Towards unification of quark and lepton flavors in modular invariance

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1 Introduction

We have a big question since the discovery of muon.

What is the principle to control flavors of quarks and leptons ?

Symmetry Approach: S_3 , A_4 , S_4 , A_5 …

New approach appears by using Modular group

- Flavor symmetry is originated from the modular invariance.
- Flavor symmetry acts non-linealy (Modular forms).
- Quark / Lepton masses and mixing depend on a modulus τ, which is stabilized by some unknown mechanism.

Talk by T. Yoshida, H. Uchida

2 Finite modular groups

$S:\tau\longrightarrow -\frac{1}{\tau},$ **Duality**

$T:\tau\longrightarrow \tau+1.$ **Dicrete shift symmetry**

$$S^2=1, \qquad (ST)^3=1.$$

Modular group

$$\Gamma\simeq\{S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I}\}$$

Modular group has series of subgroups Γ(N) level N

Imposing congruence condition
$$\Gamma(N)=\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,Z),\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\pmod{N}\}$$

$ad-bc=1$

called principal congruence subgroups

generate discrete group

$\Gamma_N\equiv\Gamma/\Gamma(N)$ quotient group finite group

$$\Gamma_N\simeq\{S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I},T^N=\mathbb{I}\}$$

$\Gamma_2\simeq S_3$ $\Gamma_3\simeq A_4$ $\Gamma_4\simeq S_4$ $\Gamma_5\simeq A_5$

We can consider effective theories with Γ_N symmetry.

$\mathcal{L}_{\rm eff} \in f(\tau)\phi^{(1)}\cdots\phi^{(n)}$ $f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

$$\boxed{f(\gamma\tau) = (c\tau + d)^k f(\tau)\ , \quad \gamma \in \Gamma(N)}$$

Modular forms have been explicitly given for some cases with fixed weight k.

On the other hand, chiral superfields are not modular forms and we have no restriction on the possible value of weight k_I, a priori.

Modular transformation of chiral superfields in MSSM

$$\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma)\phi^{(I)}$$

Modular weight

Representation matrix

Modular forms with weight 2 are given by using Dedekind eta-function.

Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod^{\infty}(1-q^n) \qquad q = e^{2\pi i\tau}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

$$Y(\alpha,\beta,\gamma,\delta|\tau) = \frac{d}{d\tau}\left(\alpha\log\eta(\tau/3) + \beta\log\eta((\tau+1)/3) + \gamma\log\eta((\tau+2)/3) + \delta\log\eta(3\tau)\right)$$

$$\alpha+\beta+\gamma+\delta = 0$$

$$S:\tau \longrightarrow -\frac{1}{\tau}, \qquad T:\tau \longrightarrow \tau+1.$$

$$S:\quad Y(\alpha,\beta,\gamma,\delta|\tau) \to \tau^2 Y(\delta,\gamma,\beta,\alpha|\tau),$$

$$T:\quad Y(\alpha,\beta,\gamma,\delta|\tau) \to Y(\gamma,\alpha,\beta,\delta|\tau).$$

Consider level N=3, T^3=1 : A_4 group

For triplet

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular forms with weight 2

S transformation **T transformation**

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \boxed{\tau^2}\rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\ Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\ Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \end{aligned}$$

$|q| \ll 1$

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$$q = e^{2\pi i\tau}$$

$$\boxed{Y_2^2 + 2Y_1Y_3 = 0}$$

3 Modular A_4 invariance as flavor symmetry

Consider both quarks and leptons in modular symmetry.

A_4 assignments: left-handed doublet 3, right-handed singlets 1, 1", 1'

Quarks

$$\begin{cases} w_u = \alpha_u u^c H_u \mathbf{Y_3^{(2)}} Q + \beta_u c^c H_u \mathbf{Y_3^{(2)}} Q + \gamma_u t^c H_u \mathbf{Y_3^{(2)}} Q \\ w_d = \alpha_d d^c H_d \mathbf{Y_3^{(2)}} Q + \beta_d s^c H_d \mathbf{Y_3^{(2)}} Q + \gamma_d b^c H_d \mathbf{Y_3^{(2)}} Q \end{cases}$$

Charged Lepton

$$w_e = \alpha_e e^c H_d Y_{\mathbf{3}}^{(2)} L + \beta_e \mu^c H_d Y_{\mathbf{3}}^{(2)} L + \gamma_e \tau^c H_d Y_{\mathbf{3}}^{(2)} L$$

$$M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

for both up- and down-quarks

Simple mass matrix

Can Modular forms reproduce CKM by fixing modulus τ ?

Real prarameters α_q, β_q, γ_q are responsible for quark mass hierarchy.

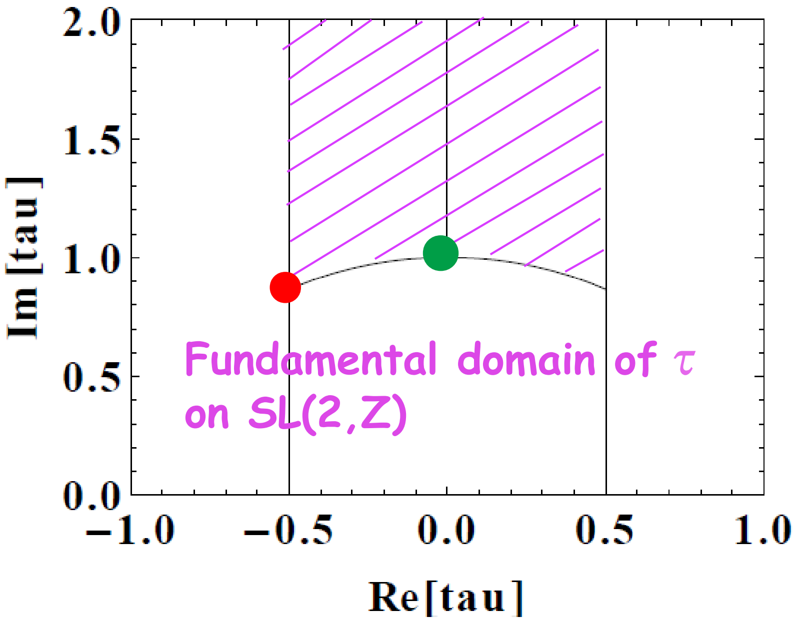
What is a Principle of fixing modulus τ ?

☆ **Modular stabilization** (non-perturbative effect, model dependent)

Some models indicate the potential minimum at the boundary of fundamental domain.

Talk by H. Uchida

☆ **Fixed point of τ** **Residual symmetry**



● $\mathbf{Z}_2$: $\tau = i$ ● $\mathbf{Z}_3$: $\tau = -1/2+\sqrt{3}/2i$

S symmetry ST symmetery

P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov,
JHEP04 (2019) 005,arXiv:1811.04933
P.P.Novichkov, S.T.Petcov and M.T,
PLB 793 (2019) 247, arXiv:1812.11289

Fixed points of τ

$T(\tau \rightarrow \tau+1)$ preserved : **$<\tau>= \infty\, i$** $(q=0)$ $(Y_1, Y_2, Y_3)=(1, 0, 0)$

Z_3: $\{1, T, T^2\}$

$S(\tau \rightarrow -1/\tau)$ preserved : **$<\tau>=i$** $(q=e^{-2\pi})$ $(Y_1, Y_2, Y_3)=Y_1(i)\,(1, 1-\sqrt{3}, -2+\sqrt{3})$

Z_2: $\{1, S\}$

$$\begin{aligned} Y_1(\tau) &= 1+12q+36q^2+12q^3+\cdots\,, \\ Y_2(\tau) &= -6q^{1/3}(1+7q+8q^2+\cdots), \\ Y_3(\tau) &= -18q^{2/3}(1+2q+5q^2+\cdots). \end{aligned}$$

$$\boxed{q=e^{2\pi i\tau}}$$

$$\boxed{\begin{aligned} Z_3:&\quad T^\dagger M_q^\dagger M_q T = M_q^\dagger M_q \quad \Rightarrow \quad [T, M_q^\dagger M_q]=0 \\ Z_2:&\quad S^\dagger M_q^\dagger M_q S = M_q^\dagger M_q \quad \Rightarrow \quad [S, M_q^\dagger M_q]=0 \end{aligned}}$$

Mixing matrices diagonalise $M_q^\dagger M_q$ also diagonalize T and S, respectively !

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}, \qquad \rho(T)=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},$$

Eigenvalues (1, -1, -1)

Define the new basis of generators, $\hat{S}$ and $\hat{T}$ by a unirary transformation as:

$$\hat{S}=USU^\dagger, \qquad \hat{T}=UTU^\dagger \quad \Longrightarrow \quad \hat{M}_{RL}=M_{RL}U^\dagger \ .$$

Since one can take the diagonal basis of S or T, both $M_q^\dagger M_q$ (q=u,d) could be diagonal if there is a residual symmetry Z_2 or Z_3 of A_4.

Hierarchical flavor structure is realized around τ =i or ∞ in general !

$$M_q=\begin{pmatrix}\alpha_q & 0 & 0\\ 0 & \beta_q & 0\\ 0 & 0 & \gamma_q\end{pmatrix}\begin{pmatrix}Y_1 & Y_3 & Y_2\\ Y_2 & Y_1 & Y_3\\ Y_3 & Y_2 & Y_1\end{pmatrix}_{RL}$$

at $<\tau>$=i $(Y_1, Y_2, Y_3)=Y_1(i)\,(1, 1-\sqrt{3}, -2+\sqrt{3})$

at $<\tau>$= i ∞ (q=0) $(Y_1, Y_2, Y_3)=(1, 0, 0)$

$$M_q^\dagger M_q=\boxed{\begin{pmatrix}0 & 0 & 0\\ 0 & \times & \times\\ 0 & \times & \times\end{pmatrix}} \quad \boxed{\begin{pmatrix}\tilde{\alpha}_q^2 & 0 & 0\\ 0 & \tilde{\beta}_q^2 & 0\\ 0 & 0 & \tilde{\gamma}_q^2\end{pmatrix}}$$

Too simple !

Vub is inconsistent with data even if τ is scanned.

Let us consider Modular forms with higher weights k=4, 6 …

# of modular forms is k+1

Weight 2
3 Modular forms

$$\mathbf{Y}_3{}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}}$$
$$\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}}$$
$$\oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}\,, \qquad \mathbf{1'} \otimes \mathbf{1'} = \mathbf{1''}\,, \qquad \mathbf{1''} \otimes \mathbf{1''} = \mathbf{1'}\,, \qquad \mathbf{1'} \otimes \mathbf{1''} = \mathbf{1}\,.$$

Modular forms with higher weights are constructed by the tensor product of modular forms of weight 2

J.T.Penedo, S.T.Petcov, Nucl.Phys.B939(2019)292

Weight 4
5 Modular forms

$$\mathbf{Y}^{(4)}_{\mathbf{1}} = Y_1^2 + 2Y_2Y_3\,, \quad \mathbf{Y}^{(4)}_{\mathbf{1'}} = Y_3^2 + 2Y_1Y_2\,, \quad \mathbf{Y}^{(4)}_{\mathbf{1''}} = Y_2^2 + 2Y_1Y_3 = 0\,,$$

$$\mathbf{Y}^{(4)}_{\mathbf{3}} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}\,,$$

Weight 6
7 Modular forms

$$\mathbf{Y}^{(6)}_{\mathbf{1}} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3\,,$$

$$\mathbf{Y}^{(6)}_{\mathbf{3}} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_1^2Y_2 + 2Y_2^2Y_3 \\ Y_1^2Y_3 + 2Y_3^2Y_2 \end{pmatrix}\,, \qquad \mathbf{Y}^{(6)}_{\mathbf{3'}} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^2Y_1 + 2Y_1^2Y_2 \\ Y_3^2Y_2 + 2Y_2^2Y_1 \end{pmatrix}$$

4 A lesson in Quarks and Leptons

Quark Sector

	Q	(d^c, s^c, b^c)	(u^c, c^c, t^c)	$H_{u,d}$	$\mathbf{Y}_{\mathbf{3}}^{(2)}$	$\mathbf{Y}_{\mathbf{3}}^{(6)},\ \mathbf{Y}_{\mathbf{3'}}^{(6)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}\ ,\quad \mathbf{3'}$
$-k_I$	-2	$\boxed{(0,\ 0,\ 0)}$	$\boxed{(-4,\ -4,\ -4)}$	0	$k=2$	$k=6$

$$M_d = \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} ,$$

Weight 2 modular forms

$$M_u = \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + \begin{pmatrix} g_{u1} & 0 & 0 \\ 0 & g_{u2} & 0 \\ 0 & 0 & g_{u3} \end{pmatrix} \begin{pmatrix} Y_1^{\prime(6)} & Y_3^{\prime(6)} & Y_2^{\prime(6)} \\ Y_2^{\prime(6)} & Y_1^{\prime(6)} & Y_3^{\prime(6)} \\ Y_3^{\prime(6)} & Y_2^{\prime(6)} & Y_1^{\prime(6)} \end{pmatrix} \right]_{RL}$$

Weight 6 modular forms

After removing parameters α_q, β_q, γ_q by inputting quark masses, we have 3 complex parameters in addition to τ (8 real parameters).

We set $g_{u1}=g_{u2}=g_{u3}$. 4 parameters ⇔4 CKM elements are reproduced.

Lepton Sector

Common modulus τ for both quarks and leptons

	L	(e^c, μ^c, τ^c)	H_u	H_d	$\mathbf{Y_r^{(2)}}, \quad \mathbf{Y_r^{(4)}}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1}, \mathbf{1''}, \mathbf{1'})$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}, \ \{\mathbf{3}, \mathbf{1}, \mathbf{1'}\}$
$-k_I$	-2	$(0,\ 0,\ 0)$ or $\boxed{(-4, -4, -4)}$	0	0	$2, \quad 4$

$$M_E^{(6)} = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \left[\begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix} + g_e \begin{pmatrix} Y_1^{\prime(6)} & Y_3^{\prime(6)} & Y_2^{\prime(6)} \\ Y_2^{\prime(6)} & Y_1^{\prime(6)} & Y_3^{\prime(6)} \\ Y_3^{\prime(6)} & Y_2^{\prime(6)} & Y_1^{\prime(6)} \end{pmatrix} \right]_{RL}$$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L \mathbf{Y_r^{(k)}})_{\mathbf{1}}$$

Weinberg operator by using weight 4 modular forms

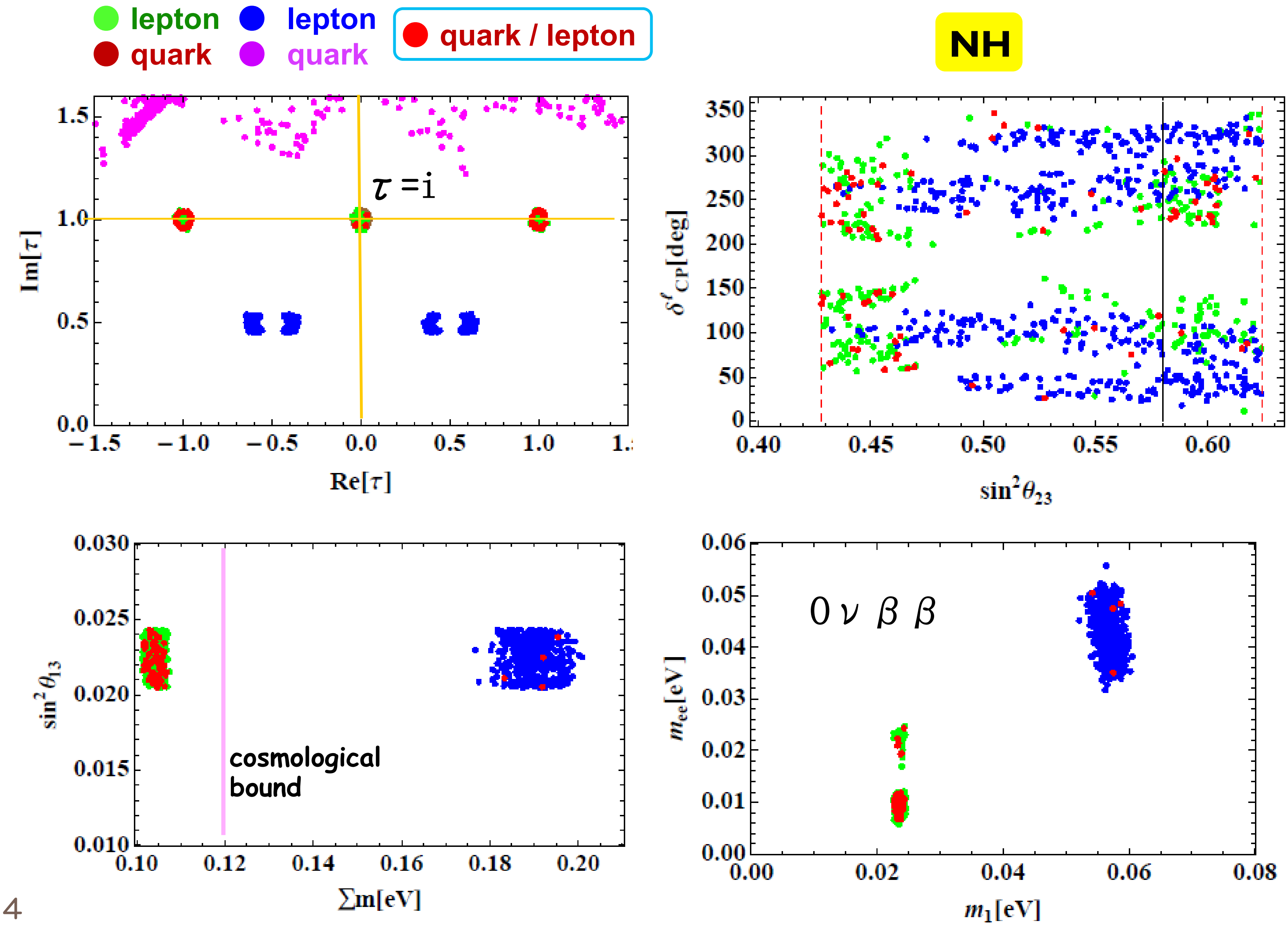
$$M_\nu = \frac{v_u^2}{\Lambda} \left[\begin{pmatrix} 2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\ -Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\ -Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} \end{pmatrix} + g_{\nu 1} \mathbf{Y_1^{(4)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\nu 2} \mathbf{Y_{1'}^{(4)}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]_{LL}$$

At τ =i (S symmetric limit)

$$\boxed{M_\nu = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}}$$

Input of 4 observed values: θ_{12}, θ_{23}, θ_{13}, $\Delta m^2_{sol} / \Delta m^2_{atm}$

output : δ_{CP}, $\langle m_{ee} \rangle$, Σm_i


● lepton
● quark
● lepton
● quark
● quark / lepton
NH
τ =i
Im[τ]
Re[τ]
1.5
1.0
0.5
0.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
$\delta^{\ell}_{\rm CP}$[deg]
$\sin^2\theta_{23}$
350
300
250
200
150
100
50
0
0.40
0.45
0.50
0.55
0.60
$\sin^2\theta_{13}$
Σm[eV]
0.030
0.025
0.020
0.015
0.010
0.10
0.12
0.14
0.16
0.18
0.20
cosmological
bound
$0\nu\beta\beta$
$m_{\rm ee}$[eV]
m_1[eV]
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.00
0.02
0.04
0.06
0.08

5 Summary

We present a A_4 modular invariant model
with common τ for quarks and leptons
by using weight 2,4 and 6 modular forms.

- Modulus τ is common in both quarks and leptons close to τ =i.
- Quark Mass matrices is consistent with observed CKM matrix.
- Lepton mass matrix is consistent with 3 observed mixing angles
  NH is favored. (IH is not allowed).

By imposing common τ for quarks and leptons,
δ_{CP}, $\langle m_{ee} \rangle$ and $\Sigma\ m_i$ are predicted.

New approach for Quark-Lepton unification with modular invariance !

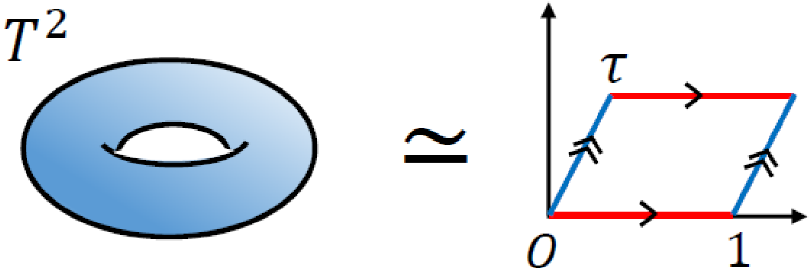
Back up slides

Alternative assignment of weight for quarks

	Q	(q_1^c, q_2^c, q_3^c)	H_q	$\mathbf{Y_3^{(k)}}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$
$-k_I$	-2	$(-4,\ -2,\ 0)$	0	$k=2,4,6$

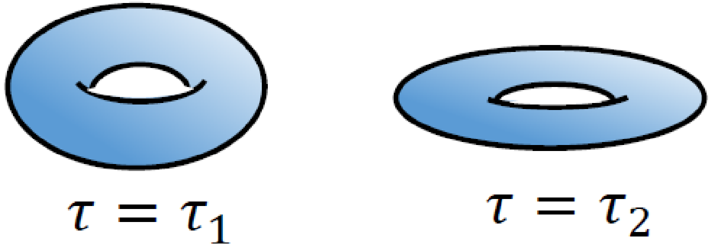
$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1^{\prime(6)} & Y_3^{(6)} + g_q Y_3^{\prime(6)} & Y_2^{(6)} + g_q Y_2^{\prime(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$2D$ torus (T^2) is equivalent to parallelogram with identification of confronted sides.



Two-dimensional torus T^2 is obtained as $T^2 = \mathbb{R}^2 / \Lambda$
Λ is two-dimensional lattice

The shape of torus is represented by a modulus $\tau \in \mathbb{C}$.



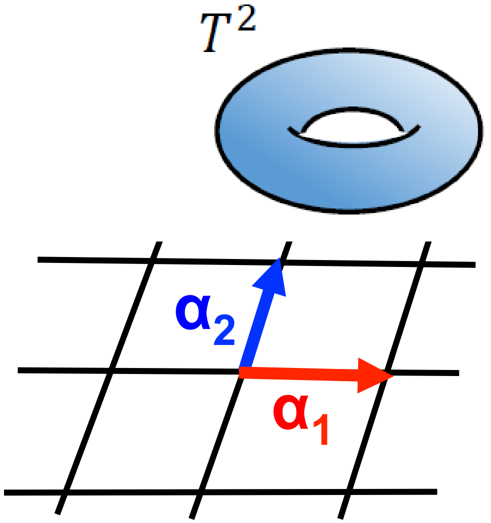
The different value of τ realize the different shape of T^2

$\mathcal{L}_{\text{eff}}$ depends on τ. e.g.) $\mathcal{L}_{\text{eff}} \supset Y(\tau)_{ij}\phi\overline{\psi_i}\psi_j + \cdots$

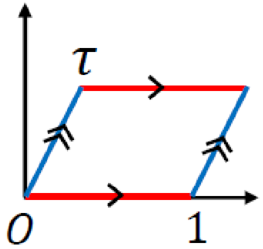
- $4D$ effective theory depends on a modulus τ

Modular transformation

The shape of a torus $T^2 \simeq$ The shape of a lattice on ℂ-plane



$\simeq$



$(x,y) \sim (x,y)+n_1\alpha_1+n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice, which is spanned by two lattice vectors

$\alpha_1=2\pi R$ and $\alpha_2=2\pi R\tau$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\boxed{\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}}$$

ad-bc=1

a,b,c,d are integer SL(2,Z)

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

$$\Downarrow$$

$\tau = \alpha_2 / \alpha_1$

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

Modular transformation

Modular transf. does not change the lattice (torus)

$$\Downarrow$$

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

The modular transformation is generated by S and T .

$$\boxed{\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

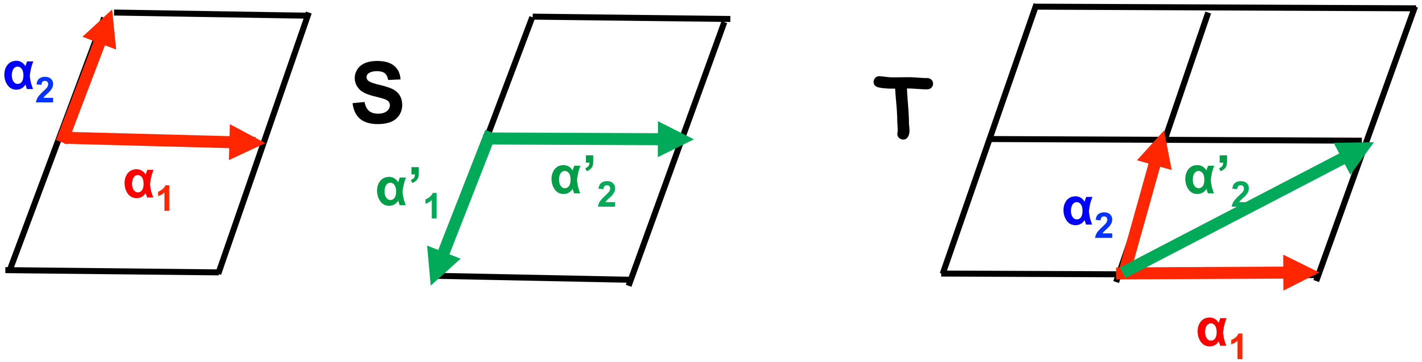
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$$



$$\tau = \alpha_2 / \alpha_1$$

Kinetic Term

Kinetic term of the modulus τ

$$\frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation

$$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

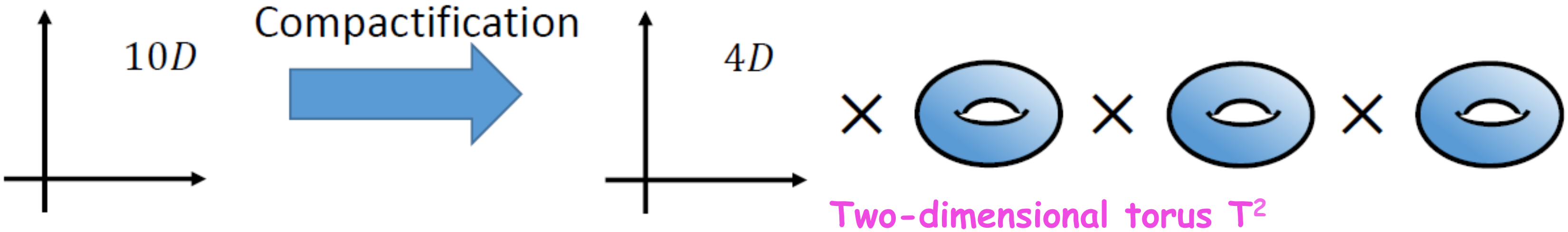
$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

➡ $$\frac{\left|\partial_\mu \tau'\right|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{\left|\partial_\mu \tau\right|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

Superstring theory 10D
Our universe is 4D

→ The extra 6D should be compactified.

Torus compactification

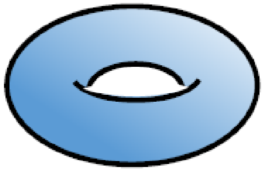


Two-dimensional torus T^2

We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\text{eff}}$$

→ $\mathcal{L}_{\text{eff}}$ depends on the structure of

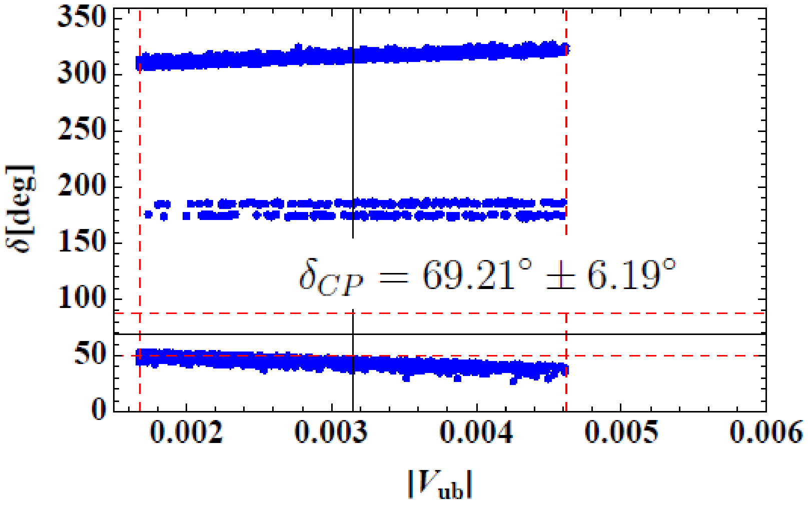


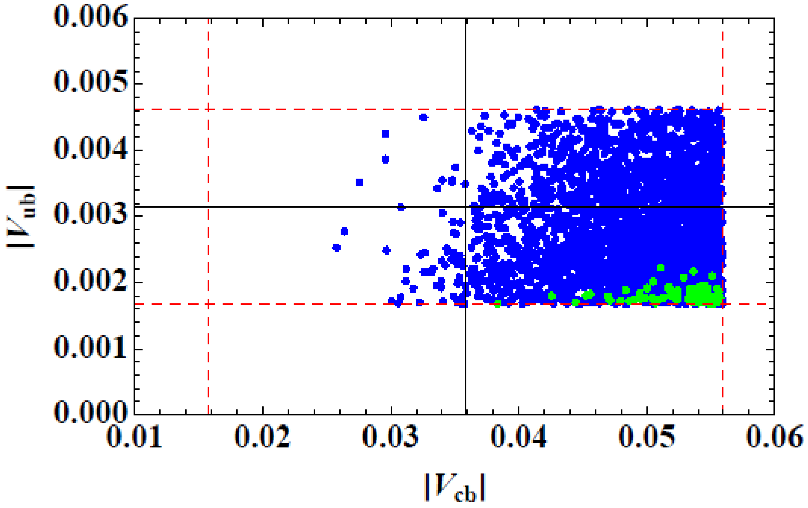
- $4D$ effective theory depends on internal space

Input : charged lepton masses and three mixing angles at GUT scale

Effect of RGE depends on tan β , M_{SUSY} , threshold effect ….

Output : CP violating phase δ_{CP} (PDG)





(tan β =5, M_{SUSY}=1 TeV)

$$|V_d| = \begin{pmatrix} 0.5537 & 0.6135 & 0.5631 \\ 0.8110 & 0.2439 & 0.5317 \\ 0.1889 & 0.7511 & 0.6326 \end{pmatrix} , \qquad |V_u| = \begin{pmatrix} 0.4857 & 0.6859 & 0.5419 \\ 0.8198 & 0.2382 & 0.5208 \\ 0.3034 & 0.6876 & 0.6596 \end{pmatrix}$$