & 0 & 0 & 0 & \checkmark \\ 0 & 0 & 0 & 0 & 0 & \checkmark & 0 & \checkmark & 0 \\ \hline 0 & 0 & \checkmark & 0 & 0 & 0 & \checkmark & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \checkmark & 0 & \checkmark & 0 \\ \checkmark & 0 & 0 & 0 & \checkmark & 0 & 0 & 0 & \checkmark \end{array}\right) \end{array}$$

21 non-zero entries

$$\boxed{C^{(a)}_{ijkl} = f_{ijkl}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)}$$

i, j = leptons
k, ℓ= quarks

59 dim.6 operators in Warsaw basis

Operator class	No symmetry, 3 Gen.		MFV $U(3)^5$, $\mathcal{O}(Y^1_{e,d,u})$		U(2) $U(2)^5$, $\mathcal{O}(V^1, \Delta^1)$	
	CP-even	CP-odd	CP-even	CP-odd	CP-even	CP-odd
X^3, H^6, H^4D^2, X^2H^2	9	6	9	6	9	6
ψ^2H^3	27	27	3	3	9	9
ψ^2XH	72	72	8	8	24	24
ψ^2H^2D	51	30	7	–	19	5
$(\bar{L}L)(\bar{L}L)$	171	126	8	–	40	17
$(\bar{L}L)(\bar{R}R)$	360	288	8	–	53	21
$(\bar{R}R)(\bar{R}R)$	255	195	9	–	29	–
$(\bar{L}R)(\bar{R}L)$ and $(\bar{L}R)(\bar{L}R)$	405	405	–	–	29	29
Total	$\boxed{1350}$	$\boxed{1149}$	52	17	212	111
Our case (a)	**693**	**953**				

Introducing Suprions

Consider a few examples of operators

not complehensive analyses

Charged current

$$C_{ijkl}Q^{(3)}_{\ell q} = C_{ijkl}(\bar{\ell}^i_L \gamma_\mu \tau^I \ell^j_L)(\bar{q}^k_L \gamma^\mu \tau^I q^l_L)$$ **b → c τ $\bar{\nu}$**

Flavor basis $$C^{(a)}_{ijkl} = f_{ijkl}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

Mass basis $$\tilde{C}^{(a)}_{pqrs} = \sum_{i,\ell=1}^{3} U_{pi} V^*_{s\ell} f_{ijkl}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

PMNS **CKM**

$$\tilde{C}^{(a)}_{pqrs} = f_{qqrr}\, U_{pq} V^*_{sr} + f_{rqrq}\, U_{pr} V^*_{sq} + \sum_{\ell=1}^{3} f_{\ell rr\ell} U_{p\ell} V^*_{s\ell}\, \delta_{qr}$$

b → c τ ν $$\tilde{C}^{(a)}_{3323} = \cancel{f_{3322}\, U_{23} V^*_{32}} + f_{2323}\, U_{32} V^*_{33}$$

b → c μ ν $$\tilde{C}^{(a)}_{2223} = \cancel{f_{2222}\, U_{22} V^*_{32} + f_{2222}\, U_{22} V^*_{32}} + \sum_{\ell=1}^{3} f_{\ell 22\ell} U_{2\ell} V^*_{3\ell}$$

b → c e ν $$\tilde{C}^{(a)}_{1123} = f_{1122}\, U_{11} V^*_{32} + f_{2121}\, U_{12} V^*_{31}$$ λ^2 **suppressed**

$$C_{ijkl}Q^{(1)}_{\ell q} = C_{ijkl}(\bar{\ell}^i_L \gamma_\mu \ell^j_L)(\bar{q}^k_L \gamma^\mu q^l_L),$$
$$C_{ijkl}Q^{(3)}_{\ell q} = C_{ijkl}(\bar{\ell}^i_L \gamma_\mu \tau^I \ell^j_L)(\bar{q}^k_L \gamma^\mu \tau^I q^l_L).$$

Neutral current

$b \to s\, \nu_i\, \bar{\nu}_j$

Flavor basis $C^{(a)}_{ijkl} = f_{ijkl}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$

Mass basis $\tilde{C}^{(a)}_{ijrs} = \sum_{k,\ell=1}^{3} V_{rk}V^*_{s\ell} f_{ijkl}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$

Taking i=j and r=2, s=3 $b \to s\, \nu_i\, \bar{\nu}_j$

$$\sum_{i=1}^{3} \tilde{C}^{(a)}_{ii23} = \sum_{i,k=1}^{3} f_{iikk} V_{2k}V^*_{3k} + \sum_{k=1}^{3} f_{kkkk} V_{2k}V^*_{3k} + \sum_{\ell=1}^{3} f_{\ell\ell\ell\ell} V_{2\ell}V^*_{3\ell} \longrightarrow 0$$

For the case $i \neq j$

$b \to s\, \nu_2\, \bar{\nu}_3$ and $b \to s\, \nu_3\, \bar{\nu}_2$ are not Cabibbo suppressed.

5 Summary and Outlook

Modular forms and Non-invertible selection rules give us new aspects in flavor physics.

These are obtained by the String (Orbifold) compactification.

Especially, aplications of non-invertible selection rules are progressive in particle physics:
(Yukawa couplings, SMEFT, Dark matter .....)

◎ Desired texture zeros of Yukawas are realized rather simply.

◎ Texture zeros in coefficients of dim.6 operators can predict the flavor structure of the new physics, which will be testable in the rare deacy processes of quarks and leptons.

Consider the mediators, gauge like bosons, Leptoquarks, Higgs-like...

B anomaly, LFV, g-2, EDM, Higgs decays