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# Group Theoretic Approach to Theory of Fermion Production

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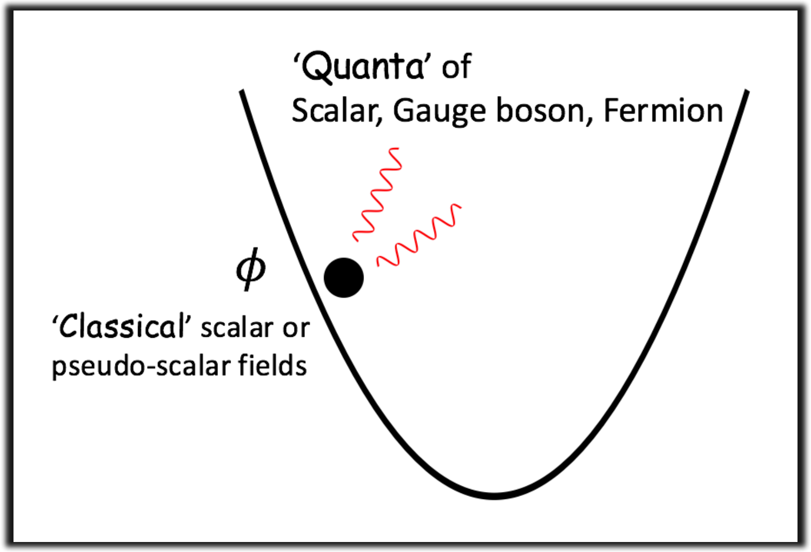
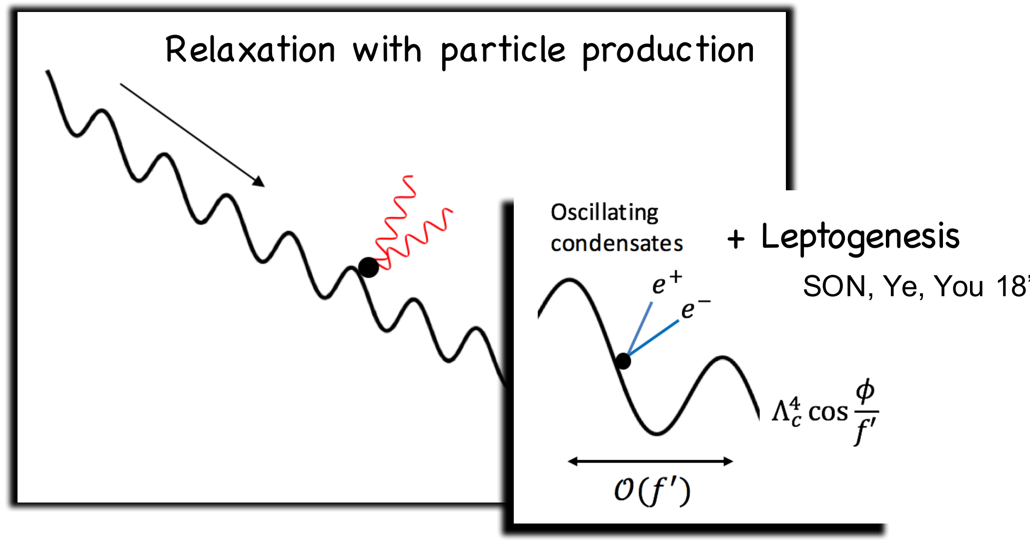
Based on
Min, SON, Suh 1808.00939

# Particle Production




- Preheating via parametric resonance or excitation in post-inflationary era

  Kofman, Linde, Starobinsky 97'

- Axion-inflation via gauge boson ($\phi F\tilde{F}$) or fermion ($\partial_\mu\phi\, j^{\mu 5}$) production

  Anbor, Sorbo 10'
  Adshead, Pearce, Peloso, Roberts, Sorbo 18'

- Gravitational waves from preheating

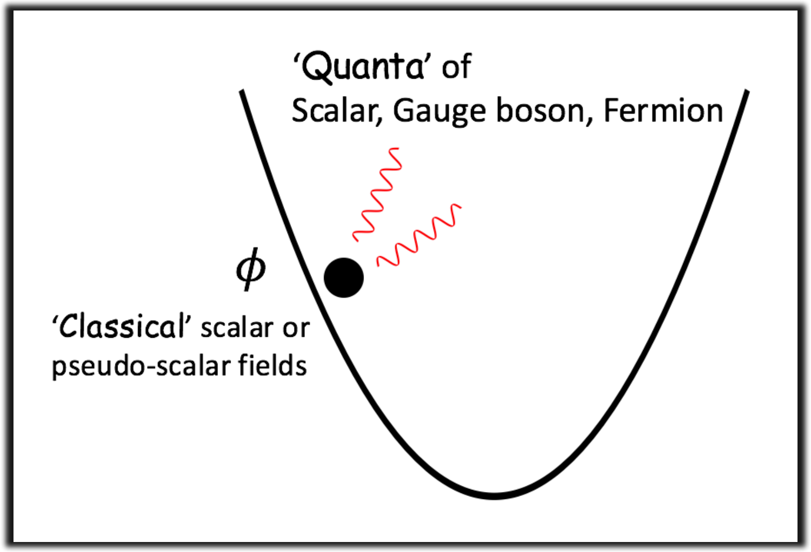
  Many literature (hard to list all here)

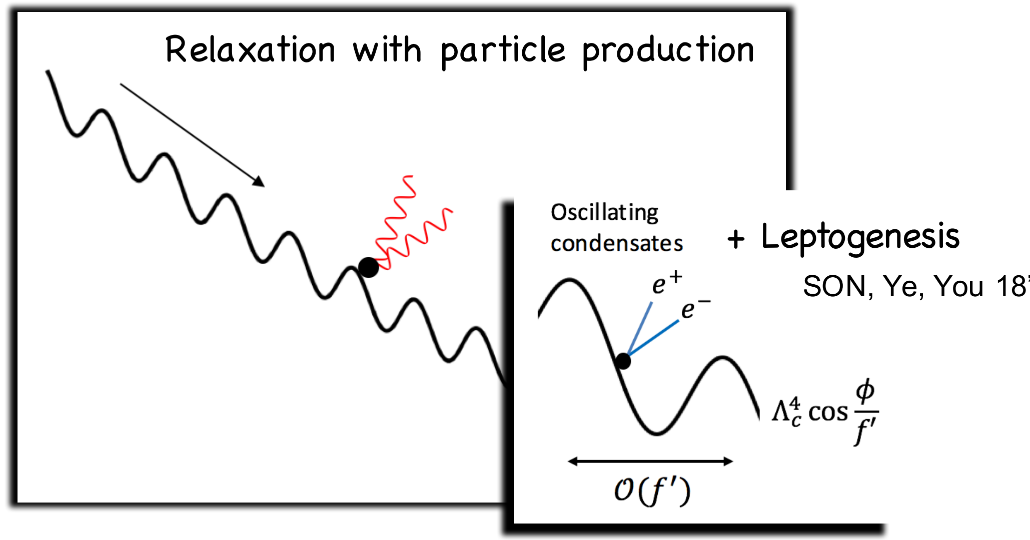
- Relaxation with particle production

  Hook, Marques-Tavares 15'
  SON, Ye, You 18'
  Fonseca, Morgante, Servant 18'
  …

List goes on …..

# Particle Production




We are going to '**Reformulate**' of theory of fermion production in a completely new manner

# Traditional Approach
# To
# Theory of Fermion Production

called technique of 'Bogoliubov' coefficient

# The model

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[ \bar{\psi} \left( i\, e^{\mu}{}_{a}\, \gamma^a D_\mu - m + g(\phi) \right) \psi + \frac{1}{2} \left( \partial_\mu \phi \right)^2 - V(\phi) \right]$$

On the metric:

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2 = a(t)^2 (d\tau^2 - d\mathbf{x}^2)$$

Under rescaling $\psi \to a^{-3/2} \psi$

$$\mathcal{L} = \bar{\psi} \left( i\, \gamma^\mu \partial_\mu - ma + \underline{g(\phi)} \right) \psi + \frac{1}{2} a^2 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - a^4 V(\phi)$$

Common Interaction type in literature

$$g(\phi) = \begin{cases} h\phi & : \text{Yukawa-type coupling} \\ \boxed{\dfrac{1}{f} \gamma^\mu \gamma^5 \partial_\mu \phi \quad : \text{derivative coupling}} \end{cases}$$

More stuff to talk about !

We will assume spatially homogenous scalar field : $\partial_\mu \phi = \dot{\phi}$

We will not distinguish $t$ and $\tau$ unless it is necessary

# Fermion Production is formulated in Hamiltonian formalism

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

A subtlety with derivative coupling

$$\Pi_{\psi} = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^{+} \qquad \Pi_{\phi} = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^{2}\dot{\phi} - \frac{1}{f}\bar{\psi}\gamma^{0}\gamma^{5}\psi$$

$$\begin{aligned}\mathcal{H} &= \Pi_{\psi}\dot{\psi} + \Pi_{\phi}\dot{\phi} - \mathcal{L} \\ &= \bar{\psi}\left(-i\,\gamma^{i}\partial_{i} + ma + \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi - \frac{1}{2a^{2}}\frac{(\bar{\psi}\gamma^{0}\gamma^{5}\psi)^{2}}{f^{2}} + \frac{1}{2a^{2}}\Pi_{\phi}^{2} + a^{5}V(\phi)\end{aligned}$$

Adshead, Sfakianakis 15'

Definition of particle number is ambiguous

Massless limit is not manifest

# A way out: field redefinition

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4 V(\phi)$$

Adshead, Sfakianakis 15'

$$\psi \to e^{-i\gamma^5\phi/f}\psi$$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu \underbrace{- ma\cos\frac{2\phi}{f}}_{= m_R} + \underbrace{i\,ma\sin\frac{2\phi}{f}}_{= m_I}\gamma^5\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4 V(\phi)$$

Adshead, Pearce, Peloso,
Roberts, Sorbo 18'

Hamiltonian formalism

$$\Pi_\psi = \frac{\delta\mathcal{L}}{\delta\dot{\psi}} = i\psi^+ \qquad \Pi_\phi = \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = a^2\dot{\phi}$$

$$\mathcal{H} = \bar{\psi}\left(-i\gamma^i\partial_i + m_R - i\,m_I\gamma^5\right)\psi + \frac{1}{2a^2}\Pi_\phi^2 + a^4 V(\phi)$$

No $\psi$ - dependence in conjugate momentum $\Pi_\phi$

Entire fermion sector is quadratic in $\psi$

: particle number is unambiguously defined

Massless limit is manifest

# Fermion production

$$\mathcal{H} = \bar{\psi}\left( -i\,\gamma^i \partial_i + m_R - i\, m_I \gamma^5 \right)\psi + \frac{1}{2a^2}\Pi_\phi^2 + a^4 V(\phi)$$

Garbrecht, Prokopec, Schmidt 02'
for generic complex mass

To estimate Fermion Production, we quantize $\psi$
while keeping pseudo-scalar as a classical field

Quantum field $\psi$

We follow notation and convention in
Adshead, Pearce, Peloso, Roberts, Sorbo 18'

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm} [\, U_r(\mathbf{k},t)\ a_r(\mathbf{k}) + V_r(-\mathbf{k},t)\ b_r^+(-\mathbf{k})\ ]$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix}$$

$$\chi_r(\mathbf{k}) = \frac{k + r\,\vec{\sigma}\cdot\mathbf{k}}{\sqrt{2k(k+k_3)}}\bar{\chi}_r \quad \text{where } \bar{\chi}_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \bar{\chi}_- = \begin{pmatrix}0\\1\end{pmatrix}$$

** helicity basis for an arbitrary $\mathbf{k}$

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}\left(|u_r|^2 - |v_r|^2\right) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\, e^{ir\varphi_k}}{2}\left[2\, m_R u_r v_r - k(u_r^2 - v_r^2) - i r m_I (u_r^2 + v_r^2)\right]$$

Fermion number density for a particle with helicity $r$

$$n_{r,k} = \langle 0 | a_r^+(\mathbf{k}; t) a_r(\mathbf{k}; t) | 0 \rangle$$

w/ $a_r(\mathbf{k}; t)$, $a_r^+(\mathbf{k}; t)$ are diagonalized $a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$ at $t \neq 0$

At $t = 0$

$$\boxed{a_r(\mathbf{k})|0\rangle = 0}$$

$a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$
$\leftrightarrow$ one-particle state
due to $B_r = 0$

At $t \neq 0$

$$\boxed{a_r(\mathbf{k}; t)|0\rangle \neq 0}$$

$a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$
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anymore due to $B_r \neq 0$

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Fermion number density for a particle with helicity $r$

$$n_{r,k} = \langle 0 | a_r^+(\mathbf{k};t) a_r(\mathbf{k};t) | 0 \rangle \boxed{= |\beta_r|^2}$$

w/ $a_r(\mathbf{k};t)$, $a_r^+(\mathbf{k};t)$ are diagonalized $a_r(\mathbf{k})$, $a_r^+(\mathbf{k})$ at $t \neq 0$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

Bogoliubov coeff.

$$a_r(\mathbf{k};t) = \alpha_r\, a_r(\mathbf{k}) - \beta_r^*\, b_r^+(\mathbf{k})$$

$$b_r^+(\mathbf{k};t) = \beta_r\, a_r(\mathbf{k}) + \alpha_r^*\, b_r^+(\mathbf{k})$$

Diag. ops at $t \neq 0$

In terms of diag. ops at $t = 0$

# looks too technical … Any simplication?

$$
\begin{aligned}
n_{r,k} &= \langle 0 | a_r^{+}(\mathbf{k}; t) a_r(\mathbf{k}; t) | 0 \rangle \\
&= \frac{1}{2} - \frac{m_R}{4\omega}\left(|u_r|^2 - |v_r|^2\right) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r\, m_I}{2\omega} Im(u_r^* v_r)
\end{aligned}
$$

Solving EOM of $u_r, v_r$ with correct initial condition is another source of confusion

# looks too technical … Any simplication?

$$n_{r,k} = \langle 0|a_r^+(\mathbf{k};t)a_r(\mathbf{k};t)|0\rangle$$
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Solving EOM of $u_r, v_r$ with correct initial condition is another source of confusion

Recall a Fourier mode in 'helicity' basis

$$\psi \sim U_r(\mathbf{k},t)a_r(\mathbf{k}) + V_r(-\mathbf{k},t)b_r^+(-\mathbf{k})$$

$$U_r = \begin{pmatrix} u_r(\mathbf{k},t)\ \chi_r(\mathbf{k}) \\ r\ v_r(\mathbf{k},t)\chi_r(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} u_r \\ r v_r \end{pmatrix} \otimes \chi_r \equiv \xi_r \otimes \chi_r$$

# looks too technical … Any simplication?

$$n_{r,k} = \langle 0 | a_r^+(\mathbf{k};t) a_r(\mathbf{k};t) | 0 \rangle$$
$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r\, m_I}{2\omega} Im(u_r^* v_r)$$

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Then we realize that

$$\zeta_{r\,1} = \frac{1}{2} r (u_r^* v_r + u_r v_r^*) = r\, Re(u_r^* v_r)$$
$$\zeta_{r\,2} = -\frac{i}{2} r (u_r^* v_r - u_r v_r^*) = r\, Im(u_r^* v_r)$$
$$\zeta_{r\,3} = \frac{1}{2}(|u_r|^2 - |v_r|^2)$$

$$\longrightarrow \quad \vec{\zeta}_r = \xi_r^+ \vec{\sigma}\, \xi_r$$

collapses into one vector

$$n_{r,k} = \langle 0 | a_r^{+}(\mathbf{k};t) a_r(\mathbf{k};t) | 0 \rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r\, m_I}{2\omega} Im(u_r^* v_r)$$

$$\mathbf{q} = rk\ \hat{x}_1 + m_I\ \hat{x}_2 + m_R\ \hat{x}_3$$

* We will see the origin of this vector later

$$\vec{\zeta}_r = \xi_r^{+} \vec{\sigma}\, \xi_r \qquad \text{w/ } \xi_r \equiv \begin{pmatrix} u_r \\ r v_r \end{pmatrix}$$

$$\zeta_{r\,1} = \frac{1}{2} r(u_r^* v_r + u_r v_r^*) = r\ Re(u_r^* v_r)$$

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$$\mathbf{q} = rk\ \hat{x}_1 + m_I\ \hat{x}_2 + m_R\ \hat{x}_3$$

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$$\zeta_{r\,1} = \frac{1}{2} r(u_r^* v_r + u_r v_r^*) = r\ Re(u_r^* v_r)$$

$$\zeta_{r\,2} = -\frac{i}{2} r(u_r^* v_r - u_r v_r^*) = r\ Im(u_r^* v_r)$$

$$\boxed{n_{r,k}(t) = \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right) = \frac{1}{2}(1 - \cos\theta)}$$

$\vec{\zeta}_r, \mathbf{q}$ behave like vector reps of SO(3)

What is this mysterious SO(3)?

# Group Theoretic Approach

# ‘Reparametrization’ Group

While $\gamma^\mu$ is fixed and only $\psi$ transforms in the Lorentz group,

$$\gamma^\mu \to \gamma^\mu, \quad \psi \to \Lambda_{1/2}\psi\,,$$

there is a freedom in choosing a representation of the gamma matrices.
This freedom is totally unphysical.

## Clifford Algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu} 1_4$$

$$\gamma^\mu \to U\gamma^\mu U^{-1} \qquad : \quad \mathrm{GL(4,C)}$$

## Dirac Theory

We assign the transformation of $\psi$, $\psi \to U\psi$

$$\mathcal{L} = \psi^+ \gamma^0 (i\gamma^\mu \partial_\mu - m)\psi$$

$$\to \mathcal{L} = \psi^+ \, U^+ U \gamma^0 U^{-1} (iU\gamma^\mu U^{-1} \partial_\mu - m) U\psi$$

$$U^+ U = UU^+ = 1 \qquad : \quad \mathrm{U(4)}$$

We consider the following subgroup of $U(4)$

$$SU(2)_1 \times SU(2)_2 \times U(1) \subset U(4)$$

The rep of subgroup is constructed as a 'tensor product' of two $SU(2)$'s and phase rotation, e.g.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{= U_1} \otimes U_2 = \begin{pmatrix} a_{11}U_2 & a_{12}U_2 \\ a_{21}U_2 & a_{22}U_2 \end{pmatrix}$$

Under $SU(2)_1 \otimes SU(2)_2$ transformation (we associate $U(1)$ with $\xi_r$)

$$\psi \sim \xi_r \otimes \chi_r \rightarrow (U_1 \otimes U_2)(\xi_r \otimes \chi_r) = \underline{(U_1 \xi_r)} \otimes (U_2 \chi_r)$$

**This is what we are looking for**

We consider the follow

$SU(2)_1 \times SU(2$

The rep of

and phase

$$\text{vo } SU(2)\text{'s}$$

$$\begin{matrix} {}_2U_2 \\ {}_2U_2 \end{matrix}\Big)$$

## Lorentz Group

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$K_i \equiv S^{i0} = \frac{i}{2}\sigma_3 \otimes \sigma_i \qquad \text{: boost}$$

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk} = \frac{1}{2}I_2 \otimes \sigma_i \qquad \text{: space rotation}$$

in Weyl Representation

$$\psi \sim \xi_r \otimes \chi_r \rightarrow \ e^{-i\vec{\theta}\cdot\vec{J}}\psi = \xi \otimes \boxed{e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\chi_r}$$

Under $SU(2)_1 \otimes SU(2$

$$\psi \sim \ \xi_r \otimes \chi_r \ \rightarrow \ (U_1 \otimes U_2)(\xi_r \otimes \chi_r) = \underline{(U_1\xi_r)} \otimes \boxed{(U_2\chi_r)}$$

Universally acts on $\psi_L$ and $\psi_R$

**This is what we are looking for**

Looks similar to space rotation of Lorentz group.

But it can not be identified with SU(2) space rotation

E.g. $\quad \bar{\psi}\gamma^\mu\psi \rightarrow \psi^+U^+U\gamma^0U^{-1}U\gamma^\mu U^{-1}U\psi = \bar{\psi}\gamma^\mu\psi$

$$\bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}\ \Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}\psi = \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\mu\psi$$

# A well-known example of $SU(2)_1$

Weyl Representation $\quad \psi_{\mathrm{Weyl}} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Dirac Representation $\quad \psi_{\mathrm{Dirac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_L + \psi_R \\ -\psi_L + \psi_R \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \sigma_3 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2$$

Two representations are related via a similarity transformation

$$\gamma^\mu_{\mathrm{Weyl}} \ \rightarrow \ U_1 \gamma^\mu_{\mathrm{Weyl}} U_1^{-1} = \gamma^\mu_{\mathrm{Dirac}}$$

$$\psi_{\mathrm{Weyl}} \ \rightarrow U_1 \psi_{\mathrm{Weyl}} = \psi_{\mathrm{Dirac}}$$

w/ $U_1(\pi/2) = e^{i\frac{\pi}{2}\frac{\sigma_y}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$SU(2)_1 \times U(1)$$

we will drop subscript from now on

is what our group theoretic approach is based on

$SU(2)_2$ does not play any important role.

# Group Theoretic Approach

Dirac equation in inertial frame

$$(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5)\psi = 0$$

EOM in tensor form for a Fourier mode can be written as (using $(\vec{\sigma}\cdot\mathbf{k})\chi_r = rk\chi_r$)

$$[(i\,\sigma_3\partial_t - irk\sigma_2 - m_R I_2 + im_I\sigma_1)\otimes I_2](\xi_r \otimes \chi_r) = 0$$

Gives rise to EOM of fundamental rep.

$$\underbrace{\partial_t\xi_r}_{\text{SU(2) fundamental}} = -i(\underbrace{\mathbf{q}\cdot\vec{\sigma}}_{\text{SU(2) embedding of SO(3) vector } \mathbf{q}})\xi_r$$

: it is called Weyl equation in condensed matter physics

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

# Group Theoretic Approach

✓ Fundamental rep. of SU(2)

$$\xi_r \equiv \binom{u_r}{r v_r}$$

- EOM of fundamental rep.

$$\partial_t \xi_r \;=\; -i\underbrace{(\mathbf{q}\cdot\vec{\sigma})}_{}\xi_r$$

SU(2) embedding
of SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

# Group Theoretic Approach

✓ Fundamental rep. of SU(2)

$$\xi_r \equiv \binom{u_r}{r v_r}$$

- EOM of fundamental rep.

$$\partial_t \xi_r = -i(\mathbf{q}\cdot\vec{\sigma})\xi_r$$

SU(2) embedding of SO(3) vector

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

✓ In terms of SO(3) ~ SU(2) reps

Bilinear of $\xi_r$ : $\xi_r^+ A\,\xi_r$

w/ $A$ = arbitrary 2×2 complex matrix

$\xi^+\,\xi\ (=1)$ : scalar

$\vec{\zeta}_r = \xi^+\vec{\sigma}\,\xi$ : vector

the only non-trivial rep.

- EOM of vector rep.

$$\partial_t \zeta_{r\,i} = \frac{1}{2}\xi_r^+[i\mathbf{q}\cdot\vec{\sigma}, \sigma_i]\xi_r = 2\epsilon_{ijk}q_j\zeta_{r\,k}$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r$$

# Analog to classical precession motion

Quantum mechanical fermion production

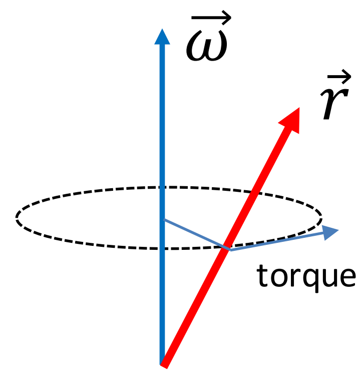
$$\frac{1}{2}\frac{d\vec{\zeta}_r}{dt} = \mathbf{q}\times\vec{\zeta}_r$$

$\mathbf{q}$ as angular velocity

$$? = \boldsymbol{q}\cdot\vec{\zeta}_r$$

Classical precession of a vector $\vec{r}$ with angular velocity $\vec{\omega}$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$




E.g. when $\vec{r} = \mathbf{M}$ (magnetization),

$$\vec{\omega} = \vec{\omega}_{\mathbf{B}} = -\gamma\mathbf{B}$$

$$\frac{d\mathbf{M}}{dt} = \vec{\omega}_{\mathbf{B}}\times\mathbf{M}$$ : called block eq.

$$E = \vec{\omega}_{\mathbf{B}}\cdot\mathbf{M}$$

# Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \frac{1}{2} - \frac{m_R}{4\omega}(|u_r|^2 - |v_r|^2) - \frac{k}{2\omega} Re(u_r^* v_r) - \frac{r m_I}{2\omega} Im(u_r^* v_r)$$

$$B_r = \frac{r\ e^{ir\varphi_k}}{2}[2\ m_R u_r v_r - k(u_r^2 - v_r^2) - irm_I(u_r^2 + v_r^2)]$$

Now it is clear that each matrix element should be a function of $\mathbf{q}$ and $\vec{\zeta}_r$ in our group theoretic approach

Diagonal element

$$A_r = \mathbf{q} \cdot \vec{\zeta}_r$$
$$= \omega \cos\theta$$

Off-diagonal element

$$|B_r| = \left|\mathbf{q} \times \vec{\zeta}_r\right|$$
$$= \omega \sin\theta$$

One can easily see why eigenvalues are $\pm\omega = \pm|\mathbf{q}|$

$$|\mathbf{q}| = \omega = \sqrt{k^2 + m^2}$$

# Particle number density

$$\mathcal{H}_\psi = \sum_{r=\pm} \int dk^3 \left(a_r^+(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r & B_r^* \\ B_r & -A_r \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^+(-\mathbf{k}) \end{pmatrix}$$

$$A_r = \mathbf{q} \cdot \vec{\zeta}_r \,, \quad |B_r| = \left|\mathbf{q} \times \vec{\zeta}_r\right|$$

In our approach, a few group properties can uniquely determine fermion number density

$$n_{r,k} = \langle 0| a_r^+(\mathbf{k};t) a_r(\mathbf{k};t) |0\rangle = |\beta_r|^2 = f(\mathbf{q} \cdot \vec{\zeta}_r, |\mathbf{q}|)$$

1. It should be at most linear in $\vec{\zeta}_r$ (note $\left|\vec{\zeta}_r\right| = 1$)

$$n_{r,k} = A \pm B \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}$$

which gives rise to inequality,

$$A - B \leq n_{r,k} \leq A + B$$

2. Pauli-blocking

$$\xrightarrow{0 \leq n_{r,k} \leq 1}$$

$$n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q} \cdot \vec{\zeta}_r}{|\mathbf{q}|}\right)$$

$'-'$ sign chosen for the consistency with the form of $A_r$

(** agrees with our explicit computation)

# Solution of EOM

Closed form of solution is available

3×3 rep. of SO(3)

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \qquad n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}\cdot\vec{\zeta}_r}{|\mathbf{q}|}\right)$$

w/ $\mathbf{q} = rk\,\hat{x}_1 + m_I\,\hat{x}_2 + m_R\,\hat{x}_3$

- Initial condition (↔ zero particle number) at $t = t_0$ is straightforward than other approach

$$\vec{\zeta}_r(t_0, t_0) = \frac{\mathbf{q}(t_0)}{|\mathbf{q}(t_0)|}$$

- Just like solving Schrödinger eq. for the unitary op., EOM can be iteratively solved

$$\vec{\zeta}_r(t, t_0) = T\exp\left(\int_{t_0}^{t} dt'\,(\mathbf{q}\cdot\mathbf{L})(t')\right)\frac{\mathbf{q}(t_0)}{|\mathbf{q}(t_0)|}$$

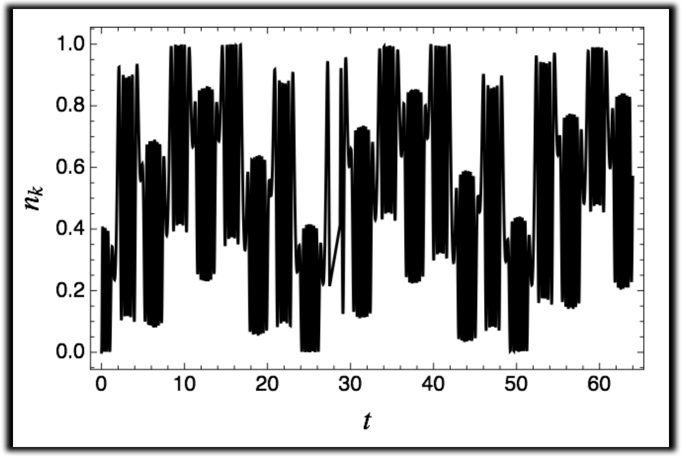
Expanding involves commutators of $\mathbf{q}\cdot\mathbf{L}$

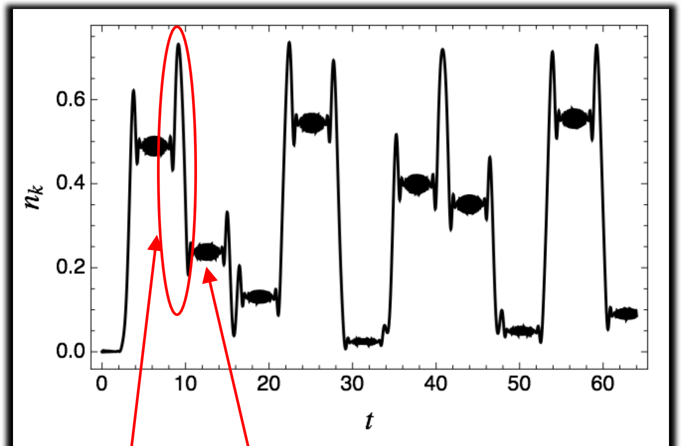
WKB solution might be the case with vanishing commutators

# Numerical example

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r \quad \text{with } \vec{\zeta}_r(0) = \frac{\mathbf{q}(0)}{|\mathbf{q}(0)|}$$

$\phi(t) = \phi_0 \sin(t)$ for chaotic potential, $V(\phi) \sim m^2\phi^2$




$m = 1, \ \frac{\phi_0}{f} = 10$

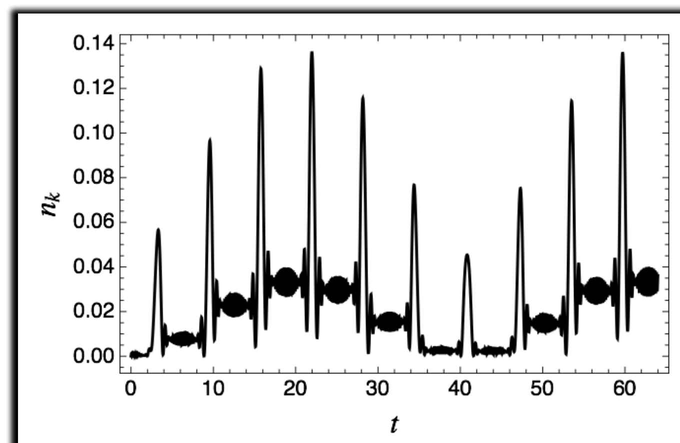
chosen for all plots

$k = 1$

Case where WKB approx. is not valid

$k = 10$

Region where WKB approx. is valid




Region where fermion production happens, and WKB approx. is not valid

$k = 12$

# `Inertial Frame' vs `Rotating Frame'

Transformation from

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^{5}\right)\psi + \cdots$$

To, via $\psi \rightarrow e^{+i\gamma^5\phi/f}\psi$,

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\,\partial_{\mu} - ma - \frac{1}{f}\gamma^{0}\gamma^{5}\dot{\phi}\right)\psi + \cdots$$

is equivalent to, in terms of $\vec{\zeta}_r$,

$$\vec{\zeta}_r \rightarrow R(t)\vec{\zeta}_r\,, \quad \text{where } R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$$

# `Inertial Frame' vs `Rotating Frame'

Transformation from

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\partial_{\mu} - m_R + i\,m_I\gamma^5\right)\psi + \cdots$$

in `Inertial Frame'

To, via $\psi \to e^{+i\gamma^5\phi/f}\psi$,

$$\mathcal{L} = \bar{\psi}\left(i\,\gamma^{\mu}\,\partial_{\mu} - ma - \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi + \cdots$$

in `Rotating Frame'

is equivalent to, in terms of $\vec{\zeta}_r$,

$$\vec{\zeta}_r \to R(t)\vec{\zeta}_r\,, \quad \text{where } R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\phi/f & -\sin 2\phi/f \\ 0 & \sin 2\phi/f & \cos 2\phi/f \end{pmatrix}$$

This rotating frame is non-inertial frame

Needs to supplement extra terms, e.g. Coriolis , centrifugal forces etc, to keep physics independent

# EOM in `Rotating Frame'

Under $\vec{\zeta}_r \to R(t)\vec{\zeta}_r$ ,

Similarly to the classical mechanics, EOM transforms like

$$\frac{1}{2}\partial_t\vec{\zeta}_r = \mathbf{q}\times\vec{\zeta}_r = (\mathbf{q}\cdot\mathbf{L})\vec{\zeta}_r \qquad \to \qquad \frac{1}{2}\partial_t(R\vec{\zeta}_r) = (\mathbf{q}\cdot\mathbf{L})(R\vec{\zeta}_r)$$

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R^T(\mathbf{q}\cdot\mathbf{L})R\;\vec{\zeta}_r - \frac{1}{2}R^T\dot{R}\vec{\zeta}_r$$

w/ $\left(R^T\dot{R}\right)_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r k}$

# EOM in `Rotating Frame'

Under $\vec{\zeta}_r \to R(t)\vec{\zeta}_r$ ,

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$$\frac{1}{2}\partial_t\vec{\zeta}_r = R^T(\mathbf{q}\cdot\mathbf{L})R\ \vec{\zeta}_r - \frac{1}{2}R^T\dot{R}\vec{\zeta}_r$$

w/ $\left(R^T\dot{R}\right)_{ij} \equiv \epsilon_{ijk}\omega_{\zeta_r k}$

EOM can be brought back to the universal form

$$\frac{1}{2}\partial_t\vec{\zeta}_r = R\mathbf{q}\times\vec{\zeta}_r + \frac{1}{2}\vec{\omega}_{\zeta_r}\times\vec{\zeta}_r = \left(R\mathbf{q} + \vec{\omega}_{\zeta_r}\right)\times\vec{\zeta}_r = \mathbf{q}'\times\vec{\zeta}_r$$

$$\mathbf{q}' = \left(rk + \frac{\dot{\phi}}{f}\right)\hat{x}_1 + ma\ \hat{x}_3$$

: different basis amounts to choose different angular velocity

# Particle number density in `Rotating (non-inertial) Frame'

Particle number density in rotating frame

$$n_{r,k} = \langle 0|a_r^+(\mathbf{k};t)a_r(\mathbf{k};t)|0\rangle = f(\mathbf{q}'\cdot\vec{\zeta}_r, |\mathbf{q}'|)$$

It should be at most linear in $\vec{\zeta}_r$.
Higher order terms should vanish to match to the one in inertial frame in $\dot{\phi} \to 0$ limit

$$\boxed{n_{r,k} = \frac{1}{2}\left(1 - \frac{\mathbf{q}'\cdot\vec{\zeta}_r}{|\mathbf{q}'|}\right)}$$

* does not take into account of quartic coupling etc..

: matches to the quadratic term

$$\mathcal{H}_\psi = \boxed{\bar{\psi}\left(-i\gamma^i\partial_i + ma + \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi} - \frac{1}{2a^2}\frac{(\bar{\psi}\gamma^0\gamma^5\psi)^2}{f^2}$$

See Adshead, Sfakianakis 15'
for a related discussion

1. It looks like particle numbers are different in two different frames.
2. Establishing the 'final' particle number as a basis-independent quantity seems very non-trivial, e.g. Inertial frame vs. Non-inertial frame

# Summary

We proposed a new group theoretic approach to theory of fermion production

1. Based on the 'Reparametrization' group of gamma matrcies

   a. Totally unphysical symmetry (that we never cared) provides us with totally different viewpoint of a very complicated process such as fermion production

2. Insightful visualization of quantum mechanical fermion production dynamics.

   a. Dynamics is analogous to the classical precession.
   b. Crystal clear initial condition unlike the traditional approach.
   c. Systematic comparison between Exact solution vs WKB solution.

3. This approach applies to any fermion system

   a. Possible extension is gravitino production, fermion production from gravitational background, fermion production in extra-dim. Spacetime
   b. Application to relaxation scenario
   c. Group theoretic approach for both fermion- and gauge boson production

# Backup slides

# Lorentz Group

Weyl Representation

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = \sigma_1 \otimes I_2 \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = i\,\sigma_2 \otimes \sigma_i \qquad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} = -\sigma_3 \otimes I_2$$

Spinor rep. satisfying Lorentz algebra

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$J_i \equiv \frac{1}{2}\epsilon_{ijk} S^{jk} = \frac{1}{2} I_2 \otimes \sigma_i \ \text{(space rotation)}, \qquad K_i \equiv S^{i0} = \frac{i}{2}\sigma_3 \otimes \sigma_i \ \text{(boost)}$$

$$\psi \sim \xi_r \otimes \chi_r \ \rightarrow \ e^{-i\vec{\theta}\cdot\vec{J}}\psi = \xi \otimes \boxed{e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\chi_r}$$

Universally acts on $\psi_L$ and $\psi_R$

On the other hand

$$(J_{L,R})_i = \frac{J_i \mp i\,K_i}{\sqrt{2}} = \frac{1}{2}(I_2 \pm \sigma_3) \otimes \frac{\sigma_i}{2} \quad : \quad SU(2)_L \times SU(2)_R$$

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

: Rep. of $SU(2)_L \times SU(2)_R$ is constructed as a 'tensor sum'