Power of $\mathbb{Z}_N$-twisted boundary condition
~Resurgence and Continuity~

Tatsuhiro MISUMI

collaboration with T. Fujimori, E. Itou, M. Nitta, N. Sakai (KeioU)
S. Kamata, Y. Tanizaki, M. Unsal (NCSU)
Y. Kikuchi (RIKEN-BNL), M. Hongo (UIC)
$Z_N$-twisted b.c. for compactified field theories

Adiabatic continuity conjecture:
Vacuum structure remains intact during decompactification with $Z_N$ twist

- Fractional instantons cause transition between classical $N$-vacua
- Makes $Z_N$ stable, leading to volume indep. of vacuum structure

't Hooft, Witten, Gonzales-arroyo, Okawa, Gross, Kitazawa...
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**Prospects and Known facts**

- Resurgent structure on \( R^{d-1} \times S^1 \) may survive on \( R^d \) \hspace{2cm} Dunne, Unsal (12)
- Weak-cplng confinement may be connected to strong-cplng one \hspace{2cm} Unsal (07)
- It is proved in 2D sigma models in a large-N limit \hspace{2cm} Sulejmanpasic (16)
- 't Hooft anomaly survives even with finite N \hspace{2cm} Tanizaki, TM, Sakai (17) Yamazaki, Yonekura (17)
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**Prospects and Known facts**

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In this talk, I discuss this conjecture and its implication on resurgence theory by using a couple of tools, including
(1) semiclassical analysis, (2) anomaly matching, and (3) lattice simulation.
Resurgence and bions in $\mathbb{CP}^{N-1}$ models

Dunne, Unsal (12), TM, Nitta, Sakai (14-16)
Fujimori, Kamata, TM, Nitta, Sakai(16-18)
CP$^1$ sigma model on $R \times S^1$

- CP$^1$ model on $R \times S^1$

\[ \mathcal{L} = \frac{1}{g^2} \frac{|\partial_\mu \varphi|^2}{(1 + |\varphi|^2)^2} + \mathcal{L}_F \]

asymptotic-free theory

- $Z_2$ Twisted boundary condition

\[ \varphi(y + L) = e^{imL} \varphi(y) \quad (m=\pi/L) \]

→ Fractional instantons ($Q=1/2, S=S_1/2$)

cf.) $m=\pi/L$

- Lee, Yi (97)
- Lee, Lu (97)
- Kraan, van Baal (97)
- Eto, et al. (04)(06)
- Bruckmann (05)
\[ \text{CP}^1 \text{ sigma model on } \mathbb{R} \times S^1 \]

- CP\(^1\) quantum mechanics \((\epsilon=1: \text{SUSY})\)

\[
L = \frac{1}{g^2} G \left[ \partial_t \varphi \partial_t \bar{\varphi} - m^2 \varphi \bar{\varphi} + i \bar{\psi} \mathcal{D}_t \psi + \epsilon m (1 + \varphi \partial_\varphi \log G) \bar{\psi} \psi \right]
\]

- Ground-state effective bosonic theory (fermion # projection)

\[
[H, \psi \bar{\psi}] = 0 \quad \rightarrow \quad \bar{\psi} |\Psi\rangle = 0
\]

\[
V = \frac{m^2}{4} \sin^2 \theta - \epsilon m g^2 \cos \theta
\]

- Two local minima
  
  North and south poles

- Instanton solution for \(\epsilon=0\)

\[
S_I = \frac{m}{g^2}
\]

Tunneling effect between two minima
Ground-state Energy in CP\(^1\) QM

\[ E^{(2)} = g^2 - m \frac{\coth \frac{m}{g^2}}{\sinh^2 \frac{m}{g^2}} \left[ \frac{\text{Ei} \left( \frac{2m}{g^2} \right) + \text{Ei} \left( -\frac{2m}{g^2} \right)}{2} - \gamma - \log \frac{2m}{g^2} \right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)} \]

- Perturbative part
  \[ E_0^{(2)} = g^2 + 2m \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}} \]

- Non-perturbative \( p \)-bion part
  \[ E_p^{(2)} = 2m \int_0^\infty dt e^{-t} \left[ \frac{(p + 1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p - 1)^2}{t + \frac{2m}{g^2}} \right] + 4mp^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{i\pi}{2} \right) \]
Ground-state Energy in CP¹ QM

\[ E^{(2)} = g^2 - m \frac{\coth \frac{m}{g^2}}{\sinh^2 \frac{m}{g^2}} \left[ \frac{\text{Ei} \left( \frac{2m}{g^2} \right) + \text{Ei} \left( -\frac{2m}{g^2} \right)}{2} - \gamma - \log \frac{2m}{g^2} \right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)} \]

- **Perturbative part**
  \[ E_0^{(2)} = g^2 + 2m \int_0^{\infty} dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}} \]
  Perturbative contribution around 0-bion background

- **Non-perturbative \( p \)-bion part** (bion : exact solution of complexified theory)
  \[ E_p^{(2)} = 2m \int_0^{\infty} dt e^{-t} \left[ \frac{(p+1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right] + 4mp^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{i\pi}{2} \right) \]
  Perturbative contribution around \( p \)-bion background
  \( p \)-bion semiclassical contribution (quasi-moduli integral)
Ground-state Energy in CP\(^1\) QM

\[
E^{(2)} = g^2 - m \frac{\coth \frac{m}{g^2}}{\sinh^2 \frac{m}{g^2}} \left[ \frac{\text{Ei} \left( \frac{2m}{g^2} \right) + \text{Ei} \left( -\frac{2m}{g^2} \right)}{2} - \gamma - \log \frac{2m}{g^2} \right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)}
\]

- **Perturbative part**

\[
E_0^{(2)} = g^2 + 2m \int_0^\infty \frac{dt}{t - \frac{2m}{g^2 \pm i0}} e^{-t} \quad \mp 2m i\pi
\]

Imaginary ambiguity of perturbation is cancelled by that of 1-bion semiclassical contribution

- **Non-perturbative \(p\)-bion part**

\[
E_p^{(2)} = 2m \int_0^\infty dt e^{-t} \left[ \frac{(p + 1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p - 1)^2}{t + \frac{2m}{g^2}} \right] + 4m p^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{i\pi}{2} \right)
\]

Fujimori, Kamata, TM, Nitta, Sakai(17)

Trans-series

\(p=1\) bion
Ground-state Energy in CP$^1$ QM

\[
E^{(2)} = g^2 - m \frac{\coth \frac{m}{g^2}}{\sinh^2 \frac{m}{g^2}} \left[ \frac{\text{Ei}\left(\frac{2m}{g^2}\right) + \text{Ei}\left(-\frac{2m}{g^2}\right)}{2} - \gamma - \log \frac{2m}{g^2} \right] = \sum_{p=0}^{\infty} e^{-\frac{2pm}{g^2}} E_p^{(2)}
\]

- Perturbative part

\[
E_0^{(2)} = g^2 + 2m \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}}
\]

- Non-perturbative $p$-bion part

\[
E_p^{(2)} = 2m \int_0^\infty dt e^{-t} \left[ \frac{(p+1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right] + 4mp^2 \left( \gamma + \log \frac{2m}{g^2} \pm \frac{i \pi}{2} \right)
\]

Cancelled

(p-1)-bion

$p$-bion

Imaginary ambiguity of perturbation around (p-1)-bion is cancelled by that of semiclassical contribution of $p$-bion!
2D CP$^{N-1}$ sigma model on R x S$^1$

Bion solution is composed of 2D 1/N fractional instantons

bare effective action in quasi moduli space

\[
S(x_r, \phi_r) = \frac{4\pi m L}{g^2} - \frac{8\pi m L}{g^2} \cos \phi_r e^{-mx_r} + 2mx_r
\]

sum over KK modes of quantum fluctuation via zeta function regularization

renormalized effective action

\[
S_R(x_r, \phi_r) = \frac{4\pi m L}{g_R^2} - \frac{8\pi m L}{g_R^2} \cos \phi_r e^{-mx_r} + 2mx_r
\]

\[
\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{1}{\pi} \log L\Lambda_0
\]

\[
\Lambda = \Lambda_0 e^{-\frac{\pi}{g^2}}
\]
dynamical scale
2D CP^{N-1} sigma model on R x S^1

Bion solution is composed of 2D 1/N fractional instantons

![Graph showing Bion solution](image)

**Bare effective action** in quasi moduli space

\[
S(x_r, \phi_r) = \frac{4\pi mL}{g^2} - \frac{8\pi mL}{g^2} \cos \phi_r e^{-mx_r} + 2mx_r
\]

**Sum over KK modes of quantum fluctuation via zeta function regularization**

**Bion contribution**

\[
E_{\text{bion}} \approx |L\Lambda|^{4mL} (\text{Re} \pm i \text{Im})
\]

**Renormalon-like Imaginary ambiguity**

**What cancels this? We hope it perturbative Borel resummation.**

See Suzuki-san, Takaura-san, Morikawa-san’s talks. But note resurgent structure we found disappear in large-N limit
’t Hooft anomaly
Use of ’t Hooft anomaly matching

’t Hooft anomaly of $G$ at UV

’t Hooft anomaly of $G$ at IR

Trivially gapped phase is prohibited

SSB of symmetry $G$ in gapped phase

CFT or Asymptotic no-free

Intrinsic topological phase
’t Hooft anomaly matching

Modern and rigorous viewpoint = Anomaly inflow

- G-Symmetry Protected Topological phase (SPT)  
  Wen, et.al., (13)

1. \( Z = e^{i\Phi} \) characterized by certain topological invariants
2. Unique ground state with trivial gap as long as \( G \) is unbroken
3. Gap should be closed in order to move to another SPT
4. Massless modes at boundary btwn two different SPTs
5. ’t Hooft anomaly cancelled btwn bulk & boundary with gauged \( G \)

All ’t Hooft anomalies are (expected to be) classified by SPTs.

Kapustin (14), Witten (15), Yonekura (16), Yonekura, Witten (19)
Recent progress in ’t Hooft anomaly

- Generalization to systems without fermions
- Generalization to higher-form symmetries
- Generalization to compactified theories

- SU($N$) YM with $\theta=\pi$
- Bifundamental gauge theories with $\theta=\pi$
- RW-symmetric QCD and QCD(adj.)
- $CP^{N-1}$ models on $R^2$ and $R \times S^1$
- QCD with $\theta=\pi$ and Dashen phase
- $Z_N$-QCD on $R^3 \times S^1$
- Yang-Mills theory on $S^3 \times R^1$
- SU($N$) spin system & Flag sigma model
- Charge-$q$ Schwinger model
- Lattice Wilson fermion & Aoki phase

Gaiotto, Kapustin, Komargodski, Seiberg (17)
Tanizaki, Kikuchi (17)
Shimizu, Yonekura (17) Yonekura (18)
Komargodski, Sharon, Thorngren, Zhou (17)
Tanizaki, TM, Sakai (17)
Gaiotto, Komargodski, Seiberg (17)
Tanizaki, TM, Sakai (17) Tanizaki, Kikuchi, TM, Sakai (17)
Yamazaki, Yonekura (17) Yamazaki (17)
Yao, Hsieh, Oshikawa (18) Tanizaki, Sulejmanpasic (18)
Hongo, TM, Tanizaki (18)
Anber, Poppitz (18) Armoni, Sugimoto (18)
TM, Tanizaki, Unsal (19)
TM, Tanizaki (19)
$Z_N$ 1-form symmetry

- 0-form symmetry = usual global symmetry, whose charged object is 0-dim point-like operator

  \[ \phi \rightarrow e^{i\theta} \phi \quad \text{U(1) 0-form symmetry} \]

  \[ \phi \rightarrow e^{i\frac{2\pi}{N}} \phi \quad \text{Z}_N \ 0\text{-form symmetry} \]

- 1-form symmetry = invariance under transf. by closed 1-form $\epsilon^{(1)}$, whose charged object is 1-dim line operator

  \[ W(C) \rightarrow \exp \left( \frac{i}{N} \int_C \epsilon^{(1)} \right) W(C) \quad a \rightarrow a + \epsilon^{(1)}/N \]

  \[ W(C) = \text{tr} \left[ iP \exp \int_C a \right] = e^{\frac{2\pi i}{N}} W(C) \quad \text{Z}_N \ 1\text{-form symmetry} \]

$SU(N)$ Yang-Mills theory has $Z_N$ 1-form center symmetry at low-T
How to gauge $Z_N$ 1-form symmetry

How to gauge such $Z_N$ 1-form symmetry

$\Rightarrow$ Background gauge field for $Z_N$ 1-form symmetry

$= \text{Pair of U}(1)$ 2-form and 1-form gauge fields $(B, C)$

$$NB = dC$$

$B$: 2-form U(1) gauge field
$C$: 1-form U(1) gauge field

• $Z_N$-gauged action with these U(1) fields

$$S = \frac{1}{2g^2} \int \text{tr}[(\tilde{G} - B) \wedge * (\tilde{G} - B)] + \frac{i\theta}{8\pi^2} \int \text{tr}[(\tilde{G} - B) \wedge (\tilde{G} - B)] + S_{\text{TFT}}$$

We note $Z_N$ 1-form symmetry itself has no 't Hooft anomaly, but CP symmetry may be broken $\Rightarrow$ Mixed 't Hooft anomaly
**SU(N) Yang-Mills theory with $\theta=\pi$**

- CP transformation

\[
S = \frac{1}{2g^2} \int \text{tr}[(\tilde{G} - B) \wedge \ast (\tilde{G} - B)] + \frac{i\theta}{8\pi^2} \int \text{tr}[(\tilde{G} - B) \wedge (\tilde{G} - B)] + S_{\text{TFT}}
\]

\[\theta \rightarrow -\theta \quad p \rightarrow -p \quad \text{with} \quad \theta = \pi\]

\[
Z[A, B] \rightarrow Z[A, B] \exp \left[ -\frac{i}{4\pi} \int \text{tr}\{\tilde{G} \wedge \tilde{G}\} - \frac{iN(2p-1)}{4\pi} \int B \wedge B \right]
\]

\[
= Z[A, B] \exp \left[ -2\pi i \mathbb{Z} \frac{2p-1}{N} \right]
\]

**Mixed ’t Hooft anomaly and Global inconsistency indicate SSB of either of CP or $Z_N$ 1-form symmetry as long as we assume a gapped phase.**
SU($N$) Yang-Mills theory with $\theta=\pi$ on $\mathbb{R}^3 \times S^1$

$$Z[A, B^{(1)}, B^{(2)}] \rightarrow Z[A, B^{(1)}, B^{(2)}] \exp \left[ -\frac{iN(2p-1)}{2\pi} \int B^{(2)} \wedge B^{(1)} \wedge L^{-1} d\chi^4 \right]$$

$$\int B^{(2)} \wedge B^{(1)}$$

Mixed 't Hooft anomaly and Global inconsistency indicate spontaneous breaking of either of CP or $Z_N$ 1-form symmetry even at finite-temperature (trivially gapped phase forbidden)!
Massless $N$-flavor QCD

Tanizaki, TM, Sakai (17)
Tanizaki, Kikuchi, TM, Sakai (17)

See also Shimizu, Yonekura (17)
and Yonekura-san's talk

\[
S = \frac{1}{2g^2} \int \text{tr}(G_c \wedge *G_c) + \int d^4x \text{ tr} \left\{ \bar{\Psi} \gamma_\mu D_\mu(a) \Psi \right\}
\]
Anomaly matching for $N_C=N_F=N$ QCD

Let us look into mixed 't Hooft anomaly between

vector $\frac{SU(N)_{\text{flavor}}}{(\mathbb{Z}_N)_{\text{color-flavor}}}$ axial $(\mathbb{Z}_{2N})_{\text{axial}}$

firstly gauge vector symmetry and perform $(\mathbb{Z}_{2N})_{\text{axial}}$ transformation

$$S_{\text{gauged}} = \frac{1}{2g^2} \int \text{tr} \left\{ (\mathcal{G}_c + B) \wedge * (\mathcal{G}_c + B) \right\} + \int d^4x \text{ tr} \left\{ \overline{\Psi} \gamma_\mu D_\mu (\vec{a}, \vec{A}) \Psi \right\}$$

$$\Delta S = \frac{i}{4\pi} \int \text{ tr} \left\{ (\mathcal{G}_c + B) \wedge (\mathcal{G}_c + B) \right\} + \frac{i}{4\pi} \int \text{ tr} \left\{ (\mathcal{G}_f + B) \wedge (\mathcal{G}_f + B) \right\} = -\frac{i2N}{4\pi} \int B \wedge B = -\frac{4\pi i}{N} \mathcal{Z}$$

For $N>2$, it has a mixed 't Hooft anomaly

$$\mathcal{Z}[(A, B)] \mapsto \mathcal{Z}[(A, B)] \exp \left( -\frac{2iN}{4\pi} \int B \wedge B \right)$$

Either of two symmetries should be broken, consistent to chiral SSB
Death of anomaly in compactified \( N \)-flavor QCD

- \( \mathbb{R}^3 \times S^1 \) compactification

\[
\begin{align*}
Z_N \text{ 1-form symmetry in 4D} & \quad \text{Z}_N \text{ 0-form symmetry in } S^1 \quad B^{(1)} \\
\text{Z}_N \text{ 1-form symmetry in 3D} & \quad B^{(2)}
\end{align*}
\]

\( \text{Z}_N \) transformation in \( S^1 \) changes quark boundary condition!

\[
\text{tr} \left[ \mathcal{P} \exp i \int_{S^1} a \right] \mapsto \omega \text{ tr} \left[ \mathcal{P} \exp i \int_{S^1} a \right] \quad \Rightarrow \quad \Psi(x, x^4 + L) = \Psi(x, x^4)e^{\frac{2\pi i}{N}}
\]

Mixed 't Hooft anomaly disappears.

\[
B^{(1)} = 0 \quad \Rightarrow \quad \mathcal{Z}[(A, B)] \mapsto \mathcal{Z}[(A, B)] \exp \left( -\frac{2iN}{4\pi} \int B \wedge B \right)
\]

What shall we do…?
**Z\(_N\)**-QCD theory on \(\mathbb{R}^3 \times S^1\)

- Introducing \(SU(N)\)\(_{\text{color}}\) \(\mathbb{Z}_N\) holonomy in \(N\)-flavor QCD

\[
\Omega = e^{i\phi} \text{diag}[1, \omega, \omega^2, \ldots, \omega^{N-1}]
\]

\(\omega \equiv e^{2\pi i/N}\)

Equivalent to flavor-dependent \(\mathbb{Z}_N\) twisted boundary condition

\[
\Psi(x, x^4 + L) = \Psi(x, x^4) \Omega \quad \xrightarrow{L \to \infty} \quad \text{N-flavor QCD on } \mathbb{R}^4
\]

\(\mathbb{Z}_N\) 0-form transformation still changes b.c. as \(\Omega \mapsto \omega \Omega\)

---

**but, we have a new intertwined \(\mathbb{Z}_N\) 0-form symmetry**

\[
\begin{align*}
\Psi & \mapsto \Psi S \\
\text{flavor rotation} & \quad + \\
\Omega & \mapsto \omega \Omega \\
\text{\(\mathbb{Z}_N\) 0-form transf.}
\end{align*}
\]

\[
S = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

named as **shift symmetry** (color-flavor center symmetry) \((\mathbb{Z}_N)_S\)
Anomaly matching $Z_N$-QCD theory on $\mathbb{R}^3 \times S^1$

Let us gauge the symmetries and perform $(Z_{2N})_{\text{axial}}$ transformation

$$Z_{\Omega}[(A_K, B^{(1)}, B^{(2)})] = Z[(A_K + B^{(1)} + A_{\text{cl}}, B^{(2)} + B^{(1)} \wedge L^{-1} dx^4)]$$

$$Z_{\Omega}[(A_K, B^{(1)}, B^{(2)})] \rightarrow Z_{\Omega}[(A_K, B^{(1)}, B^{(2)})] \exp \left( -\frac{2iN}{2\pi} \int B^{(2)} \wedge B^{(1)} \right)$$

Mixed ’t Hooft anomaly among

$$(Z_N)_S \quad U(1)^{N-1}/(Z_N)_{\text{color-flavor}} \quad (Z_{2N})_{\text{axial}}$$

shift sym. \quad flavor sym. \quad axial sym.

one of them should be broken if gap assumed

We can make constraints on finite-$(T, \mu)$ phase diagram of $Z_N$-QCD consistent with lattice results, Iritani, Itou, TM (15)
Comparison with lattice $Z_N$-QCD on $\mathbb{R}^3 \times S^1$

- Polyakov-loop distribution plot
- Hadron spectrum

\begin{align*}
\text{Z}_3\text{-QCD} & \quad \text{N}_f=3 \text{ QCD} \\
\begin{array}{c}
\beta = 1.70 \\
\beta = 2.00 \\
\beta = 2.20 \\
\end{array}
& \\
\begin{array}{c}
\beta = 1.70 \\
\beta = 2.00 \\
\beta = 2.20 \\
\end{array}
\end{align*}

\begin{align*}
\text{Im } L & \quad \text{Im } L \\
\text{Re } L & \quad \text{Re } L \\
\text{mass } [a^{-1}] & \\
\beta = 1.5, \kappa = 0.1400 \\
\text{PS} & \quad \text{V} \\
\text{Z}_3\text{-QCD} & \quad \text{N}_f = 3 \\
\text{PCAC} & \\
\end{align*}

Low-T: around the origin $\sim \text{Z}_3$-sym
High-T: equiv. 3 vacua $\sim$ SSB of $\text{Z}_3$
$\rightarrow \text{Z}_3$ at the action level

Low-T: on the real axis
High-T: on the real axis
$\rightarrow$ Explicit $\text{Z}_3$ breaking

At zero temperature, hadron spectrum agrees with that of usual QCD
Comparison with lattice $Z_N$-QCD on $R^3 \times S^1$

$\beta$ dependence of Polyakov-loop & chiral condensate

Polyakov loop for $(Z_N)_S$

Chiral condensate for $(Z_{2N})_{\text{axial}}$

Shift and Chiral transitions seem to occur at the same temperature (consistent with absence of trivially-gapped phase)

The analysis on anomaly matching is consistent with the lattice result
$S = \int d^2x \left[ \frac{1}{2} |(\partial_\mu + ia_\mu)\bar{z}|^2 + \frac{\lambda}{4} (|\bar{z}|^2 - \mu^2)^2 \right] - \frac{i\theta}{2\pi} \int da$
Anomaly matching for \( \mathbb{CP}^{N-1} \) models

Our goal is to show mixed ’t Hooft anomaly between

flavor \( SU(N)/\mathbb{Z}_N \) \hspace{2cm} \text{time reversal} \quad \mathcal{T}

firstly gauge flavor sym. and perform \( \mathcal{T} \) transformation

\[
\mathcal{Z}_\pi[\mathcal{T} \cdot (A, B)] \exp \left( -ik \int T \cdot B \right) = \mathcal{Z}_\pi[(A, B)] \exp \left( -ik \int B \right) e^{i(2k-1) \int B} \frac{2\pi \mathbb{Z}}{N}
\]

For even \( N \), it has a mixed ’t Hooft anomaly
For odd \( N \), it has global inconsistency between \( \theta = 0, \pi \)

\( \rightarrow \) It indicates spontaneous \( \mathcal{T} \) breaking at \( \theta = \pi \) on \( \mathbb{R}^2 \)

On \( \mathbb{R} \times S^1 \), \( \mathbb{Z}_N \) 0-form symmetry disappears… \( \rightarrow \) no ’t Hooft anomaly
**\( Z_N \)-twisted \( \mathbb{CP}^{N-1} \) model at \( \theta = \pi \) on \( \mathbb{R} \times \mathbb{S}^1 \)**

\( Z_N \) twisted boundary condition in \( \mathbb{S}^1 \) direction

\[ \tilde{z}(x^1, x^2 + L) = \Omega \tilde{z}(x^1, x^2) \quad \Omega = \text{diag}(1, \omega, \ldots, \omega^{N-1}) \quad (\omega = e^{2\pi i/N}) \]

\[ \rightarrow \quad \text{we again have intertwined} \ Z_N \ 0\text{-form shift symmetry} \ (Z_N)_S \]

\[ \tilde{z} \rightarrow S \tilde{z} \quad \& \quad \Omega \rightarrow \omega \Omega \]

\[ \text{flavor rotation} \quad \text{Z}_N \ 0\text{-form transf.} \]

\[ s = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad S\Omega S^{-1} = \omega \Omega \]

\[ \rightarrow \quad \text{U}(1) \ 1\text{-form field} \ B^{(1)} \]

\[ B = B^{(1)} \wedge L^{-1} dx^2 \]

\[ Z_{\pi,\Omega}[T \cdot B^{(1)}] = Z_{\pi,\Omega}[B^{(1)}] \exp \left( -i \int B^{(1)} \right) \]

Mixed anomaly or global inconsistency between \( Z_N \) and \( T \) survives on \( \mathbb{R} \times \mathbb{S}^1 \).

It indicates \( Z_N \) tbc works to maintain the vacuum structure of \( \mathbb{R}^2 \).
Lattice simulation for $Z_N$-twisted $\mathbb{C}P^{N-1}$ model

What we want to check is the following conjecture:

- $\langle P \rangle \neq 0$ for large $\beta$
- $\langle P \rangle \sim 0$ for large $\beta$

We will show quite suggestive results on fractional instantons and adiabatic continuity.
Setup of lattice simulation

cf.) Berg,Luscher(81), Campostrini,et.al.(92), Alles,et.al.(00), Flynn,et.al.(15), Abe,et.al.(18)

• Lattice formulation
  \[ S = -N\beta \sum_{n,\mu} \left( \bar{z}_{n+\mu} \cdot z_n \lambda_{n,\mu} + \bar{z}_n \cdot z_{n+\mu} \bar{\lambda}_{n,\mu} - 2 \right) \]

Vector field \( \Phi \) is introduced:

\[
\phi_{2j} = \Re[z_{n,j}], \quad \phi_{2j+1} = \Im[z_{n,j}], \quad j = 0, \ldots, N - 1
\]

\[
\phi^R = \Re[\lambda_n], \quad \phi^I = \Im[\lambda_n],
\]

\[ s_{\phi} = -N\beta \phi \cdot F_{\phi} = -N\beta |F_{\phi}| \cos \theta \quad \text{updated just by updating } \theta \]

Over heat-bath algorithm is adopted to update this \( \theta \)

• Parameters and quantities
  \( N_x = 40-400, \quad N_t = 8,12, \quad \beta = 0.1-4.0, \quad N = 3-20, \quad N_{\text{sweep}} = 200000,400000 \)
  
  • Expectation values of Polyakov loop and its susceptibility
  
  • Thermal entropy \( s = \beta(N\tau)^2(<T_{xx}> - <T_{tt}>) \)
Distribution plot of P-loop

Im[\(P\)] \(N=5, \beta=0.1\)

\(|<P>| \sim 0\)

Re[\(P\)]

High-\(\beta\) : Transition between N vacua
\(\rightarrow\) quantum \(Z_N\) symmetry

Low-\(\beta\) : around the origin
\(\rightarrow\) \(Z_N\) symmetry at the action level
Distribution plot of P-loop

Im\[P\]  \(N=5, \ \beta=0.1\)

\[\langle P \rangle \sim 0\]

Re\[P\]

\[\langle P \rangle \sim 0\]

Low-\(\beta\) : around the origin \(\rightarrow\) 
\(Z_N\) symmetry at the action level

High-\(\beta\) : Transition between \(N\) vacua
\(\rightarrow\) quantum \(Z_N\) symmetry
Distribution plot of P-loop

\[ \text{Im}[P] \quad N=5, \quad \beta=0.1 \]

\[ \text{Re}[P] \quad |\langle P\rangle| \sim 0 \]

\[ N=5, \quad \beta=1.8 \]

\[ |\langle P\rangle| \neq 0 \]

Low-\(\beta\) : around the origin \(\rightarrow\) \[ Z_N \] symmetry at the action level

Quite high-\(\beta\) : polygon shape broken, but larger statistics restore it
VEV of Polyakov loop \(|<P>|\)

\(N=5\)

\[\begin{array}{c}
\text{\(\beta=1.4\)} \\
\text{\(\beta=1.6\)} \\
\text{\(\beta=1.8\)} \\
\text{\(\beta=2.0\)}
\end{array}\]

\[\begin{array}{c}
|<P>| \\
|<P>| \\
|<P>| \\
|<P>| \\
\end{array}\]

\(N=5\)

- Low \(\beta\) \(\rightarrow\) \(|<P>| = 0\) : distribution around origin
- High \(\beta\) \(\rightarrow\) \(|<P>| \sim 0\) : distribution forms polygons
- Quite high \(\beta\) \(\rightarrow\) suddenly \(|<P>| \neq 0\) : but more stat. restore polygon

This peculiar P-loop could imply something special (\(Z_N\) stability?). We still need larger volume or more statistics to judge continuity.
VEV of Polyakov loop $|\langle P \rangle|$  

$N=20$  

- Low $\beta \rightarrow |\langle P \rangle| = 0$: distribution around origin  
- High $\beta \rightarrow |\langle P \rangle| \sim 0$: distribution forms polygons  
- Quite high $\beta \rightarrow$ suddenly $|\langle P \rangle| \neq 0$: but more stat. restore polygon  

This peculiar P-loop could imply something special ($Z_N$ stability?). We still need larger volume or more statistics to judge continuity.
Polygon-shaped distributions of Polyakov loop ($|<P>| \sim 0$) appear more often with more statistics.

It may indicate $Z_N$ stability (continuity)....

To check this, let us go to larger volume.
Distribution plot of P-loop (very high $\beta$, large volume)

Independent configurations for very high $\beta$ ($\beta=4.0$) with large volume include a quantum $Z_N$ symmetric case as below!

$\text{Im}[P] \quad N=3, \beta=4.0, (400 \times 12)$

$\text{Re}[P] \quad |<P>| \sim 0$

Hysteresis of $\text{arg}[P]$

Any of $Z_N$ vacua is not selected

Very high-$\beta$ : quantum $Z_N$ symmetric case found with certain probability

it implies we need larger volume or more statistics for $Z_N$ continuity....
Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\arg[P]$.

$\arg[P]$ implies fractional instantons cause transition between classical vacua at high $\beta$, which lead to quantum $\mathbb{Z}_N$ symmetry and could yield adiabatic continuity.
Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\arg[P]$.

$\arg[P]$

1/3 fractional antiinstanton + 1/3 fractional instanton = bion

3 $\times$ 1/3 fractional instantons = instanton

implies fractional instantons cause transition between classical vacua at high $\beta$, which lead to quantum $\mathbb{Z}_N$ symmetry and could yield adiabatic continuity
Fractional instantons

Pick up two of configurations and look into the $x$-dependence of $\text{arg}[P]$

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implies fractional instantons cause transition between classical vacua at high $\beta$, which lead to quantum $\mathbb{Z}_N$ symmetry and could yield adiabatic continuity
Fractional instantons
Charge-\(q\) \(N\)-flavor Schwinger model on \(R \times S^1\)

\[
S = \frac{1}{2e^2} \int_{M_2} |da|^2 + \frac{i\theta}{2\pi} \int_{M_2} da + \sum_{f=1}^N \int_{M_2} d^2x \, \bar{\psi}^f \gamma^\mu (\partial_\mu + qa_\mu) \psi^f
\]

- \(q=2\) on domain-wall of \(\mathcal{N}=1\) SU(2) SYM
  Anber, Poppitz (18)
- O\(1\) - D\(1\) system : \(q=2, \, N=8\)
  Sugimoto, Takahashi (04) Armoni, Sugimoto (18)
\( G_{\text{sub}} = \mathbb{Z}_q^{[1]} \times \frac{SU(N)_V}{(\mathbb{Z}_N)_V} \times (\mathbb{Z}_{qN})_R \subset G \) has mixed 't Hooft anomaly

\[ S_3[A_{\text{sub}}] = \int_{M_3} \frac{qN}{2\pi} A^{(1)}_\chi \wedge B^{(2)}_V \] quantized to \( \mathbb{Z}_{qN} \) phase

it indicates \( (\mathbb{Z}_{qN})_R \) SSB or CFT behavior at IR

Non-abelian bosonization on \( R^2 \) teaches us

<table>
<thead>
<tr>
<th>((q, N))</th>
<th>Mass gap</th>
<th>Symmetry breaking</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1))</td>
<td>(e^2/\pi^2)</td>
<td>No symmetry</td>
</tr>
<tr>
<td>((q, 1))</td>
<td>(q^2e^2/\pi^2)</td>
<td>((\mathbb{Z}_q)_R \rightarrow 1) with condensate ( \langle \bar{\psi}\psi \rangle )</td>
</tr>
<tr>
<td>((1, N))</td>
<td>(SU(N)_1 ) WZW CFT</td>
<td>Symmetry is unbroken</td>
</tr>
<tr>
<td>((q, N))</td>
<td>(SU(N)_1 ) WZW CFT</td>
<td>((\mathbb{Z}_{Nq})_R \rightarrow (\mathbb{Z}_N)_R) with condensate ( \langle \det(\bar{\psi}<em>f \psi</em>{f'}) \rangle )</td>
</tr>
</tbody>
</table>

where 't Hooft anomaly is matched by SSB or CFT
Compactification on $\mathbb{R} \times S^1$

- **Thermal compactification**
  \[ M_2 = M_1 \times S^1 \ni (x^1, x^2) =: (\tau, x) \]
  \[ \psi^f(\tau, x + L) = -\psi^f(\tau, x + L), \]
  \[ a(\tau, x + L) = a(\tau, x) + d\lambda(\tau) \]

- **Global symmetries**
  \[ G_{1d} = \mathbb{Z}_q^0 \times \frac{SU(N)_L \times SU(N)_R \times (\mathbb{Z}_q N)_R}{(\mathbb{Z}_N)_V \times (\mathbb{Z}_N)_R} \]
  \[ P(\tau) \mapsto e^{2\pi i/q} P(\tau) \]

  0-form symmetry

- **'t Hooft anomaly**
  \[ B_V^{(2)} = A_{c(q)} \wedge \frac{dx}{L} \quad \rightarrow \quad \int \Omega_3^0 = \frac{q}{2\pi} \int (N A_X^{(1)}) \wedge A_{c(q)} \quad \rightarrow \quad \mathbb{Z}_q \text{ phase} \]

\[ \mathbb{Z}_q^0 \times (\mathbb{Z}_q)_R \text{ has } \mathbb{Z}_q \text{ mixed 't Hooft anomaly, indicating } \mathbb{Z}_q \text{ SSB} \]

There is no remnant of 2d WZW
Compactification on $\mathbb{R} \times S^1$

- **$Z_N$ twisted compactification**: 2d $Z_N$-QED
  \[ \psi^f(\tau, x + L) = e^{2\pi i f/N} \psi^f(\tau, x) \]

- **Global symmetries**
  \[ Z_{Nq}^{[0]} \times \frac{U(1)^{N-1}_L \times U(1)^{N-1}_R \times (\mathbb{Z}_{Nq})_R}{(\mathbb{Z}_N)_V \times (\mathbb{Z}_N)_R} \]
  \[ a_2 \mapsto a_2 + \frac{2\pi}{NqL} \]
  \[ \psi^f \mapsto \psi^{f+1} \]
  0-form symmetry

- **’t Hooft anomaly**
  \[ B_V^{(2)} = A_{c(Nq)} \wedge \frac{dx}{L} \quad \xrightarrow{\quad} \quad \frac{Nq}{2\pi} \int A^{(1)}_X \wedge A_{c(Nq)} \quad \rightarrow \quad \mathbb{Z}_{Nq} \text{ phase} \]

$Z_{Nq}^{[0]} \times (\mathbb{Z}_{Nq})_R$ has $Z_{Nq}$ mixed ’t Hooft anomaly, indicating $Z_{Nq}$ SSB

$Z_N$-QED has remnant of 2d WZW!
Chiral condensate in $\mathbb{R} \times S^1$

- $q=2$, $N=3$ (pbc)

$$\frac{\langle \theta, k | \text{det} \bar{\psi}_L^f \psi_R^f | \theta, k \rangle}{\langle \theta, k | \theta, k \rangle} = e^{i \frac{\theta + 2\pi k}{q}} \frac{N!}{L^N} \exp \left( - \frac{N\pi}{Lm_\gamma} \right)$$

$$ (\mathbb{Z}_{Nq})_R \rightarrow (\mathbb{Z}_N)_R $$

chiral sym. breaking, but no remnant of WZW

- $q=2$, $N=3$ ($\mathbb{Z}_N$-tbc)

$$\frac{\langle \theta, k | \bar{\psi}_L^f \psi_R^f | \theta, k \rangle}{\langle \theta, k | \theta, k \rangle} = \frac{1}{NL} e^{i \frac{\theta + 2\pi k}{Nq}} \exp \left( - \frac{\pi}{NLm_\gamma} \right)$$

$$ (\mathbb{Z}_{Nq})_R \rightarrow 1 $$

chiral sym. breaking with remnant of WZW!
### Semiclassics in $\mathbb{R} \times S^1$

<table>
<thead>
<tr>
<th></th>
<th>Barrier height</th>
<th>Width $\Delta a$</th>
<th>Fracton action</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal</td>
<td>$O(N)$</td>
<td>$2\pi/q$</td>
<td>$S_F = \frac{\pi N}{m_\gamma L}$</td>
</tr>
<tr>
<td>$\Omega_F$ twisted</td>
<td>$O(N^{-1})$</td>
<td>$2\pi/qN$</td>
<td>$S_F = \frac{\pi}{Nm_\gamma L}$</td>
</tr>
</tbody>
</table>

**Graphs:**
- **Thermal (periodic):**
  - Peaks at $\pi$ and $2\pi$.
- **Flavor-twisted:**
  - Peaks at $\pi$, $2\pi$, $3\pi$, etc., with decreasing amplitude.
Chiral condensate in $R \times S^1$

- Exact results of $L$ dependence

(a) $(q, N) = (1, 1)$

(b) $(q, N) = (1, 3)$ with $\Omega_F$-twist
Chiral condensate in $R \times S^1$

- **Exact results of $L$ dependence**

\[
\frac{|\langle \bar{\psi}_L \psi_R \rangle|}{m_\gamma} = \frac{1}{NL} \left( \frac{L m_\gamma e^\gamma}{4\pi} \right)^{1/N} \exp \left[ -\frac{I(Lm_\gamma)}{N} \right] \quad \text{SU(N) WZW scaling dim.}
\]

\[
Z_N\text{-twisted theory correctly keeps 2-dimensional CFT properties!}
\]
SU(3)/U(1)^2 flag sigma model on $\mathbb{R} \times S^1$

$$S = -\sum_{\ell=1}^{3} \int \left[ \frac{1}{2g} |(d + ia_\ell)\phi_\ell|^2 - \frac{i\theta_\ell}{2\pi} da_\ell \right]$$

$$\phi_\ell = (\phi_{1,\ell}, \phi_{2,\ell}, \phi_{3,\ell})^t \in \mathbb{C}^3 \ (\ell = 1, 2, 3)$$

$$\overline{\phi}_\ell \cdot \phi_k = \delta_{\ell k}$$  
$$a_1 + a_2 + a_3 = 0$$

- Another extension of spin chain systems
- Another generalization of O(3) or $\text{CP}^1$ nonlinear sigma model

Bykov (11), Lajko, et.al. (17)  
Ohmori, Seiberg, Shao (18)
\[ SU(3)/U(1)^2 \text{ flag sigma model} \]

- **Relevant symmetries**

  - **\( SU(3)/\mathbb{Z}_3 \) flavor symmetry**
    \[
    \phi_{\ell} \mapsto U \phi_{\ell} \quad \text{with} \quad U \in SU(3)
    \]

  - **\( \mathbb{Z}_3 \) permutation symmetry**
    \[
    \mathbb{Z}_3 \text{ permutation} : \begin{cases} 
    \phi_{\ell} \mapsto \phi_{\ell+1}, \\
    a_{\ell} \mapsto a_{\ell+1}, 
    \end{cases}
    \]

- **Charge conjugation symmetry \( \mathcal{C} \)**

  \[
  \mathcal{C}_k : \begin{cases} 
  \phi_{\ell} \mapsto -\bar{\phi}_{-\ell-k}, \\
  a_{\ell} \mapsto -a_{-\ell-k}, 
  \end{cases} \quad (k = 1, 2, 3)
  \]

  \[
  \text{\( \mathcal{C}_1 \)-invariant points} : \theta_2 = 2\theta_1 \mod 2\pi, \\
  \text{\( \mathcal{C}_2 \)-invariant points} : \theta_1 = 2\theta_2 \mod 2\pi, \\
  \text{\( \mathcal{C}_3 \)-invariant points} : \theta_1 = -\theta_2 \mod 2\pi.
  \]

- **\( \theta_{\ell} = \frac{2\pi p}{3} \ell \mod 2\pi \quad p \in \mathbb{Z} \)**
’t Hooft anomaly matching

Action with gauged \( SU(3)/\mathbb{Z}_3 \) flavor symmetry

\[
S_{\text{gauged}} = \sum_{\ell=1}^{3} \int_{M_2} \left[ -\frac{1}{2g} \left| (d + i\alpha_{\ell} + i\tilde{A}) \phi_{\ell} \right|^2 + \frac{i\theta_{\ell}}{2\pi} (d\alpha_{\ell} + B) \right] \int_{M_2} B \in \frac{2\pi}{3} \mathbb{Z}
\]

• \( SU(3)/\mathbb{Z}_3 - (\mathbb{Z}_3)_{\text{perm.}} \) anomaly

\((\mathbb{Z}_3)_{\text{permutation}} : Z[(A, B)] \mapsto Z[(A, B)] \exp \left( -ip \int_{M_2} B \right) \) at \( \theta_{\ell} = \frac{2\pi p}{3} \ell \)

\( \rightarrow \) SSB of \((\mathbb{Z}_3)_{\text{perm.}}\) or conformal behavior

• \( SU(3)/\mathbb{Z}_3-C \) global inconsistency

\( C_3 : Z_n[(A, B)] \mapsto Z_n[(A, B)] \exp \left( -2i(n + k) \int_{M_2} B \right) \quad \theta_1 = -\theta_2 + 2\pi k \) (\( k \in \mathbb{Z} \))

Counter terms to avoid anomaly are distinct between \( k \) and \( k+1 \)

\( \rightarrow \) (1) either of them is not trivially gapped (e.g. SSB of \( C \))

(2) both are trivially gapped but separated by quantum transition
't Hooft anomaly matching

- Conjectured phase diagram
  
  Red blobs are points with SSB of $Z_3$ permutation symmetry. Lines between differently-colored SPT phases have SSB of $C$. 

Lajko, et.al. (17) Tanizaki, Sulejmanpasic (18)
Anomaly matching for $Z_3$ twist

$Z_3$ tbc on $R \times S^1$ \[
\begin{align*}
\Phi(x, t + L) &= C\Phi(x, t), \\
a_\ell(x, t + L) &= a_\ell(x, t), \quad \text{with} \quad C = \text{diag}(1, e^{2\pi i/3}, e^{4\pi i/3})
\end{align*}
\]

$\rightarrow$ Exact $Z_3$ symmetry intertwined of $Z_3$ center and shift \((Z_3)_{\text{shift}}\)

- \text{(Z3)$_{\text{shift}}$- (Z3)$_{\text{permutation}}$ anomaly}

\[
Z_{M_1 \times S^1}[B^{(1)}] \mapsto Z_{M_1 \times S^1}[B^{(1)}] \exp \left( -ip \int_{M_1} B^{(1)} \right) \quad \text{at} \quad \theta_\ell = \frac{2\pi p}{3} \ell
\]

spontaneously broken \((Z_3)_{\text{perm.}}\) or conformal behavior.

- \text{(Z3)$_{\text{shift}}$-C global inconsistency}

\[
C_3 : Z_n[B^{(1)}] \rightarrow Z_n[B^{(1)}] \exp \left( -2i(n + k) \int_{M_1} B^{(1)} \right) \quad \theta_1 = -\theta_2 + 2\pi k \ (k \in \mathbb{Z})
\]

Counter terms to avoid anomaly are distinct between $k$ and $k+1$

$\rightarrow$ (1) either is not trivially gapped or (2) separated by quantum transition
DIGA and phase diagram

- **BPS solutions satisfying $Z_3$ tbc**

  \[ \phi_1^T = \frac{1}{\sqrt{1 + |e^{z-z_0}|^2}} \begin{pmatrix} 1 \\ e^{z-z_0} \\ 0 \end{pmatrix}, \quad \phi_2^T = \frac{1}{\sqrt{1 + |e^{z_0-z}|^2}} \begin{pmatrix} -1 \\ e^{z_0-z} \\ 0 \end{pmatrix}, \quad \phi_3^T = \begin{pmatrix} 0 \\ 0 \\ e^{\frac{2\pi i}{3L}(t_0-t)} \end{pmatrix} \]

  Twice of fractional instanton action in $\text{CP}^2$

- **DIGA gives six lowest eigenenergies**

  \[ E_{k\pm}(\theta_1, \theta_2) = \pm Ke^{-S I} \left| e^{\frac{i}{3}(\theta_1-\theta_2)} + e^{\frac{i}{3}(\theta_2+2\pi k)} + e^{-\frac{i}{3}(\theta_1+2\pi k)} \right| \]

Three-branch ground state
• Triple degeneracy at red blobs $\rightarrow Z_3$ perm. symmetry can be broken
• Double degeneracy at lines $\rightarrow C$ can be broken at quantum critical lines
• The mixed ’t Hooft anomalies are realized within this Hilbert subspace

$Z_N$-twisted theory correctly keeps 2d vacuum structure!
Summary

- Bion contributions at field theoretical levels with $\mathbb{Z}_N$-twisted boundary condition yield renormalon-like imaginary ambiguity.
- The 't Hooft anomalies survive in the compactified theory with $\mathbb{Z}_N$-twisted boundary condition.
- Lattice simulation exhibits special behavior of Polyakov loop even at high temperature, which could imply adiabatic continuity.
- Our results indicate $\mathbb{Z}_N$ twisted boundary condition leads to volume independence of the vacuum and phase structures.