

Schwinger-Keldysh Formalism for Lattice Gauge Theories

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26 Jun 2019, XQCD2019 in Tokyo

Nonequilibrium processes in particle physics
(ex. the creation and evolution of QGP)

—→ **Nonperturbative framework**

- The standard method

⇒ The imaginary-time formalism of lattice QCD

[Taniguchi *et al.*; 1901.01666, Asakawa *et al.*; hep-lat/0011040]

- Analytic continuation cannot be performed

⇒ We need to infer spectral functions.

(cf. ansatzes, MEM etc.)

The Schwinger-Keldysh formalism

Real-time Green's functions directly

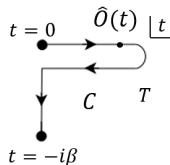
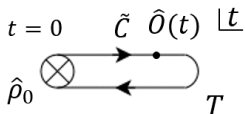
- Formulation for lattice scalar field
- Study of a sign problem with generalized Lefschetz thimble method

[Alexandru *et al.*; 1605.08040, 1704.06404]

[Berges *et al.*; hep-lat/0609058, Müller *et al.*; 1606.00342]

⇒ Schwinger-Keldysh formalism for Lattice QCD

Schwinger-Keldysh formalism



An observable for nonequilibrium processes:

$$\begin{aligned}
 \langle \hat{O}(t) \rangle &\equiv \text{Tr}\{\hat{\rho}_0 \hat{O}(t)\} \\
 &= \text{Tr}\{\hat{\rho}_0 e^{i\hat{H}T} e^{-i\hat{H}(T-t)} \hat{O} e^{-i\hat{H}t}\} \\
 &= \int \mathcal{D}\phi \langle \phi_0 | \hat{\rho}_0 | \phi'_0 \rangle O(t) e^{i \int_{\tilde{C}} dt \mathcal{L}[\phi, \partial\phi]} \\
 &\rightarrow \int \mathcal{D}\phi O(t) e^{i \int_C dt \mathcal{L}[\phi, \partial\phi]} \Big|_{\phi(0)=\pm\phi(-i\beta)} \\
 (\hat{\rho}_0 = e^{-\beta\hat{H}}/Z)
 \end{aligned}$$

Strategies for lattice formulation of SK formalism (1)

- Continuum theories

$$\langle \hat{\phi}(t)\hat{\phi}(t') \rangle_{\beta} = \text{Tr} \left\{ e^{-\beta\hat{H}} e^{i\hat{H}T} e^{-i\hat{H}(T-t)} \hat{\phi} e^{-i\hat{H}(t-t')} \hat{\phi} e^{-i\hat{H}t'} \right\} / Z.$$

- Lattice theories

$$Z = \text{Tr}\{\hat{\rho}_0\} = \text{Tr}\{(\hat{T}_{+1})^{N_{\beta}}\}, \quad \text{where } \hat{T}_{+1} = e^{-a_0\hat{H}_{\text{latt}}}.$$

$$\langle \hat{\phi}_{n_t}\hat{\phi}_{n_{t'}} \rangle_{\beta} \equiv \text{Tr}\{ (\hat{T}_{+1})^{N_{\beta}} (\hat{T}_{-i})^{N_T} (\hat{T}_{+i})^{N_T-n_t} \hat{\phi} (\hat{T}_{+i})^{n_t-n_{t'}} \hat{\phi} (\hat{T}_{+i})^{n_{t'}} \} / Z.$$

- The ideal method

$$\hat{T}_{+1} = e^{-a_0\hat{H}_{\text{latt}}} \longrightarrow \hat{T}_{\pm i} \equiv e^{\mp ia_0\hat{H}_{\text{latt}}}$$

- A typical imaginary-time TM

$$\hat{T}_{+1} = e^{-a_0 \sum_n \frac{V(\hat{\phi}(x))}{2}} e^{-\sum_n \frac{a_0}{2} \hat{\Pi}^2(x)} e^{-a_0 \sum_n \frac{V(\hat{\phi}(x))}{2}}$$

Simple replacement $a_0 \rightarrow ia_0 \implies$ unitary, but $[\hat{T}_{+1}, \hat{T}_{\pm i}] \neq 0$.

Strategies for lattice formulation of SK formalism (2)

Our conditions for real-time TMs

- Natural extensions of the imaginary-time TMs
- Unitarity condition

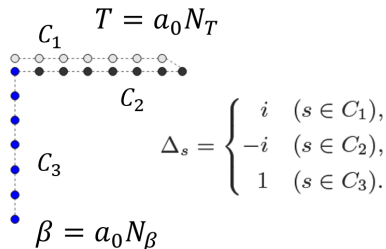
$$\hat{T}_{+i}\hat{T}_{-i} = 1, \quad \hat{T}_{\pm i} = \hat{A}\hat{U}_{\pm i}\hat{A}^{-1}, \quad \hat{U}_{\pm i}^\dagger = \hat{U}_{\mp i} = \hat{U}_{\pm i}^{-1}.$$

- Commutativity condition

$$[\hat{T}_{+1}, \hat{T}_{\pm i}] = O(a).$$

A partition function and action of the lattice QCD in the SK formalism.

$$\begin{aligned} Z &= \text{Tr} \left\{ (\hat{T}_{+1})^{N_\beta} (\hat{T}_{-i})^{N_T} (\hat{T}_{+i})^{N_T} \right\} \\ &= \int [\mathcal{D}\phi] e^{-S_{\text{SK}/\text{latt}}} \end{aligned}$$



TMs and SK formalism for lattice QCD

- Imaginary-time TM of Gauge bosons $(U_\tau = U_i(\tau, \mathbf{n}))$

$$T_{+1}^G(U_\tau, U_{\tau+1}) = e^{-\frac{a}{2}(V(U_\tau) + V(U_{\tau+1}))} \\ \times \prod_{i, \mathbf{n}} e^{-6N/g_0^2 a} \sum_R d_R (L_R(1/g_0^2 a)) \times \text{Tr}_R(U_\tau U_{\tau+1}^\dagger)$$

$$V(U_\tau) = \sum_{k,l,\mathbf{n}} [1 - (U_{kl} + U_{kl}^\dagger)/2]/g_0^2 a, \quad U_{kl}: \text{plaquettes}$$

- Imaginary-time TM of Wilson fermions

$$T_{+1}^F = A_{\tau+1} (1 - \mathcal{H}_{\tau+1}/2) \frac{1}{1 + \mathcal{H}_\tau/2} A_\tau^{-1}, \quad \text{w/ } A_\tau = B_\tau^{-1/2} [(2 + aD_{3W})\gamma_0]_\tau$$

$$\mathcal{H}_\tau = \gamma_0 a_0 D_{3W,\tau} \frac{1}{2 + a_0 D_{3W,\tau}}, \quad B_\tau = \delta_{\mathbf{n}, \mathbf{n}'} + a_0 (m_0 \delta_{\mathbf{n}, \mathbf{n}'} + \sum_k \frac{\nabla_k \nabla_k^\dagger}{2})$$

- TM for SK formalism for lattice QCD $\hat{T}_{+1} = \hat{T}_{+1}^G \otimes \hat{T}_{+1}^F$

$$Z_{\text{latt}}^{\text{QCD}}[\beta] = \text{Tr}\{(\hat{T}_{+1})^{N_\beta}\} \\ = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-(S_G[U] + S_F[\psi, \bar{\psi}, \mu])}$$

TMs and SK formalism for lattice QCD

- Real-time TM of Gauge bosons $(U_s = U_i(s, \mathbf{n}))$

$$T_{\Delta}^G(U_s, U_{s+1}) = e^{-\frac{\Delta_s a}{2}(V(U_s) + V(U_{s+1}))} \\ \times \prod_{i, \mathbf{n}} e^{-6N/\Delta_s g_0^2 a} \sum_R d_R (L_R(1/g_0^2 a))^{\Delta_s} \text{Tr}_R(U_s U_{s+1}^{\dagger})$$

$$V(U_s) = \sum_{k, l, \mathbf{n}} [1 - (U_{kl} + U_{kl}^{\dagger})/2]/g_0^2 a, \quad U_{kl}: \text{plaquettes}$$

- Real-time TM of Wilson fermions

$$T_{\Delta}^F = A_{s+1} (1 - \Delta_{s+1} \mathcal{H}_{s+1}/2) \frac{1}{1 + \Delta_{s+1} \mathcal{H}_s/2} A_s^{-1}, \quad \text{w/ } A_s = B_s^{-1/2} [(2 + aD_{3W})\gamma_0]_s$$

$$\mathcal{H}_s = \gamma_0 a_0 D_{3W, s} \frac{1}{2 + a_0 D_{3W, s}}, \quad B_s = \delta_{\mathbf{n}, \mathbf{n}'} + a_0 (m_0 \delta_{\mathbf{n}, \mathbf{n}'} + \sum_k \frac{\nabla_k \nabla_k^{\dagger}}{2})$$

- TM for SK formalism for lattice QCD $\hat{T}_{\Delta} \equiv \hat{T}_{\Delta}^G \otimes \hat{T}_{\Delta}^F$

$$Z_{\text{SK/latt}}^{\text{QCD}}[\beta, T] = \text{Tr}\{(\hat{T}_{+1})^{N_{\beta}} (\hat{T}_{-i})^{N_T} (\hat{T}_{+i})^{N_T}\} \quad (\alpha_{s\Delta_s} = [\Delta_s D_{3W, s} + B_s^{1/2} A_s]/2) \\ = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-(S_{G/\text{SK}}[U] + S_{F/\text{SK}}[\psi, \bar{\psi}, \mu])} \times \prod_{s=1}^{2N_T} \{\det \alpha_{s\Delta_s}\}^{-1}.$$

The action for gauge fields and fermionic fields

- $U_0(s, \mathbf{n}) = 1$ except $s = 2N_T + N_\beta - 1$ (Polyakov loops)

SK action for the lattice gauge bosons

$$S_{G/SK} = \sum_s \left\{ \frac{\Delta_s a}{2} (V(U_s) + V(U_{s+1})) + K_{\Delta_s}(U_s, U_{s+1}) \right\}$$

$$e^{-K_{\Delta_s}(U_s, U_{s+1})} = \begin{cases} \exp \left[- \sum_{\mathbf{n}, i} \frac{1}{g_0^2 a} \text{Tr} [2 - U_s U_{s+1}^\dagger - U_{s+1} U_s^\dagger] \right], & (s \in C_3) \\ \prod_{\mathbf{n}, i} \left(e^{-\frac{2N}{g_0^2 \Delta_s a} \sum_R d_R (L_R(1/g_0^2 a))^{\Delta_s} \text{Tr}_R (U_s U_{s+1}^\dagger)} \right), & (s \in C_1 \cup C_2) \end{cases}$$

SK action for the Wilson fermions

$$S_{F/SK} = \sum_{\mathbf{n}, s, s'} \bar{\psi}(s, \mathbf{n}) \left[- \left(\frac{1 - \gamma_0}{2} \right) \nabla_0 - \left(\frac{1 + \gamma_0}{2} \right) \nabla_0^\dagger + a_0 D_{3W} V_{s, s'} \right] \psi(s', \mathbf{n})$$

$$V_{s, s'} = \begin{pmatrix} \frac{1 + \Delta_s}{2} \delta_{s, s'} - \frac{1 - \Delta_s}{2} \delta_{s, s'+1} & 0 \\ 0 & \frac{1 + \Delta_{s-1}}{2} \delta_{s, s'} - \frac{1 - \Delta_s}{2} \delta_{s+1, s'} \end{pmatrix}.$$

Linear response theory and the spectral function

- Response to time-dependent external sources

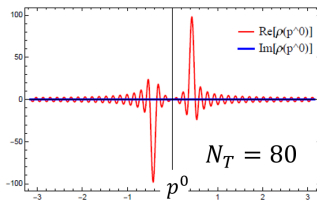
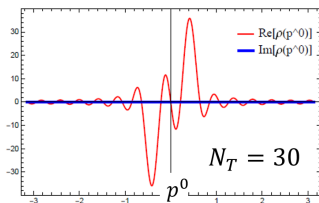
ex. EM field ($U(1)$ link field) $V_\mu(t, \mathbf{n}) = e^{ia_0 A_\mu(t, \mathbf{n})}$

$$\Delta \langle \hat{J}_k(t, \mathbf{n}) \rangle_\beta = ia_0 \sum_{t'=0}^t a^d \sum_{\mathbf{n}'} \langle [\hat{J}(t, \mathbf{n}), \hat{J}(t', \mathbf{n}')] \rangle_\beta A_0(t', \mathbf{n}'),$$

$$\sigma = \frac{1}{d} \sum_{t=0}^T \sum_{s=t}^{-N_\beta-t} \sum_{\mathbf{n}} \langle \hat{J}_k(t, \mathbf{n}) \hat{J}_k(s, \mathbf{n}) \rangle_\beta = \frac{1}{d} N_\beta \sum_{t=0}^T G_{kk}^{\text{Kubo}}(n, n')$$

- Spectral function

$$\rho(p; N_T, N_\beta, a_0) := \sum_{t=0}^{N_T} e^{ip^0(t-N_T/2)} \langle [\hat{\phi}(t), \hat{\phi}(N_T/2)] \rangle.$$

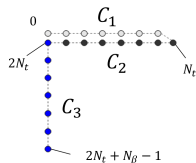


$$a_0 = 1, N_\beta = 8, \hat{m}_0 = 0, \hat{p}_1 = 0.4267$$

Exact expressions of scalar Green's functions in LSK

$$G_C(s, s') = \begin{pmatrix} \Delta_F(t, t') & \Delta(t, \tau') \\ \Delta^T(\tau, t') & \Delta_\beta(\tau, \tau') \end{pmatrix},$$

$$s \in C_1 \cap C_2 \cap C_3, \\ t \in C_1 \cap C_2, \tau \in C_3, \quad t = \begin{cases} \tilde{t} & (t \in C_1) \\ 2N_T - \tilde{t} & (t \in C_2) \end{cases}$$



$$\Delta_F(t, t') = A(E') \left(e^{-iE\tilde{t}} \left[\alpha_+^2 e^{iE\tilde{t}'} + \alpha_+ \alpha_- e^{-iE\tilde{t}'} + \alpha_+ \alpha_- e^{-NE} e^{iE\tilde{t}'} + \alpha_-^2 e^{-NE} e^{iE\tilde{t}'} \right] \right. \\ \left. + e^{iE\tilde{t}} \left[\alpha_-^2 e^{-iE\tilde{t}'} + \alpha_+ \alpha_- e^{iE\tilde{t}'} + \alpha_+ \alpha_- e^{-NE} e^{iE\tilde{t}'} + \alpha_+^2 e^{-NE} e^{-iE\tilde{t}'} \right] \theta_C(t - t') \right. \\ \left. + (t \leftrightarrow t') \right)$$

$$\Delta_\beta(\tau, \tau') = A(E') \left\{ e^{-E'|\tau - \tau'|} + e^{-NE'} e^{E'|\tau - \tau'|} \right\}$$

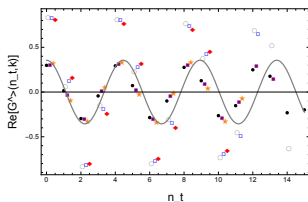
$$\Delta(t, \tau') = A(E') \left\{ e^{-iE\tilde{t}} \left[\alpha_+ e^{-E'(N_\beta - \tau')} + \alpha_- e^{-E'\tau'} \right] \right. \\ \left. + e^{iE\tilde{t}} \left[\alpha_- e^{-E'(N_\beta - \tau')} + \alpha_+ e^{-E'\tau'} \right] \right\}$$

$$A(E') = \frac{1}{2 \sinh E'} \frac{1}{1 - e^{-NE'}},$$

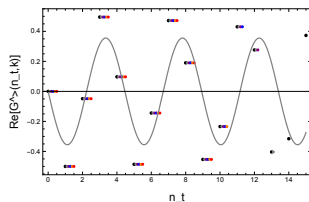
$$\alpha_\pm = \frac{1}{2} \left(1 \pm \frac{\sinh E'}{\sin E'} \right), \quad \cos E = 1 - \frac{w^2}{2}, \quad \cosh E' = 1 + \frac{w^2}{2}$$

Time translation properties of $G^>$ in equilibrium

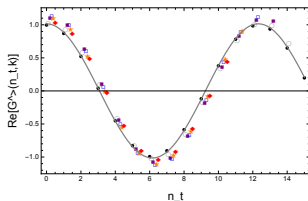
$$G^>(n_t, n'_t) = \langle \hat{\phi}(n_t) \hat{\phi}(n'_t) \rangle \quad (g \rightarrow 0, \text{ various reference times})$$



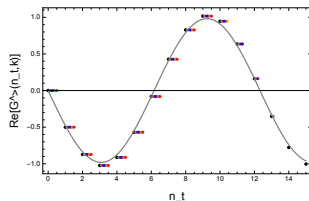
(a) $\hat{m}_0 = 1, \hat{p}_1 = 0.99$ (Re)



(b) $\hat{m}_0 = 1, \hat{p}_1 = 0.99$ (Im)



(a) $\hat{m}_0 = 0.1, \hat{p}_1 = 0.5$ (Re)



(b) $\hat{m}_0 = 0.1, \hat{p}_1 = 0.5$ (Im)

Summary and future works

Summary

- We have constructed the real-time TMs for the link fields and the fermionic fields, and formulated the SK formalism for lattice QCD.
- We have investigated linear response theory in this formalism, and defined the spectral function nonperturbatively.

Future works

- Renormalizability of the real-time Green's functions
- Construction of energy momentum tensor in the SK formalism (cf. gradient flow method [Suzuki;1304.0533])
- Applications of the Lefschetz thimble and the complex Langevin methods

Backup

Free Green's functions and the spectral function

- We get analytic solutions of free Green's functions from the action.
- **Definition of spectral functions on a lattice with finite lattice size**

$$\begin{aligned}\rho(p; N_T, N_\beta, a_0) &:= \sum_{n_t=0}^{N_T} e^{ip^0(n_t - N_T/2)} \left\langle [\hat{\phi}(n_t), \hat{\phi}(N_T/2)] \right\rangle \\ &= \frac{1}{2\sin E} \sum_{n'_t=-N_T/2}^{N_T/2} (\cos(p^0 - E)n'_t - \cos(p^0 + E)n'_t) \\ &\xrightarrow{N_T \rightarrow \infty} 2\pi \operatorname{sign}(p^0) \delta(4\sin^2 \frac{p^0}{2} - \sum_{i=1}^d 4\sin^2 \frac{p^i}{2} - m_0^2) \\ &\xrightarrow{a \rightarrow 0} 2\pi \operatorname{sign}(p^0) \delta((p^0)^2 - \vec{p}^2 - m_0^2) \\ \cosh E &= 1 + \frac{w^2}{2}, \quad w^2 = \sum_{i=1}^d 4\sin^2 \frac{p^i}{2} + m_0^2\end{aligned}$$

The action and TMs for the scalar field

SK action for the lattice scalar boson [Alexandru *et al.*;1704.06404]

$$S_{SC/SK} = \sum_{s,\mathbf{n}} a^4 \left[\frac{(\phi_{s+1,\mathbf{n}} - \phi_{s,\mathbf{n}})^2}{2\Delta_s a^2} + \frac{\Delta_s}{2} \sum_{\mathbf{k}} \left(\frac{(\phi_{s+1,\mathbf{n}+\hat{\mathbf{k}}} - \phi_{s+1,\mathbf{n}})^2}{2a^2} + \frac{(\phi_{s,\mathbf{n}+\mathbf{k}} - \phi_{s,\mathbf{n}})^2}{2a^2} \right) - \frac{\Delta_s}{2} (V(\phi_{s,\mathbf{n}}) + V(\phi_{s+1,\mathbf{n}})) \right]$$

•Description in TMs

$$Z = \text{Tr} \left\{ (\hat{T}_{+1}^{SC})^{N_\beta} (\hat{T}_{-i}^{SC})^{N_T} (\hat{T}_{+i}^{SC})^{N_T} \right\} = \int \mathcal{D}[\phi] e^{-S_{SC/SK}}$$

$$\hat{T}_{\Delta}^{SC} \equiv e^{-a\Delta_s \frac{V(\hat{\phi}(x))}{2}} e^{-\frac{a\Delta_s}{2} \hat{\Pi}^2(x)} e^{-a\Delta_s \frac{V(\hat{\phi}(x))}{2}}$$

$$T_{+i}^{SC} T_{-i}^{SC} = 1, \quad (T_{+i}^{SC})^\dagger = T_{-i}^{SC}, \quad [T_{+1}^{SC}, T_{\pm i}^{SC}] \neq 0$$

The action for gauge fields

SK action for the lattice gauge boson

$$S_{G/SK} = \sum_s \left\{ \frac{\Delta_s a}{2} (V(U_s) + V(U_{s+1})) + K_{\Delta_s}(U_s, U_{s+1}) \right\}$$

$$V(U_s) = \sum_{k,l,n} [1 - (U_{kl}(n) + U_{kl}^\dagger)/2] / g_0^2 a$$

$$e^{-K_{\Delta_s}(U_s, U_{s+1})} = \begin{cases} \exp \left[- \sum_{\mathbf{n}, i} \frac{1}{g_0^2 a} \text{Tr} [2 - U_s U_{s+1}^\dagger - U_{s+1} U_s^\dagger] \right], & (s \in C_3) \\ \prod_{\mathbf{n}, i} \left(e^{-\frac{2N}{g_0^2 \Delta_s a} \sum_R d_R (L_R(1/g_0^2 a))^{\Delta_s} \text{Tr}_R(U_s U_{s+1}^\dagger)} \right), & (s \in C_1 \cup C_2) \end{cases}$$

Character expansion ($U \in \text{SU}(N)$, R, d_R : a label of irreducible representation and its dimension)

$$\exp \left[\frac{1}{g_0^2 a} \text{Tr}(U + U^\dagger) \right] = \sum_R d_R L_R(1/g_0^2 a) \text{Tr}(U),$$
$$L_R(1/g_0^2 a) \equiv \frac{1}{d_R} \int dU \text{Tr}(U_R) \exp \left[\frac{1}{g_0^2 a} \text{Tr}(U + U^\dagger) \right]$$

The action for fermionic fields

SK action for the Wilson fermions

$$S_{F/SK} = \sum_{n,n'} \bar{\psi}(n) \left[- \left(\frac{1-\gamma_0}{2} \right) \delta_{s+1,s'} \delta_{\mathbf{n},\mathbf{n}'} - \left(\frac{1+\gamma_0}{2} \right) \delta_{s,s'+1} \delta_{\mathbf{n},\mathbf{n}'} \right. \\ \left. + 1 \dot{\delta}_{s,s'} \delta_{\mathbf{n},\mathbf{n}'} + a V_{s,s'} D_{3W} \right] \psi(n')$$

$$D_{3W,\tau} = \sum_k \left(\frac{\nabla_k \nabla_k^\dagger}{2} + \gamma_k \frac{\nabla_k - \nabla_k^\dagger}{2} \right) + m_0 \delta_{\mathbf{n},\mathbf{n}'},$$

$$\nabla_k = \delta_{\mathbf{n}+\hat{\mathbf{k}},\mathbf{n}'} U_k(n) - \delta_{\mathbf{n},\mathbf{n}'}, \quad \nabla_k^\dagger = \delta_{\mathbf{n},\mathbf{n}'+\hat{\mathbf{k}}} U_k^\dagger(n) - \delta_{\mathbf{n},\mathbf{n}'},$$

$$\frac{\nabla_k \nabla_k^\dagger}{2} \equiv - \frac{\nabla_k + \nabla_k^\dagger}{2}$$

$$V_{s,s'} = \begin{pmatrix} \frac{1+\Delta_s}{2} \delta_{s,s'} - \frac{1-\Delta_s}{2} \delta_{s,s'+1} & 0 \\ 0 & \frac{1+\Delta_{s-1}}{2} \delta_{s,s'} - \frac{1-\Delta_s}{2} \delta_{s+1,s'} \end{pmatrix}.$$