Foundation of Normal Conducting Accelerator

The 2nd International School on Beam Dynamics and Accelerator Technology (ISBA19)

Masao KURIKI
Graduate School of Advanced Sciences of Matter,
Hiroshima University

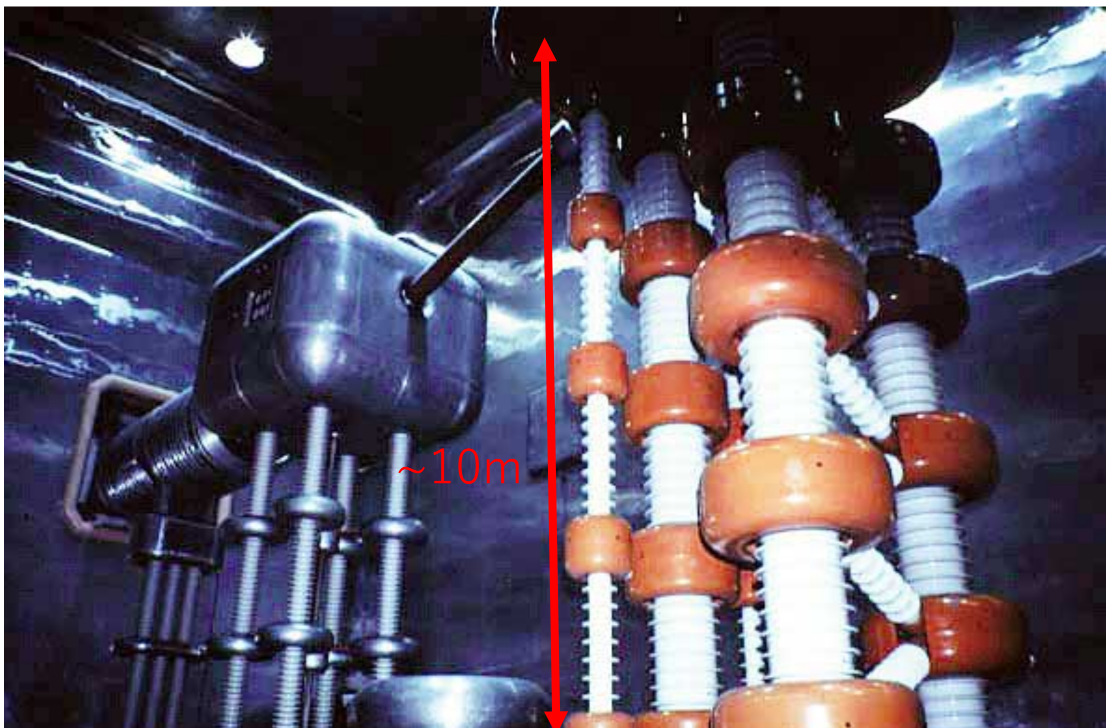
Contents

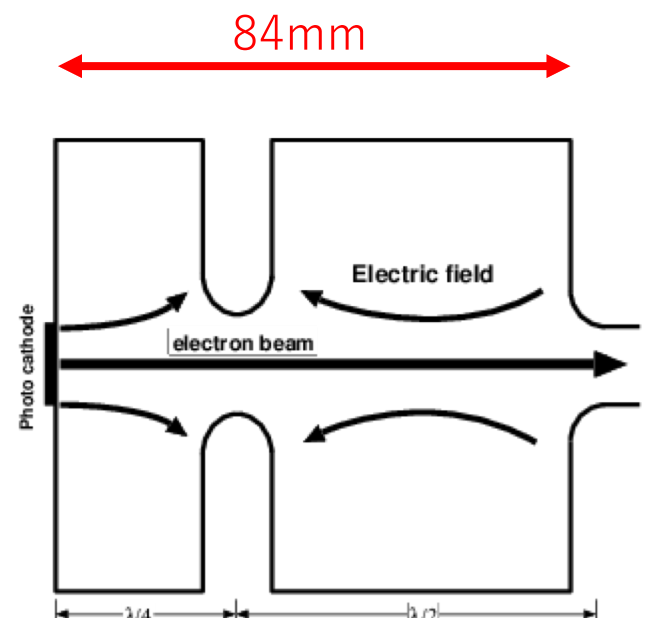
- The greatest Invention in Accelerator science, RF accelerator.
- Foundation of RF acceleration.
- Accelerator Structures.
- Beam loading control.

The greatest Invention in Accelerator science, RF accelerator

Static and RF fields

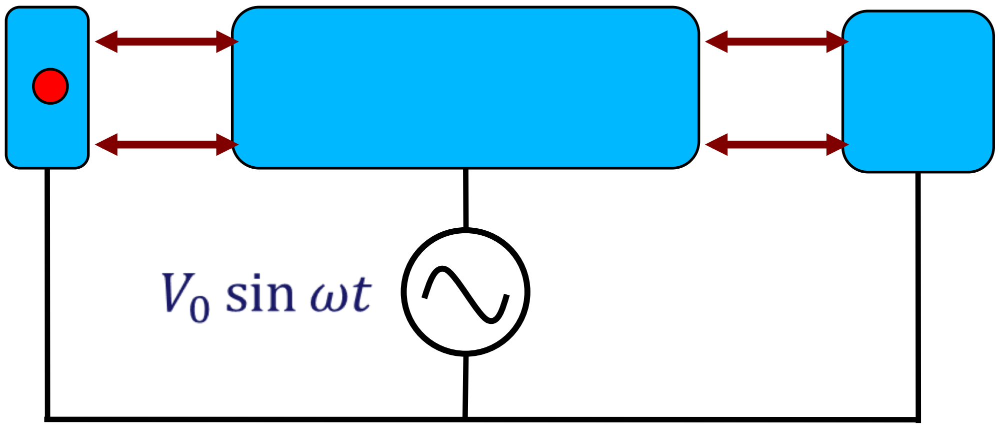
- Cockcrof Walton（KEK-PS Injector）： 700kV/15 x 15 x 15 (m^3) = 2.1e+2 V/m^3.
- RF electron gun：4MV/0.2x0.2x0.2 (m^3)=5.0e+8 V/m^3.

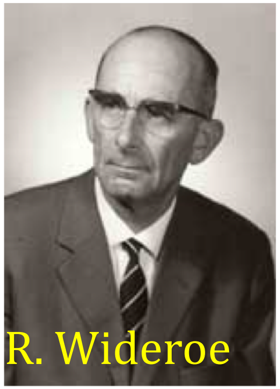




The greatest Invention

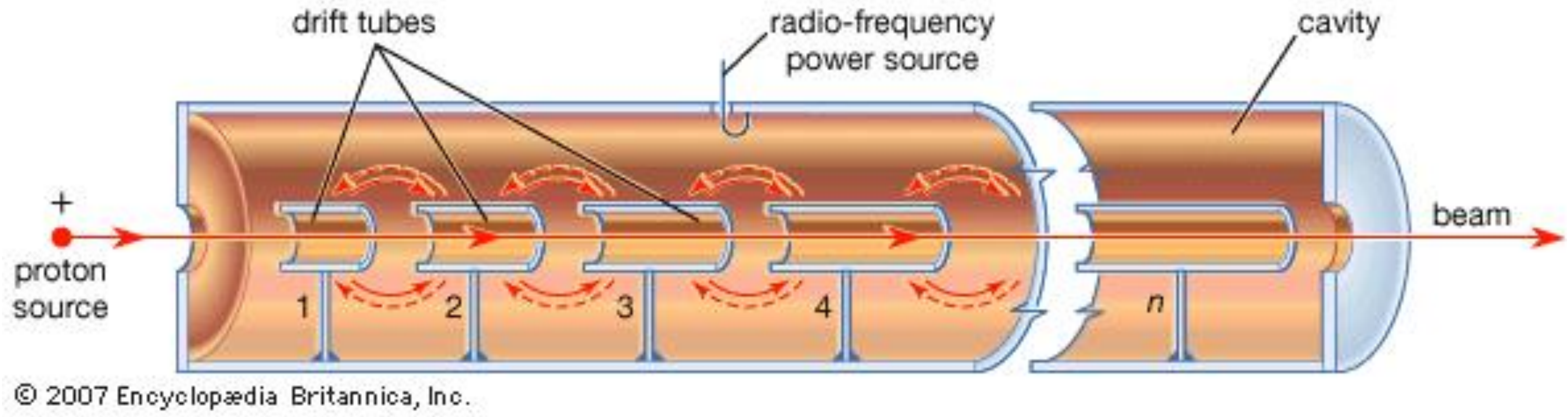
- R. Wideroe invented principle of AC acceleration （RF acceleration）. This is the greatest invention in the accelerator science.
- By the repetitive acceleration employing temporally varied EM field, unlimited acceleration becomes possible.





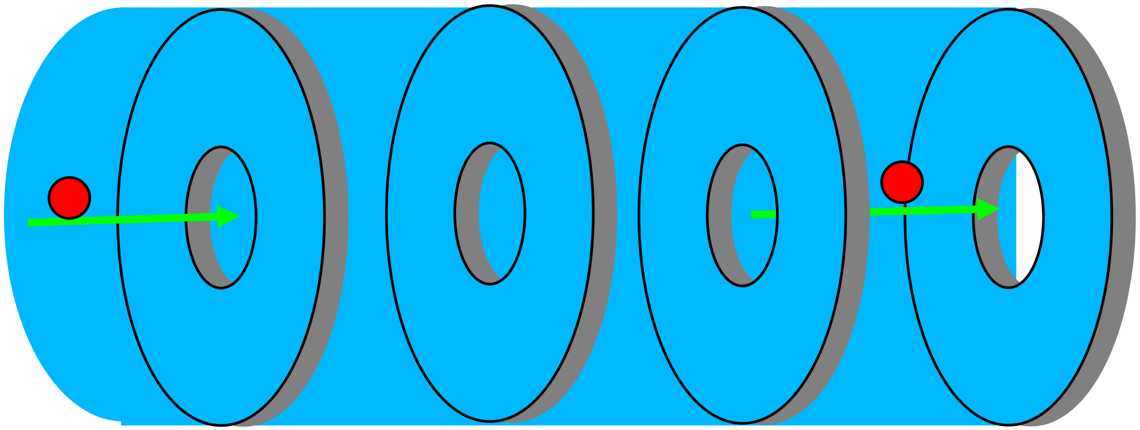
Alvarez-Linac:The first resonant cavity.

- Resonant structure surrounded by metal wall.
- No limitation on the acceleration frequency.
- Alvarez type, DTL(Drift Tube Linac) is still used in the hadron accelerator.



Disk-loaded Linac

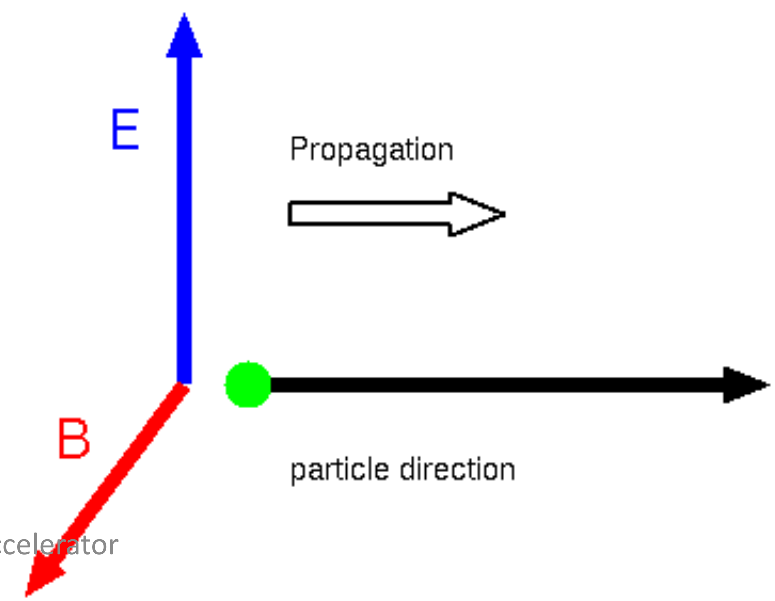
- In a cylindrical wave guide, disks with a hole is placed.
- The particle passes through the hole.
- Phase velocity and group velocity are controlled by the disk shape and interval.



Foundation of RF Acceleration

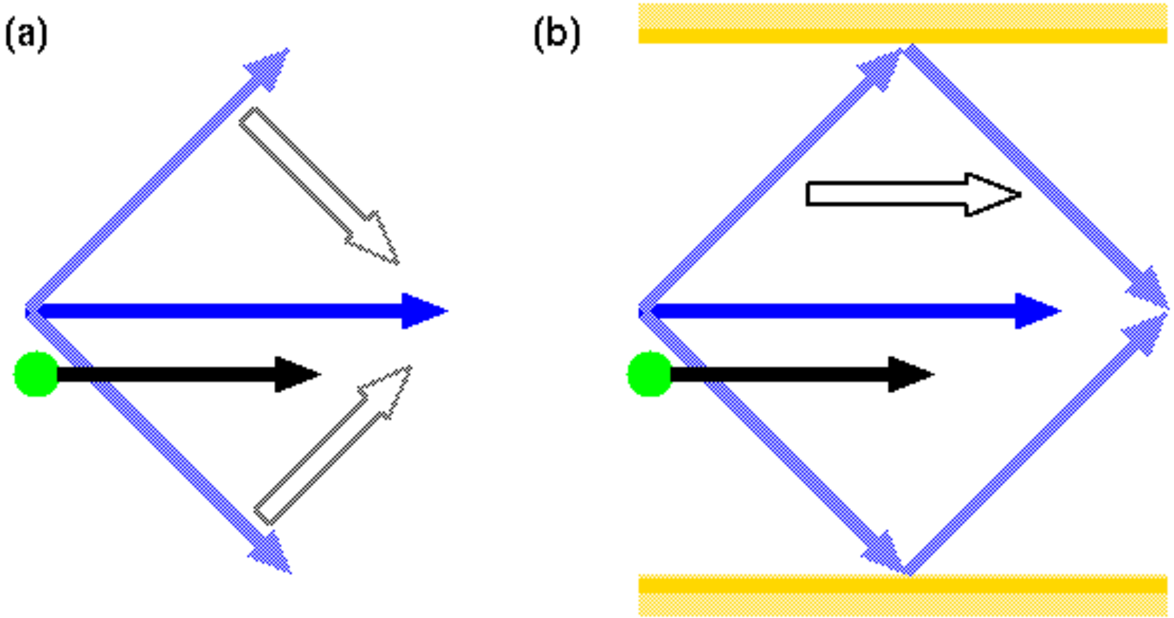
Question

- All of modern high energy accelerator is based on RF acceleration.
- Radio Frequency field is composed from plane wave.
- Plane wave propagation direction is perpendicular to Electric field.
- Question: How to realize a continuous acceleration?



Answer

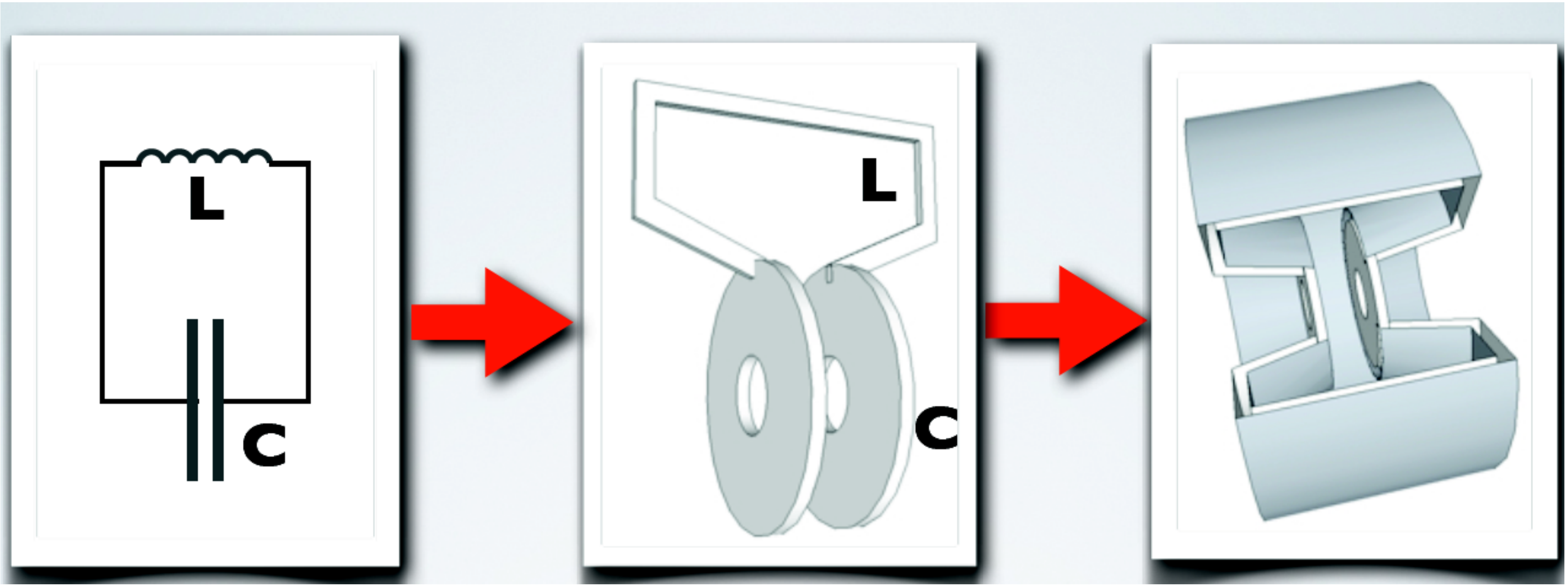
- Superposition of plane waves + reflection : E-field and propagation becomes same direction.
- Reflection is by metal wall -> cavity structure.



Pill Box Cavity

- The simplest accelerator structure.
- It is equivalent with a LC circuit.
- Frequency is determined by Geometry.

$$\omega = \frac{1}{\sqrt{LC}}$$



Starting from Maxwell Equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \mathrm{t}}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

In vacuum,

$$\Delta \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0,$$

$$\Delta \boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0,$$

with

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Helmholz Equation

Assume temporal oscillation,

$$\boldsymbol{E} = \boldsymbol{E_s} e^{\imath\omega t}$$
$$\boldsymbol{B} = \boldsymbol{B_s} e^{\imath\omega t}$$

Equation is simplified as

$$\Delta \boldsymbol{E_s} + k^2 \boldsymbol{E_s} = 0,$$
$$\Delta \boldsymbol{B_s} + k^2 \boldsymbol{B_s} = 0,$$

with

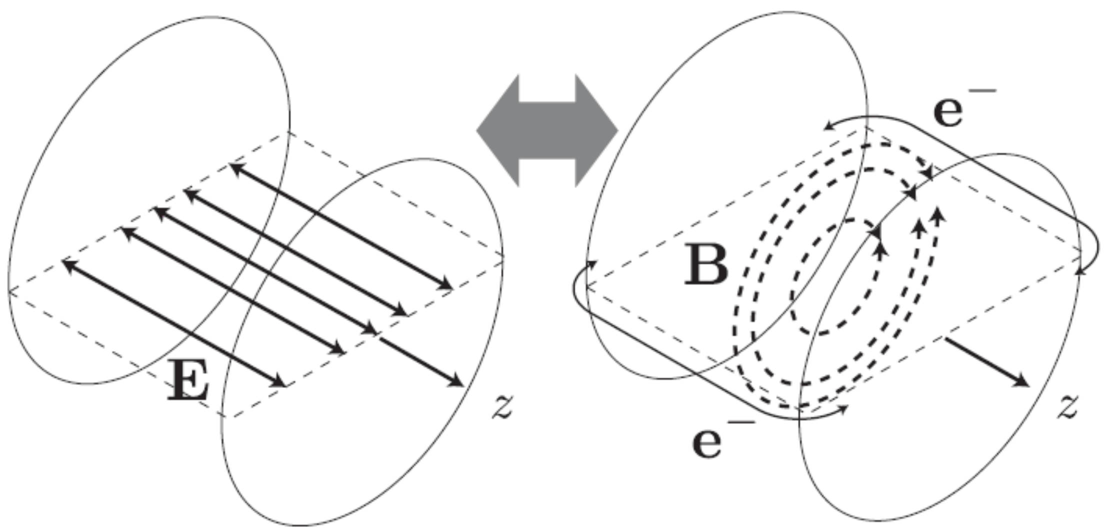
$$k^2 \equiv \frac{\omega^2}{c^2} = \varepsilon_0 \mu_0 \omega^2$$

Solving Helmholz Equation

- Assuming TM mode (Only transverse B field)
- Other components are zero.

$$E_z = E_0 J_0(kr)\cos\omega t$$
$$H_\phi = -\frac{E_0}{Z_0} J_1(kr)\sin\omega t,$$

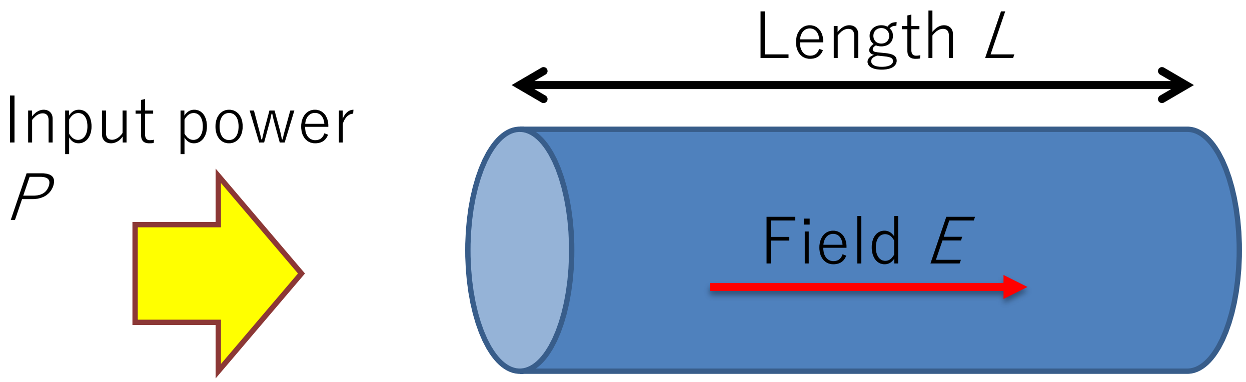
$Z_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega,\ k = 2\pi/\lambda = p_{01}/b,\ p_{01}$ は Bessel 関数の根で 2.405



How much gain?

- Shunt impedance *r* (Ω/m) gives the gain,
  - *E*: field
  - *P*: dissipated power
  - *L*: length

$$r = \frac{(EL)^2}{PL} = \frac{E^2}{P/L}$$



Transit Time Factor

Time variation of RF field gives less acceleration than *EL*

$$V = \int_{\mathrm{L}/2}^{L/2} E_0 \cos\frac{\omega s}{\lambda} ds = LE_0 \frac{\sin\frac{\omega L}{2\lambda}}{\frac{\omega L}{2\lambda}} = LE_0 T$$

where *T* is Transit time factor

$$T \equiv \frac{\sin\frac{\omega L}{2\lambda}}{\frac{\omega L}{2\lambda}} < 1$$

The effective shunt impedance is defined with *T* as

$$r = \frac{(LE_0 T)^2}{PL} = \frac{(E_0 T)^2}{P/L}$$

Shunt Impedance of Pill Box Cavity

$$r = \frac{G_1 G_2 T^2}{\lambda R_s}$$

Surface Resistance

Wave length

A physics constant

$$G_2 = \frac{4Z_0}{P_{01}^2 J_1^2(P_{01})} = 967[\Omega],$$

$$G_1 = \frac{P_{01}}{2}\left(\frac{L}{b+L}\right) Z_0 = 453\left(\frac{L}{b+L}\right)[\Omega],$$

A geometrical factor

$$R_s = \frac{\rho}{2\pi B \delta} \Rightarrow R_s \propto \sqrt{\omega}$$

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}}$$

$$\lambda \propto \frac{1}{\omega}$$

$$\Rightarrow Z_{sh} \propto \sqrt{\omega}$$

Higher ω is better

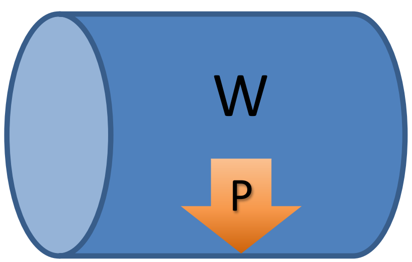
Q value

Q-value :numbers of oscillation to damp the stored energy *W*.

$$
\begin{aligned}
Q &= \frac{\omega W}{P} \\
&= \frac{\omega(=ck=cP_{01}/b)Z_0\varepsilon(=\sqrt{\mu\varepsilon}=1/c)Z_0 bL}{2R_s(b+L)} \\
&= \frac{P_{01}Z_0L}{2R_s(b+L)} \\
&= \frac{G_1}{R_s}
\end{aligned}
$$

A geometrical factor (G_1)

Surface resistance (R_s)



What is r/Q ?

- r/Q is the ratio of the shunt impedance and Q value.
- The shunt impedance is product of r/Q and Q. So what???

$$\left(\frac{r}{Q}\right) = \frac{r}{Q}$$

$$r = Q\left(\frac{r}{Q}\right)$$

This is r/Q.

$$\frac{r}{Q} = \frac{V^2}{\omega W L}$$
$$= \frac{2(E_0 T L)^2}{\omega \pi \varepsilon b^2 L^2 E_0^2 J_1^2(P_{01})}$$
$$= \frac{2T^2 Z_0}{\omega(= cP_{01}/b)\pi(= P_{01}\lambda/(2b))Z_0\varepsilon(= 1/c)b^2 J_1^2(P_{01})}$$
$$= \frac{4T^2 Z_0}{P_{01}^2 J_1^2(P_{01})\lambda}$$
$$= \frac{G_2 T^2}{\lambda},$$

A Constant

Transit time factor

Wave length

(r/Q) is determined by the geometry.

Shunt impedance again

- *r* is composed from two parts : (r/Q) and Q.

$$r = \left(\frac{r}{Q}\right) Q = \left(\frac{G_2 T^2}{\lambda}\right) \left(\frac{G_1}{R_s}\right)$$

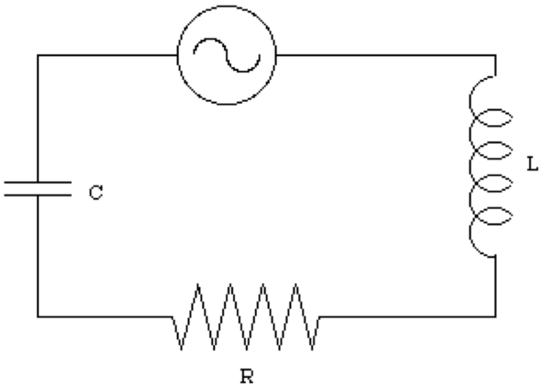
Geometry

Material property

- r and Q values are measurable. The cavity shape and electrical quality can be evaluated with (r/Q) and Q values, respectively.

Equivalent Circuit :Pillbox

- Let us start with a simple L, C, R circuit.
- Impedance is



$$Z = R + \jmath(\omega L - \frac{1}{\omega C}),$$

- It can be rewritten as

$$Z = R\left[1 + \jmath\sqrt{\frac{L}{C}}\frac{1}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right],$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Equivalent Circuit :Pillbox

- We define Q as

$$Q = \sqrt{\frac{L}{C}}\frac{1}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0}CR,$$

- With Q, Z is rewritten as

$$Z = R\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right],$$

$$Z \sim R\left(1 + jQ\frac{2\delta f}{f_0}\right),$$

$$\delta f = \frac{\omega - \omega_0}{2\pi}$$

Equivalent Circuit :Pillbox

- On resonance (δ_f=0), the impedance is

$$Z = R\left(1 + jQ\frac{2\delta_f}{f_0}\right) = R$$

- Off resonance ($\delta_f = f_0/(2Q)$), the impedance is

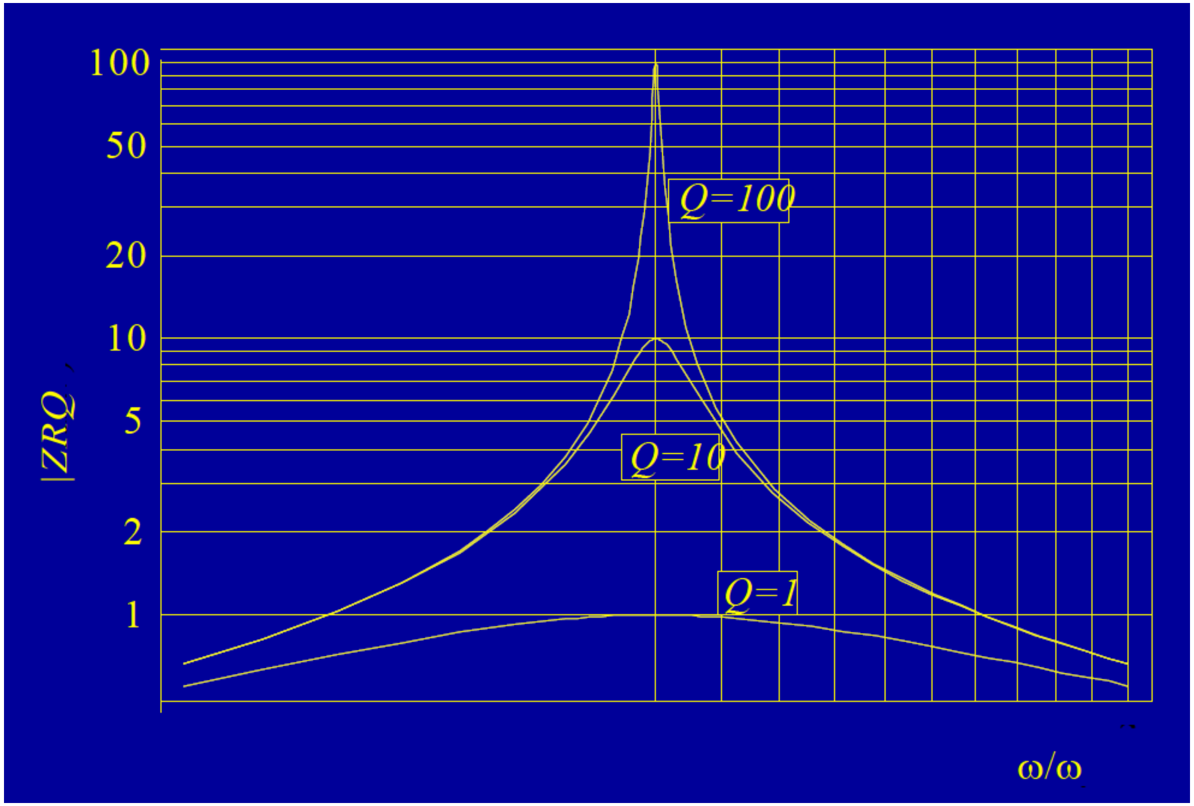
$$Z = R\left(1 + jQ\frac{2\delta_f}{f_0}\right) = R(1 + j)$$

	Tuned	Detuned
Impedance	R	$\sqrt{2}R$
Field	E	$E/\sqrt{2}$
Stored Energy	W	$W/2$

Resonance Curve

Find δ_f where the resonance curve becomes ½ of the peak.

$$Q = \frac{f_0}{2\delta_f}$$



Other Definition of Q-value

$$W_C = \frac{1}{2}CV_C^2 = \frac{C}{2}\left(\frac{I}{\jmath\omega_0 C}\right)^2 = \frac{1}{2}LI^2,$$

$$P = \frac{1}{2}RI^2,$$

$$Q = \omega_0\frac{L}{R} = \omega_0\frac{1/2LI^2}{1/2RI^2} = \omega_0\frac{W}{P},$$

The last definition of Q-value

Q is a time constant of decay curve of W.

$$\frac{dW}{dt} = -P = -\frac{\omega_0}{Q}W$$

because $Q = \omega_0 \frac{W}{P}$. It leads

$$W(t) = W_0 e^{-\frac{\omega_0}{Q}t}$$

Multi-cell cavity

The multi-cell cavity is LCR circuits with mutual inductance L_c

nth cell

$$\left(\frac{1}{j\omega C}+R+j\omega L\right)i_n+j\omega L_c(i_n-i_{n+1})+j\omega L_c(i_n-i_{n-1})=0,$$

we assume a solution in a form of $i_n=i_0e^{j(\omega t+n\phi)},$

which gives

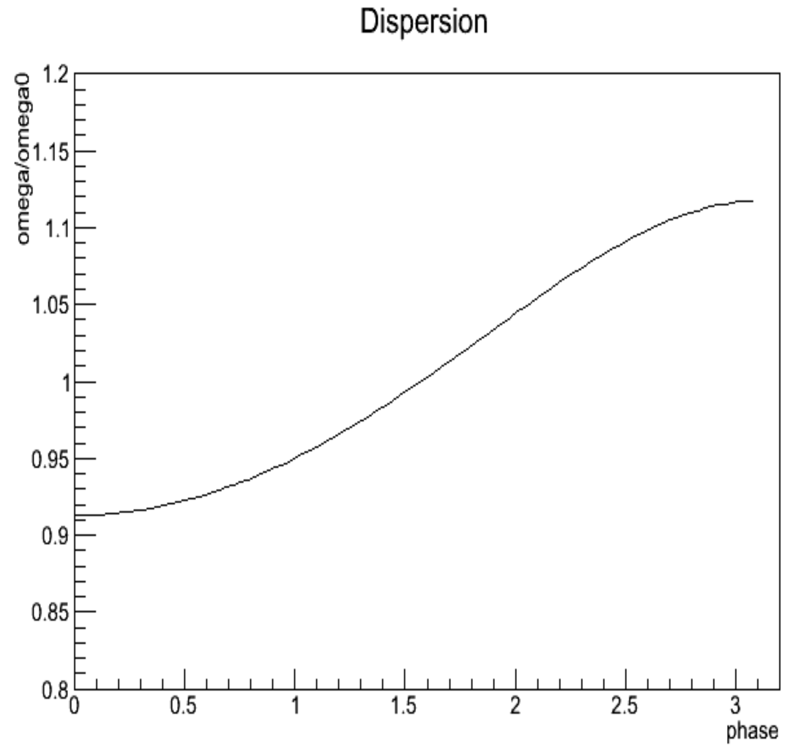
$$\left(\frac{1}{j\omega C}+R+j\omega L\right)i_n+j\omega 2L_c(1-\cos\phi)i_n=0,$$

Dispersion Relation

$$\frac{1}{\jmath\omega C} + R + \jmath\omega\left[L + 2L_c(1 - \cos\phi)\right] = 0,$$

$$\omega_0 = 1/\sqrt{C(L + 2L_c)}$$

$$\omega = \omega_0\left[1 - \frac{2L_c}{L}\cos\phi\right]^{-1/2}$$



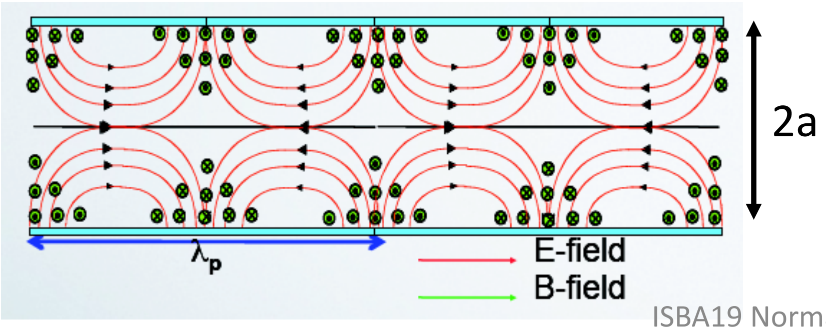
EM field in a cylinder

TM$_{010}$ solution

$$E_z = E_0 J_0(k_c r) e^{-jk_z z} e^{j\omega t}$$

$$E_r = j\frac{k_z}{k_c} E_0 J_1(k_c r) e^{-jk_z z} e^{j\omega t}$$

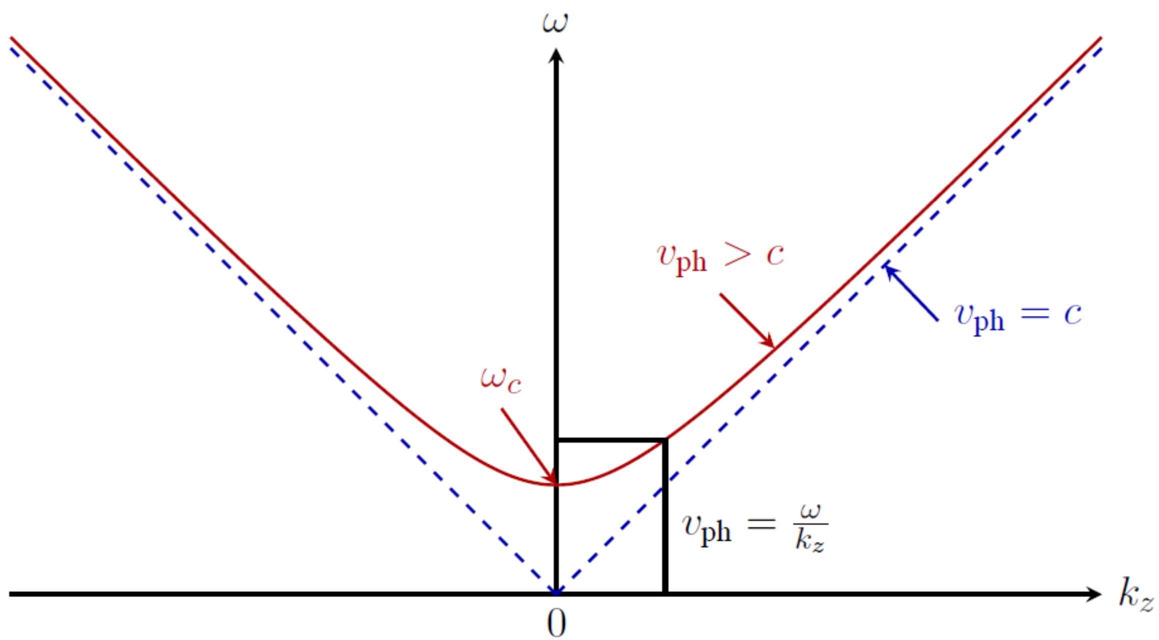
$$H_\phi = j\frac{k}{Z_0 k_c} E_0 J_1(k_c r) e^{-jk_z z} e^{j\omega t}$$



$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \qquad {k_z}^2 = k^2 - {k_c}^2$$

$$k_c = \frac{2\pi}{\lambda_c} = \frac{\omega_c}{c} \qquad \lambda_c = 2.61a$$

Dispersion Relation



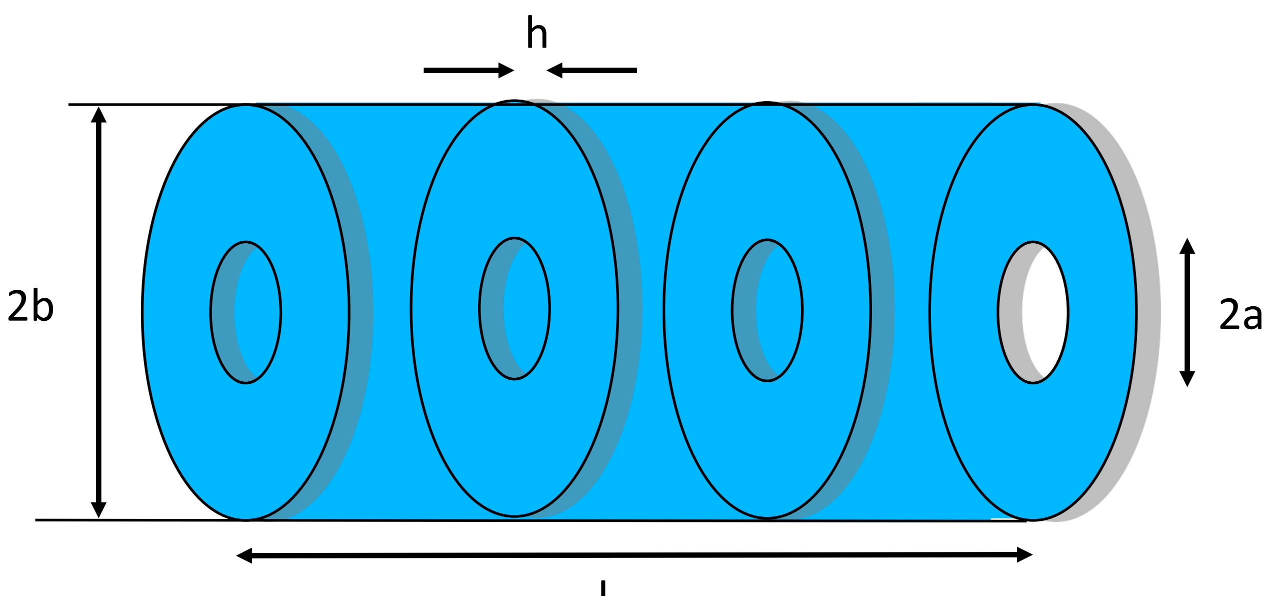
$$\omega^2 = \omega_c^2 + c^2 k_z^2$$

$$v_{ph} = \frac{\omega}{k_z} = \sqrt{\frac{\omega^2}{k_z^2} + c^2} > c$$

v_{ph} is always faster than c.
No physical particle can synchronize with the EM field in a cylinder.

Disk-loaded Structure

- Periodic boundary condition by disks.
- Dispersion relation is modified by the periodic condition.
- A part of forward wave is reflected by the disks and a backward wave is induced.

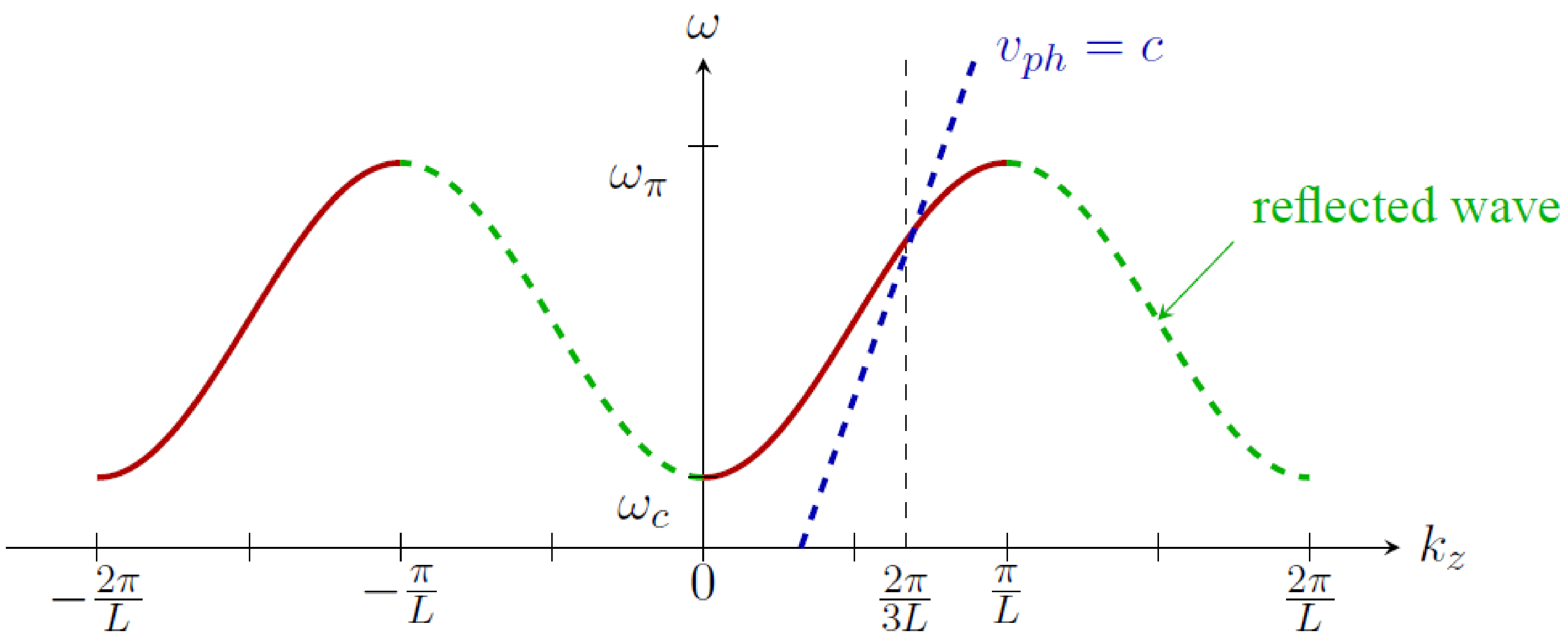


Dispersion Relation of Disk-loaded structure

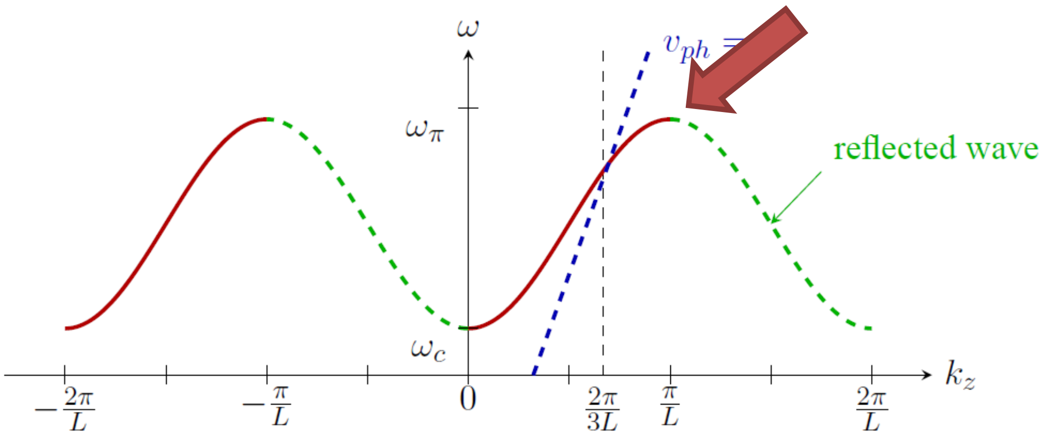
$$\omega = \frac{2.405c}{b}\sqrt{1 + \kappa(1 - \cos(k_z L)e^{-\alpha h})}$$

$$\kappa = \frac{4a^3}{3\pi {J_1}^2(2.405)b^2 L} \ll 1$$

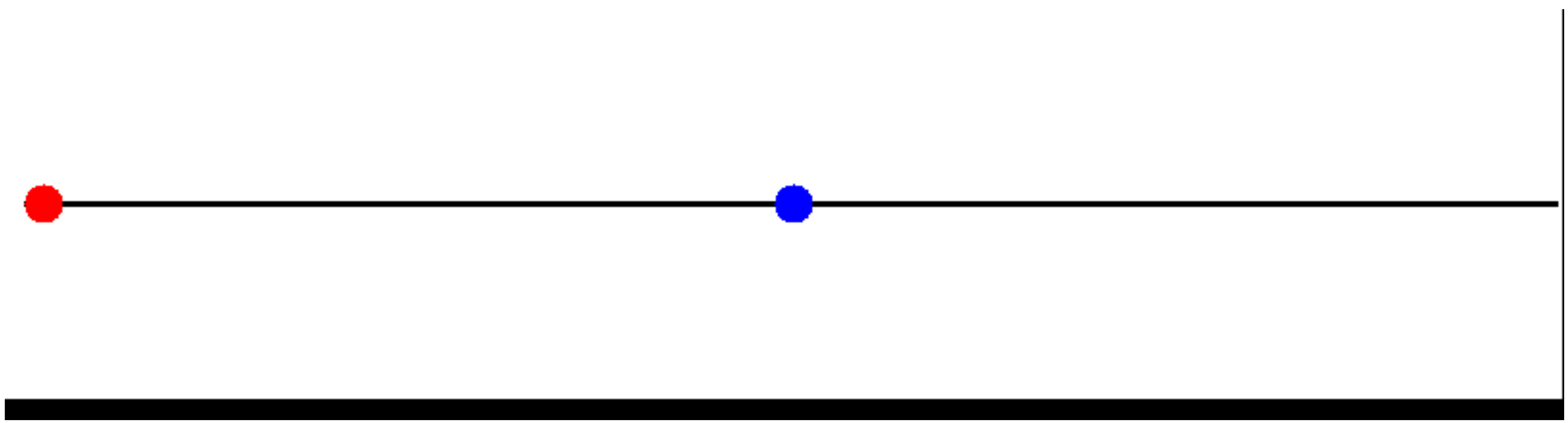
$$\alpha = \frac{2.405}{a}$$



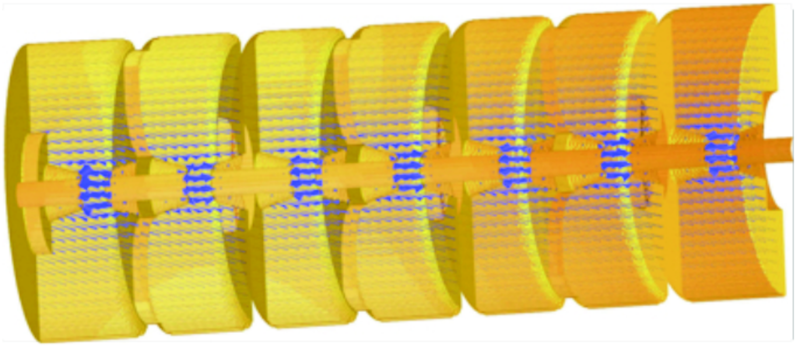
Standing Wave Linac



$$v_g = \frac{\partial \omega}{\partial k_z} = 0$$



Standing Wave Linac

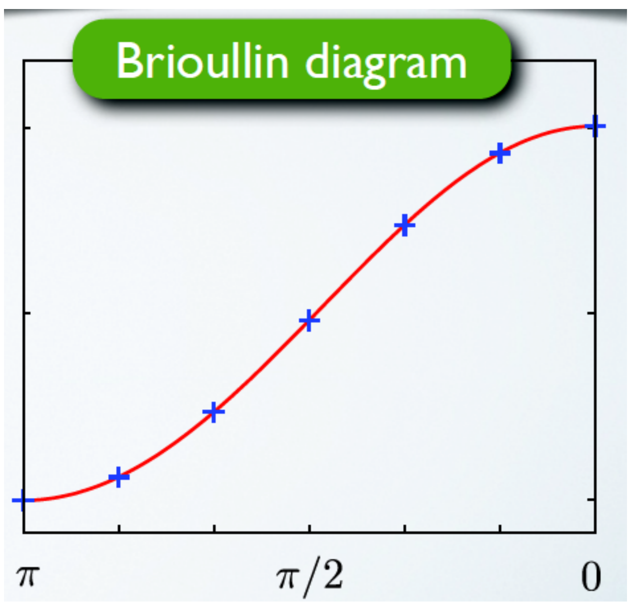


- **Operated in Standing Wave.**
- **Phase advance per cell is**

$$\phi_n = \frac{n\pi}{N},\ n = 1, 2, \cdots, N$$

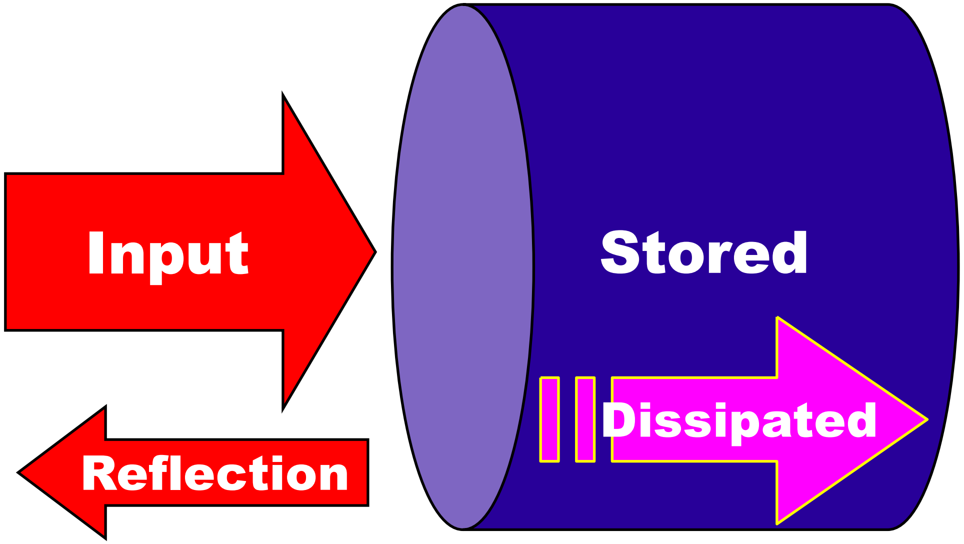
$$f_n = \frac{f_0}{\sqrt{\;- k\cos(n\pi/N)}}$$

$$\frac{f_0}{\sqrt{\;- k}}$$



$$\frac{f_0}{\sqrt{\;+ k}}$$

SW Cavity Power Balance



Matching

$$E_i + E_r = E_{cav}$$

Start

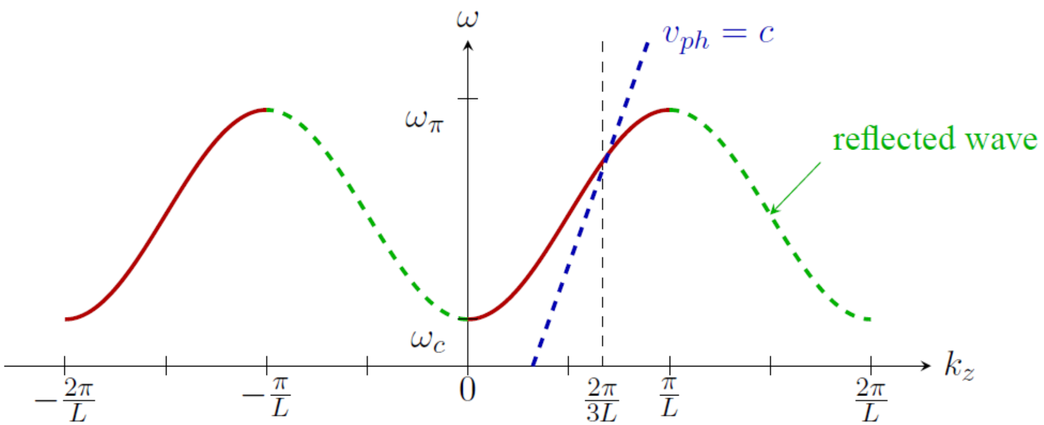
$$E_i + E_r = 0$$

End

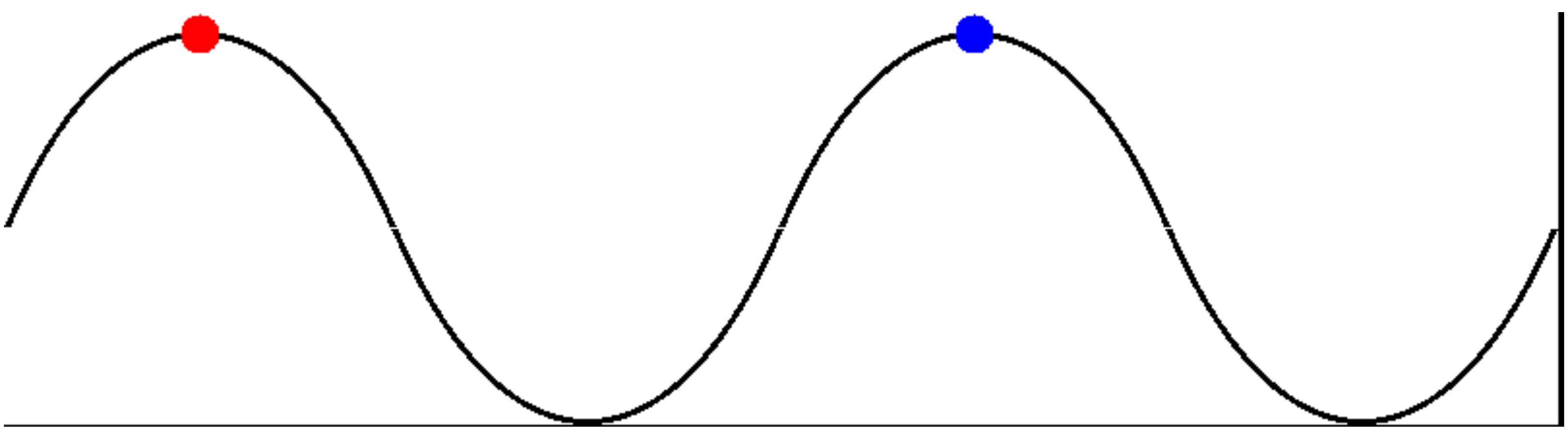
$$E_i = E_{cav}$$

- The field should be matched at the coupling window.
- Starting from 100% reflection, the cavity power is grown up to 0% reflection.

Traveling Wave Cavity

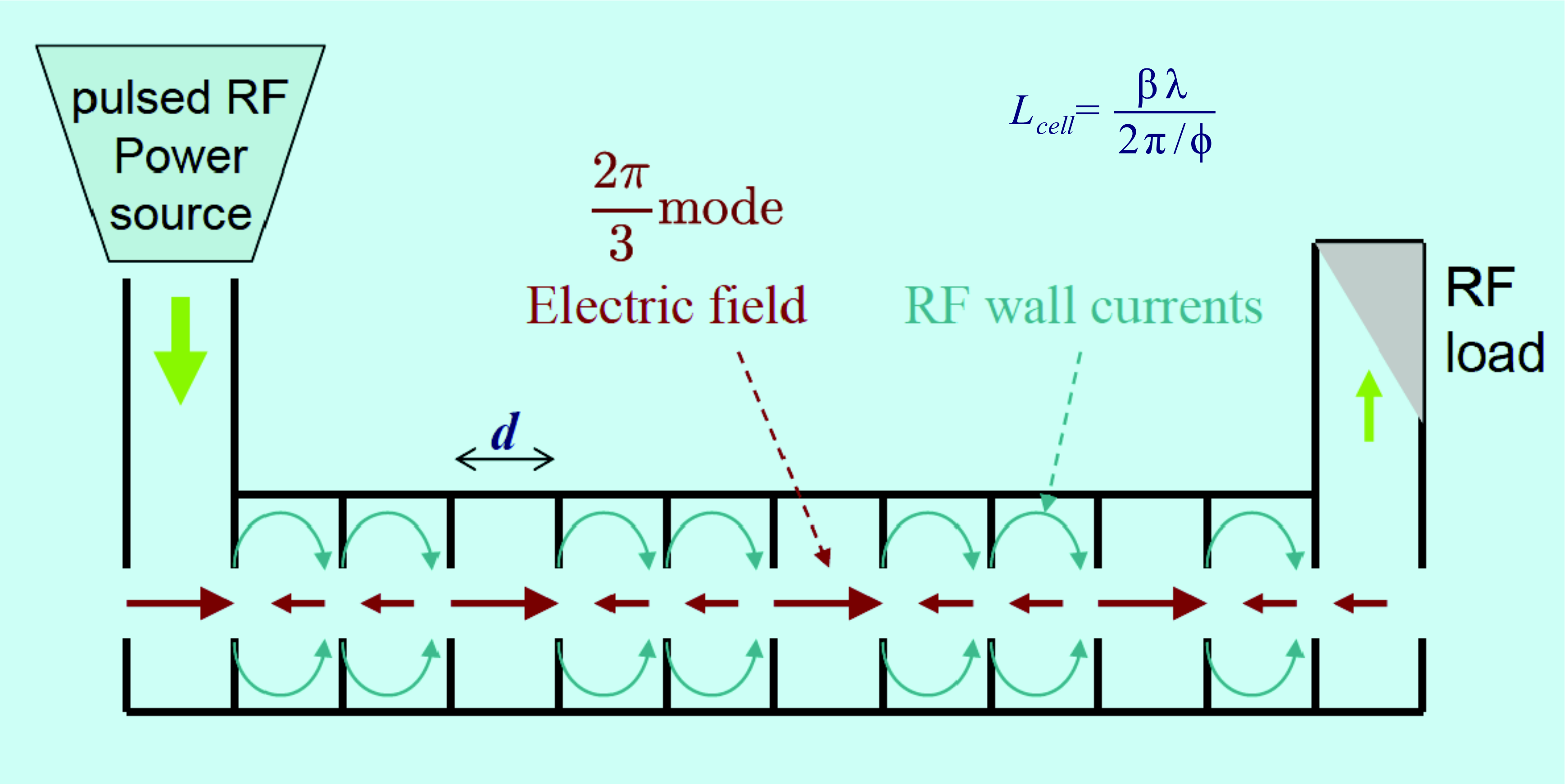


$$v_g = \frac{\partial \omega}{\partial k_z} \neq 0$$

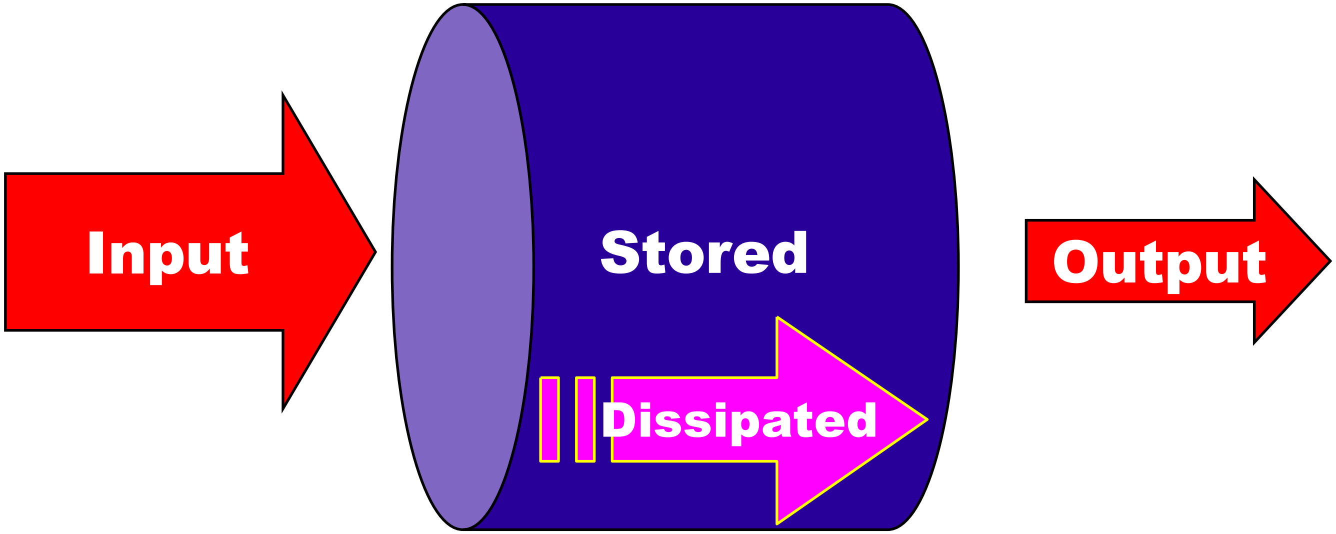


Traveling Wave Linac

- EM power flows through accelerator



TW Cavity Power Balance

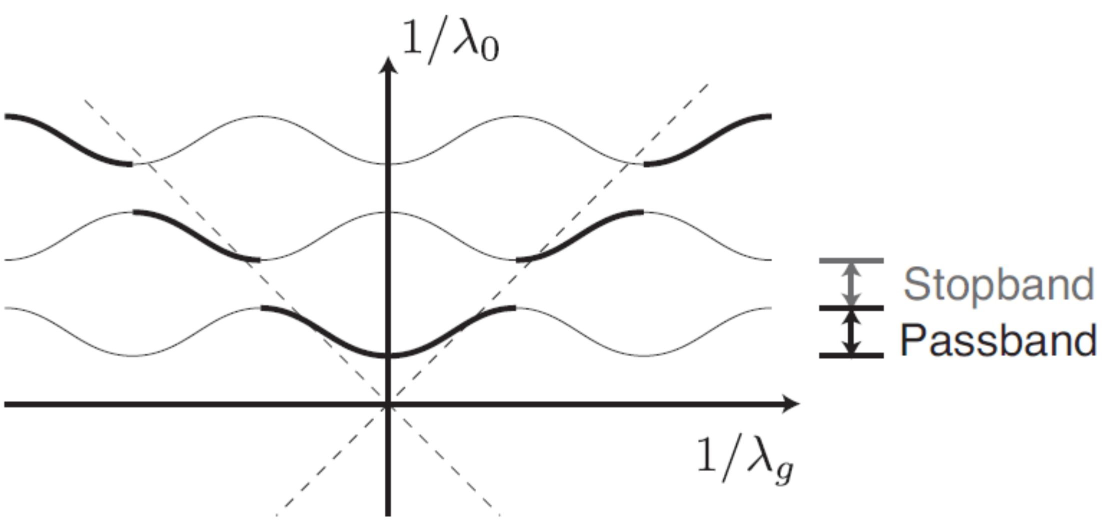


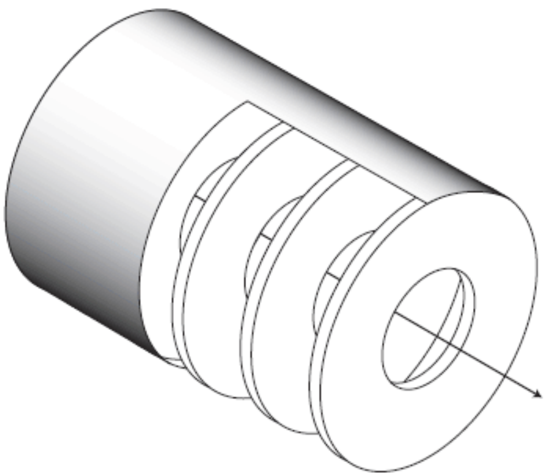
- Basically no reflection.
- Power flows through cavities.
- Rest of power is extracted and dumped.

Property of TW structure

Disk-Loaded TW Linac

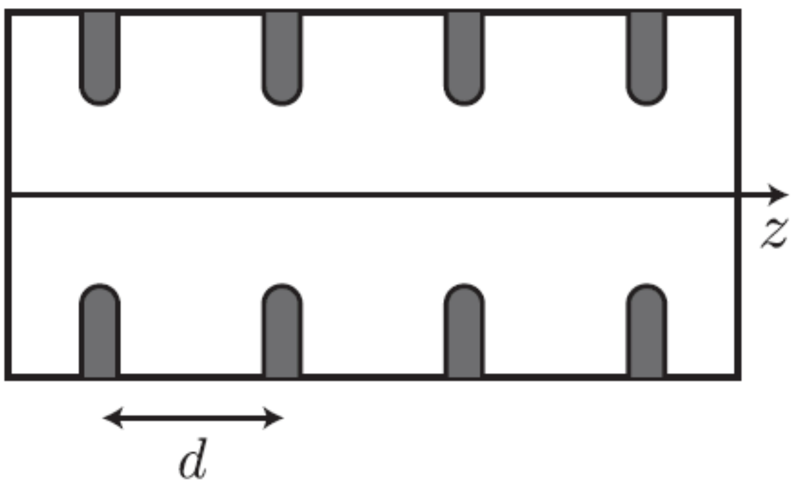
- By inserting disks periodically to the cylindrical wave guide, the phase velocity can be decreased.
- Synchronous frequency is determined from the dispersion relation.



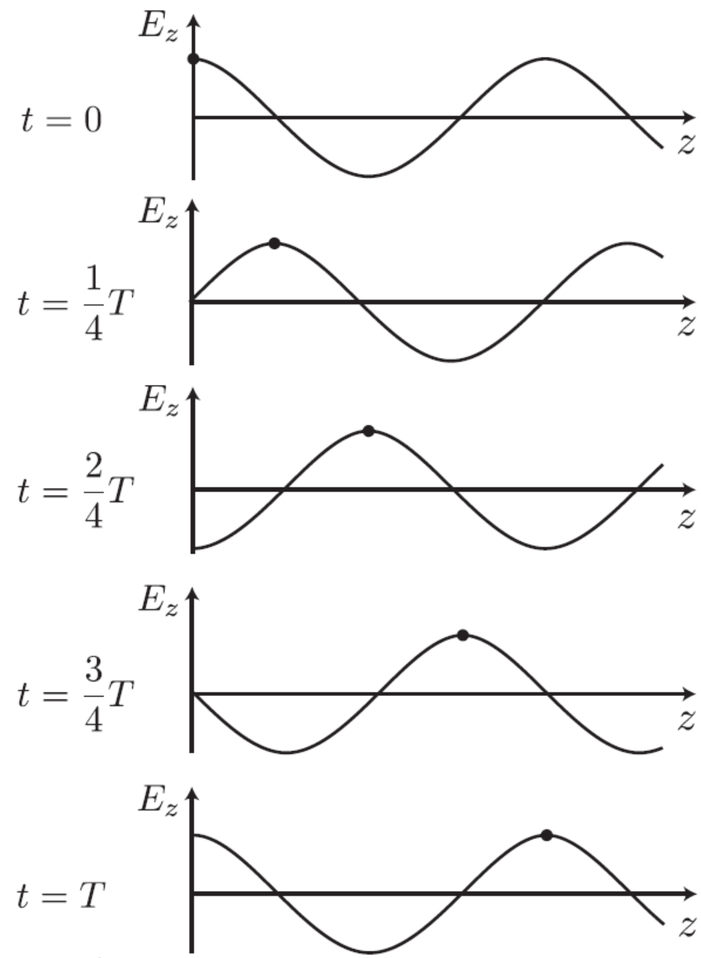


Disk-Loaded TW Linac

By operating it with the synchronous frequency, the particle sits always on a same phase along the structure.

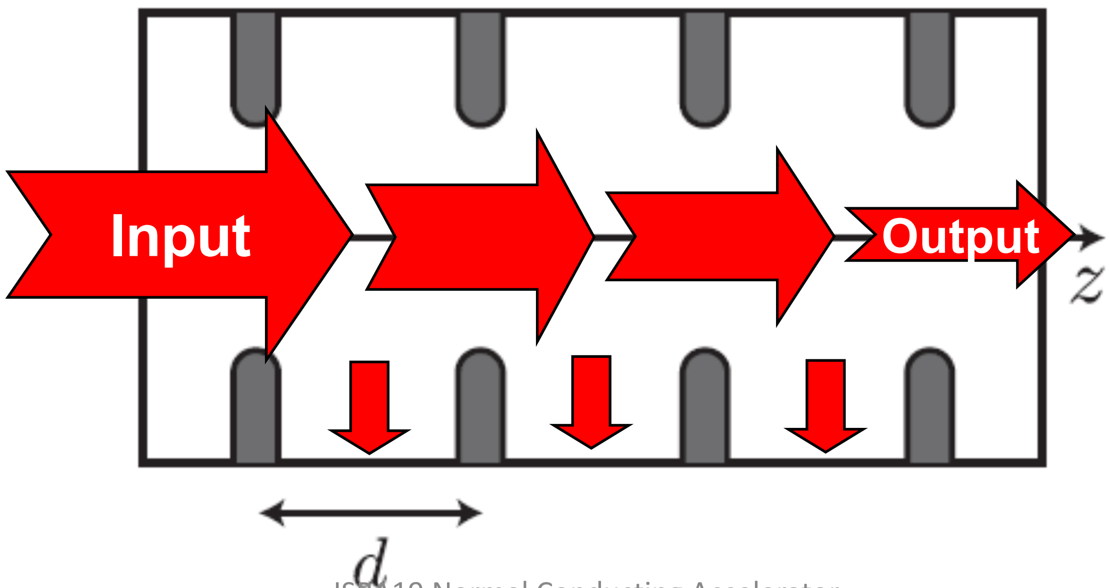


$$d = \frac{\varphi}{2\pi}\lambda$$



Disk-Loaded TW Linac

- The structure can be parametrized with a similar way to the pill box.
- The biggest difference from the SW cavity is that the power flows through the TW structure.



Parameters

Shunt Impedance: $r = -\frac{E^2}{dP/dz}$

Q vaule : $Q = -\frac{\omega W}{dP/dz}$

r/Q : $Q = -\frac{E^2}{\omega W}$

Group velocity : $v_g = \frac{P}{W} = \frac{\omega P}{QdP/dz}$

Attenuation parameter α : $\alpha = -\frac{1}{2P}\frac{dP}{dz}$

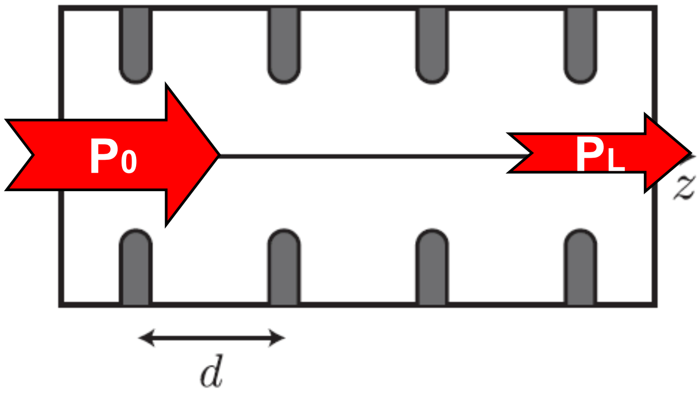
Filling time t_f: $t_f = \int_0^L \frac{dz}{v_g}$

Attenuation constant

$$\frac{dP}{dz} = -2\alpha P$$

P(z) is

$$P(z) = P_0 e^{\left(-2\int_0^z \alpha(z')dz'\right)}$$



Output power

$$P(L) = P_0 e^{\left(-2\int_0^L \alpha(z')dz'\right)} = P_0 e^{(-2\tau)}$$

With attenuation constant τ as

$$\tau = \int_0^L \alpha(z')dz'$$

Which can be measured as

$$\tau = -\frac{1}{2} ln\left(\frac{P_L}{P_0}\right)$$

Constant Impedance Structure

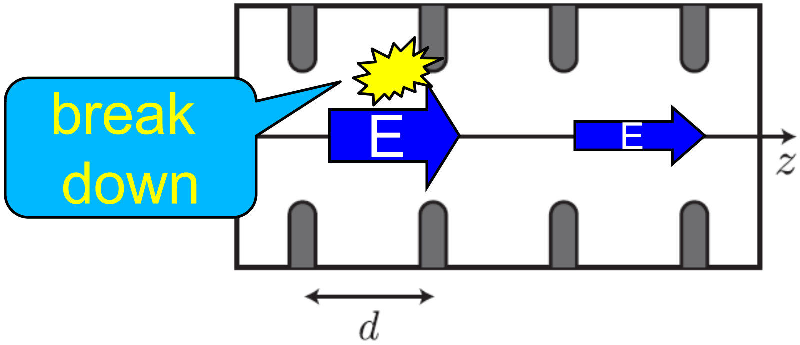
Constant v_g and r along the structure.

$$\tau = \alpha L$$

$$E(z) = E_0 e^{-\alpha z}$$

$$P(z) = P_0 e^{-2\alpha z}$$

Field is not constant along z. Because the maximum field is higher than the average, frequent discharge.



Constant Gradient

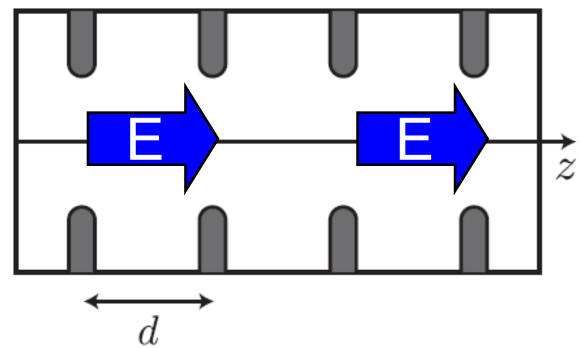
- In the CG structure, α is determined to give a constant field along the structure.
- α is adjusted by changing the iris size (aperture, *a*).
  - *r* depends not strongly on *a*.
  - $dP(z)/dz$ should be constant.

$$P(z) = P_0 - \frac{P_0 - P_L}{L} z = P_0 \left(1 - \frac{1 - e^{-2\tau t}}{L} z \right)$$

$$\frac{dP}{dz} = -\frac{P_0 - P_L}{L} = -P_0 \frac{1 - e^{-2\tau t}}{L}$$

The gradient of CG structure is a constant as expected.

$$E^2 = -r\frac{dP}{dz} = rP_0\frac{1-e^{-2\tau t}}{L}$$



v_g and t_f are

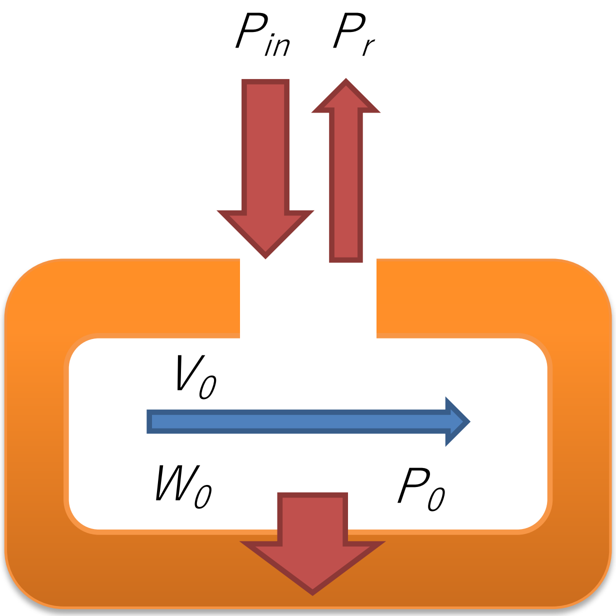
$$v_g = \frac{\omega}{Q}\frac{L-(1-e^{-2\tau t})}{1-e^{-2\tau t}}z$$

$$t_f = \frac{2Q\tau}{\omega}$$

Because the maximum field is same as the average, discharge rate is much suppressed.

Transient Property and Beam Loading Compensation

Standing Wave Accelerator



Power balance

$$\frac{dW_0}{dt} = P_{in} - P_r - P_0$$

Express in V_0

$$\frac{dW_0}{dt} = \frac{Q_0}{\omega}\frac{dP_0}{dt} = \frac{Q_0}{\omega}2G_0V_0\frac{dV_0}{dt}$$

Admittance G_0

$$P_0 = G_0{V_0}^2$$

With the coupling beta β, admittance of wage guide is

$$G_{wg} = \beta G_0$$

$$P_{in} = \beta G_0 {V_{in}}^2 \qquad P_r = \beta G_0 {V_r}^2$$

Matching condition at the coupling window

$$V_{in} + V_r = V_0$$

$$\frac{Q_0}{\omega} 2 G_0 V_0 \frac{dV_0}{dt} = P_{in} - P_r - P_0 = -\beta G_0 {V_0}^2 + 2\beta G_0 V_0 V_{in} - G_0 {V_0}^2$$

The equation can be summarized with $Q_0 = (1+\beta) Q_L$

$$\frac{2Q_L}{\omega} \frac{dV_0}{dt} = -V_0 + \frac{2\beta}{1+\beta} V_{in}$$

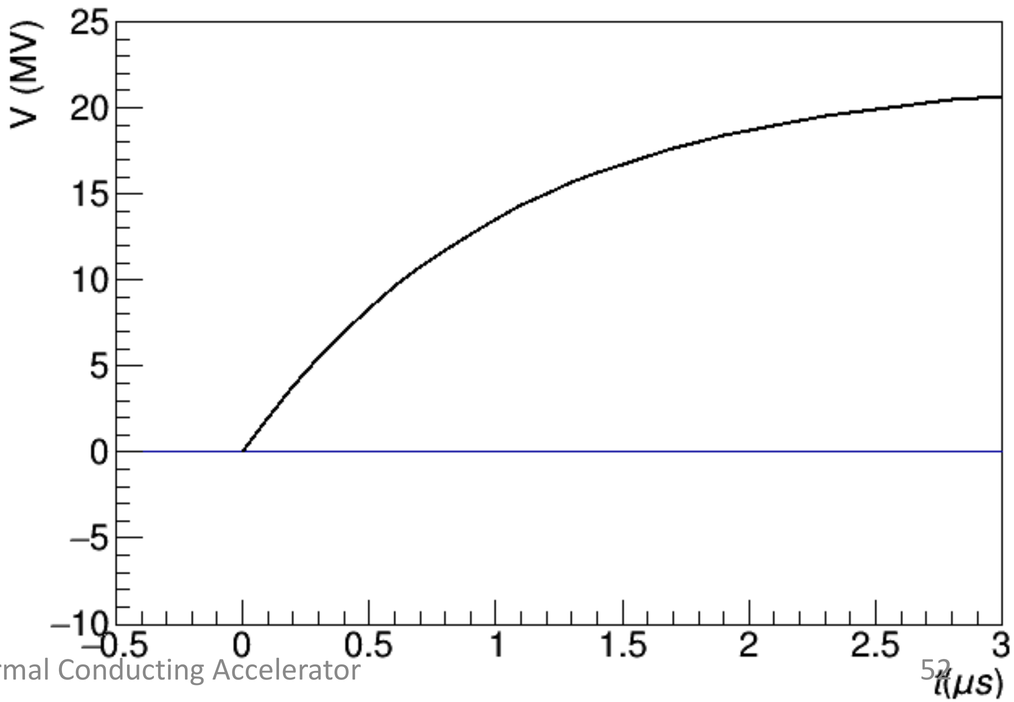
Differential equation of SW accelerator

$$\tau \frac{dV_0}{dt} = -V_0 + \alpha V_{in}$$

$$\alpha \equiv \frac{2\beta}{1+\beta} \qquad \tau \equiv \frac{2Q_L}{\omega}$$

Solution with $V_0 = 0$ at $t = 0$

$$V_0(t) = \frac{2\beta}{1+\beta}\sqrt{\frac{P_{in}}{\beta G_0}}\left(1 - e^{-\frac{t}{\tau}}\right)$$



Introduce Beam

- Cavity voltage is determined by not only Input power, but also beam current.
- Beam consume the cavity stored energy as

$$P_{beam} = \frac{1}{T}\int_0^T dt\, IV_0 \cos(\omega t)$$

If the beam is delta function on crest,

$$P_{beam} = IV_0$$

- Power balance is

$$\frac{dW_0}{dt} = P_{in} - P_r - P_0 - \mathrm{I}V_0$$

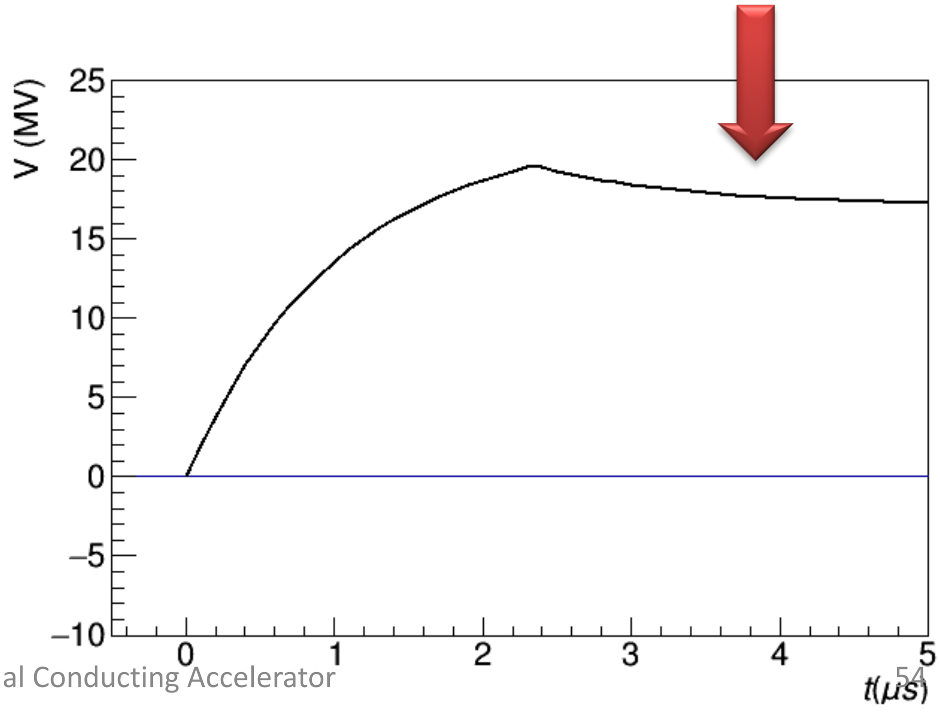
Differential equation of SW accelerator with Beam

$$\tau \frac{dV_0}{dt} = -V_0 + \alpha V_{in} - \frac{I}{(1+\beta)G_0}$$

Solution with
$V_0=0$ at $t=0$,
Beam start at t= t_b

$$V_0(t) = \frac{2\beta}{1+\beta}\sqrt{\frac{P_{in}}{\beta G_0}}\left(1 - e^{-\frac{t}{\tau}}\right) - \frac{I}{(1+\beta)G_0}\left(1 - e^{-\frac{t-t_b}{\tau}}\right)$$

Voltage reduction by beam

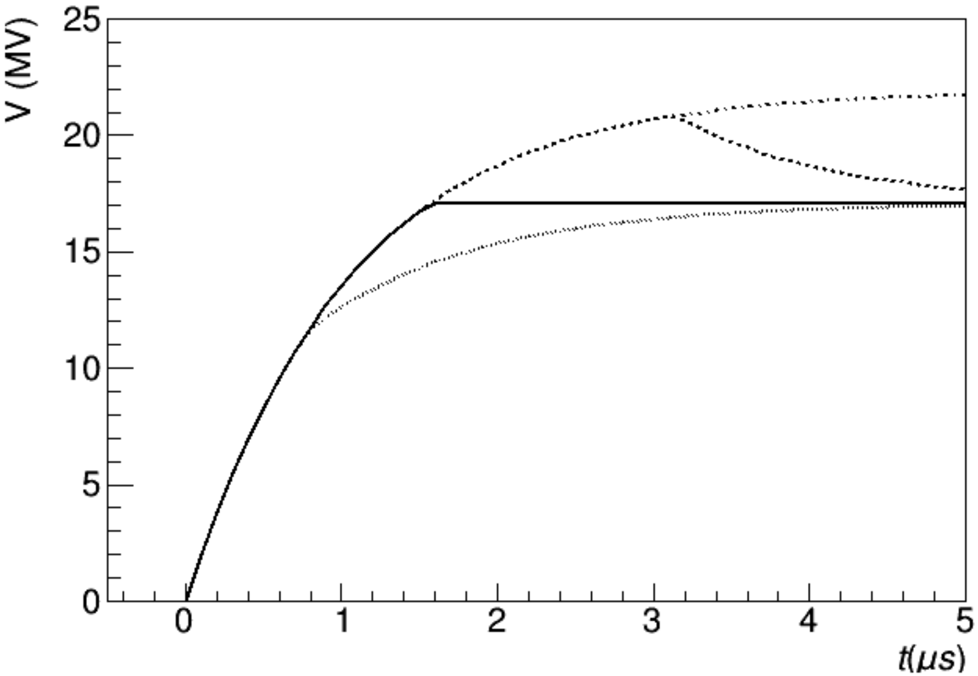


Beam Loading Compensation

Since RF and Beam term have the same time constant τ, V_0 can be constant by adjusting amplitude and timing.

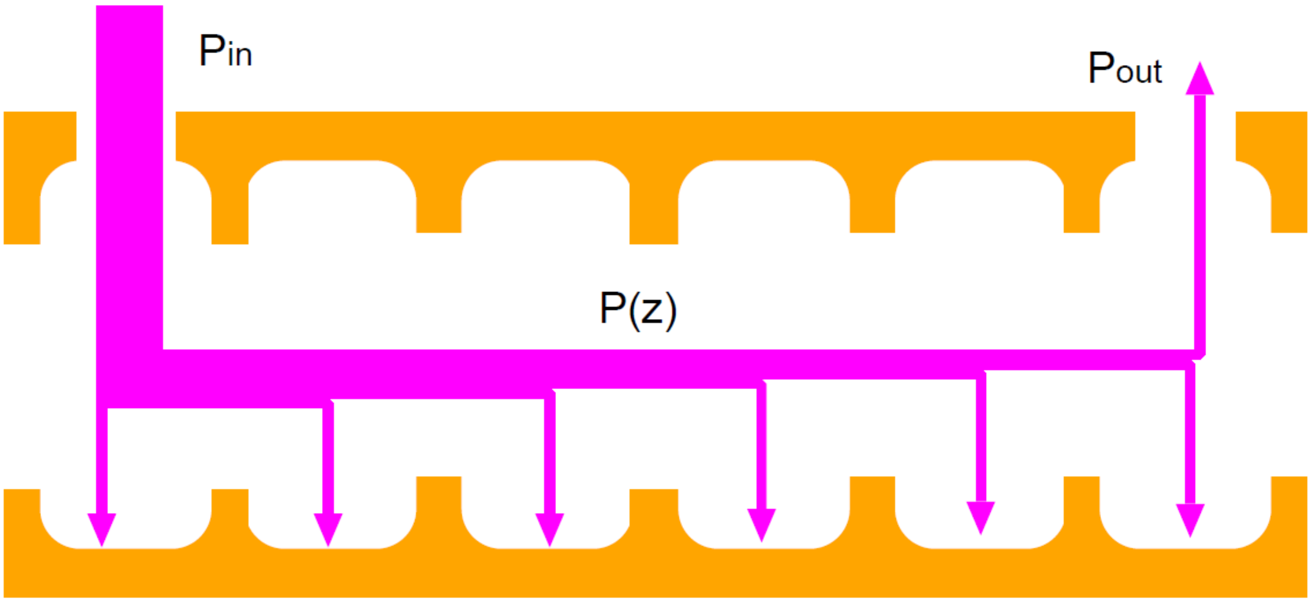
$$V_0(t) = \frac{2\beta}{1+\beta}\sqrt{\frac{P_{in}}{\beta G_0}}\left(1 - e^{-\frac{t}{\tau}}\right) - \frac{I}{(1+\beta)G_0}\left(1 - e^{-\frac{t-t_b}{\tau}}\right)$$

$$t_b = -\tau ln\left(\frac{I}{2}\sqrt{\frac{rL}{\beta P_{in}}}\right)$$



Traveling Wave Accelerator

- Accelerator with a finite group velocity.
- The power flows through the accelerator.
- The beam loading can be suppressed with an active method.



Power Balance

Assume quasi-constant power flow in TW accelerator.

$$\frac{dP}{dz} = \left(\frac{dP}{dz}\right)_{wall} + \left(\frac{dP}{dz}\right)_{beam} = -2\alpha(z)P(z,t) - IE(z,t)$$

Expansion with partial derivative

$$\frac{dP}{dz} = \frac{\partial P}{\partial z} + \frac{\partial P}{\partial t}\frac{dt}{dz} = \frac{\partial P}{\partial z} + \frac{1}{v_g}\frac{\partial P}{\partial t}$$

$$\frac{\partial P}{\partial z} + \frac{1}{v_g}\frac{\partial P}{\partial t} - 2\alpha(z)P(z,t) - IE(z,t) = 0$$

Expression in field,

$$\frac{\partial E}{\partial z} - \frac{E}{2\alpha}\frac{d\alpha}{dz} + \frac{1}{v_g}\frac{\partial E}{\partial t} + \alpha(z)E(z,t) + I\alpha r = 0$$

$$P = \frac{E^2}{2\alpha r}$$

r: shunt impedance

Constant Gradient condition $\frac{d\alpha}{dz} = -2\alpha$

$$\frac{\partial E(z,t)}{\partial z} + \frac{1}{v_g}\frac{\partial E(z,t)}{\partial t} + I(t)\alpha r = 0$$

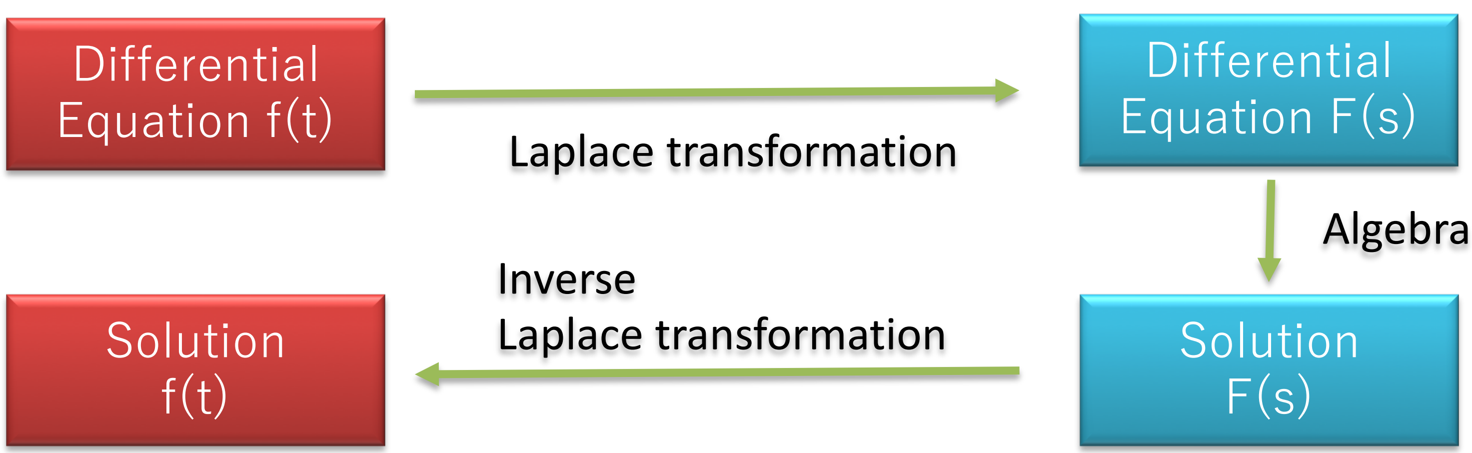
Laplace Transformation

$$\mathcal{L}[f(t)] \equiv F(s) = \int_0^\infty dt f(t) e^{-st}$$

In s-domain, differential and integral operation becomes algebraic operation.

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}F(s)$$



$$\frac{\partial E(z,t)}{\partial z}+\frac{1}{v_g}\frac{\partial E(z,t)}{\partial t}+I(t)\alpha r=0$$

Laplace transformation

$$\frac{\partial E(z,s)}{\partial z}+\frac{s}{v_g}\frac{\partial E(z,s)}{\partial t}+I(s)\alpha r=0$$

By integrating with z,

$$E(z,s)-E(0,s)e^{st_z}+\ I(s)r\int_0^z \alpha dz=0$$

By integrating with z again,

$$V(s)=\frac{\omega L}{Q(1-e^{-2\tau})}\frac{1}{s+\frac{\omega}{Q}}E(s)\left[1-e^{-\left(s+\frac{\omega}{Q}\right)t_f}\right]$$

$$-\frac{\omega r L I(s)}{2Q(1-e^{-2\tau})s}\left[1-e^{-\frac{\omega}{Q}t_f}-\frac{\omega\left(1-e^{-st_f-2\tau}\right)}{Q(s+\omega/Q)}\right]$$

Input RF E(t) and Beam current,

$$E(t) = E_0 u(t), \qquad I(t) = I_0 u(t - t_f)$$

Giving

$$E(s) = \frac{E_0}{s}, \qquad I(s) = \frac{I_0}{s} e^{-st_f}$$

$$V(s) = \frac{\omega L}{Q(1 - e^{-2\tau})} \frac{1}{s + \frac{\omega}{Q}} \frac{E_0}{s} \left[1 - e^{-\left(s + \frac{\omega}{Q}\right) t_f}\right]$$

$$- \frac{\omega r L I(s)}{2Q(1 - e^{-2\tau})} \frac{I_0 e^{-st_f}}{s^2} \left[1 - e^{-\frac{\omega}{Q} t_f} - \frac{\omega\left(1 - e^{-st_f - 2\tau}\right)}{Q(s + \omega/Q)}\right]$$

In t-domain,

$$V(t) = \frac{L E_0}{1 - e^{-2\tau}} \left[\left(1 - e^{-\frac{\omega}{Q} t}\right) u(t) - e^{-2\tau}\left(1 - e^{-\frac{\omega}{Q}(t - t_f)}\right) u(t - t_f)\right]$$

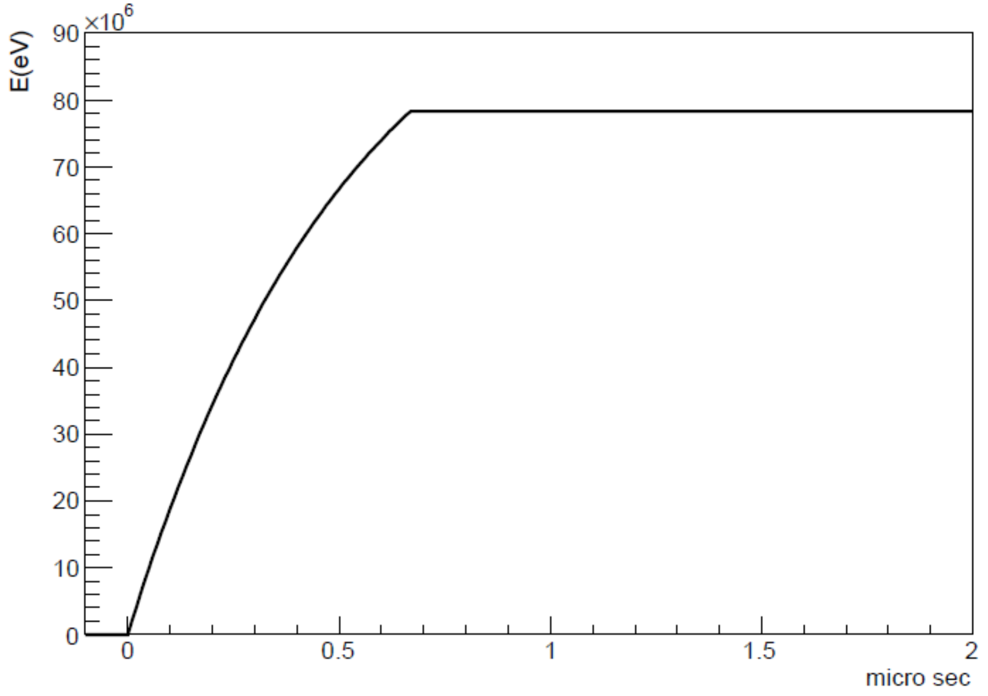
$$- \frac{r L I_0}{2} \left[-\frac{\omega}{Q} \frac{e^{-2\tau}}{1 - e^{-2\tau}} (t - t_f) + \frac{1 - e^{-\frac{\omega}{Q}(t - t_f)}}{1 - e^{-2\tau}}\right] u(t - t_f)$$

$0 < t < t_f$

$$V(t) = \frac{LE_0}{1 - e^{-2\tau}}\left(1 - e^{-\frac{\omega}{Q}t}\right)$$

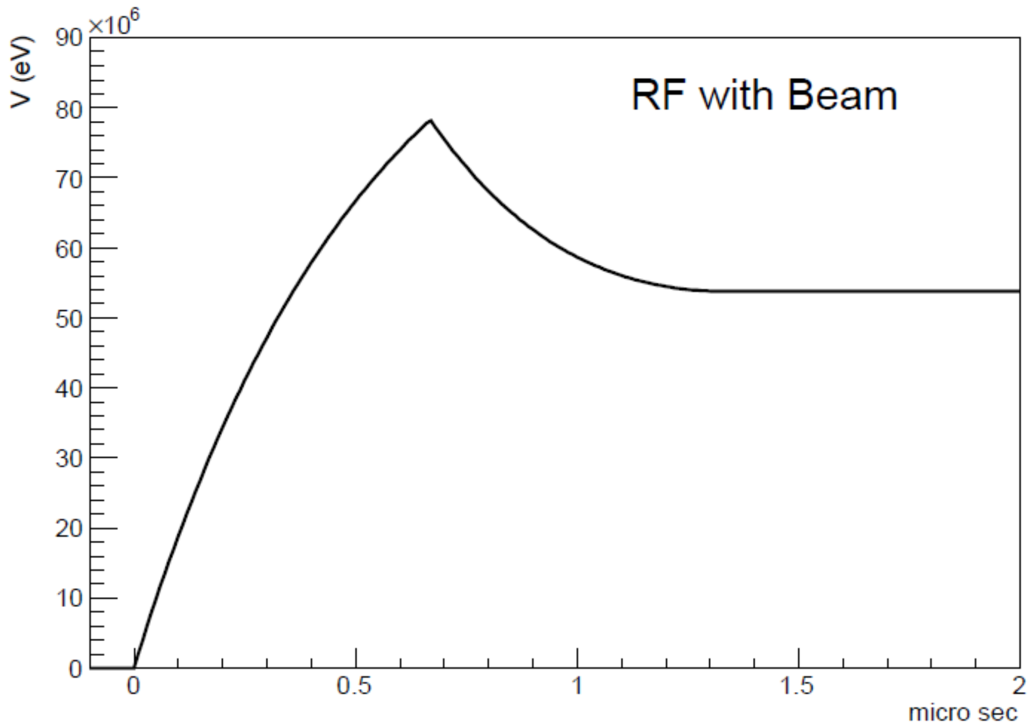
$t > t_f \; and \; I_0 = 0,$

$$V(t) = \frac{LE_0}{1 - e^{-2\tau}}\left(1 - e^{-\frac{\omega}{Q}t_f}\right) = LE_0$$



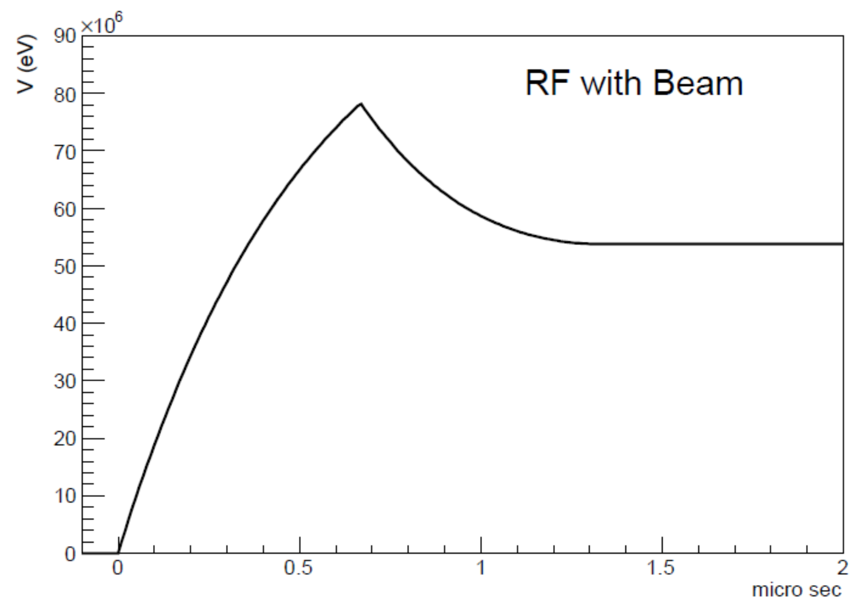
$t > t_f$ and $I_0 \neq 0$,

$$V(t) = LE_0 - \frac{rLI_0}{2}\left[-\frac{\omega}{Q}\frac{e^{-2\tau}}{1-e^{-2\tau}}(t-t_f) + \frac{1-e^{-\frac{\omega}{Q}(t-t_f)}}{1-e^{-2\tau}}\right],$$



Beam Loading Compensation

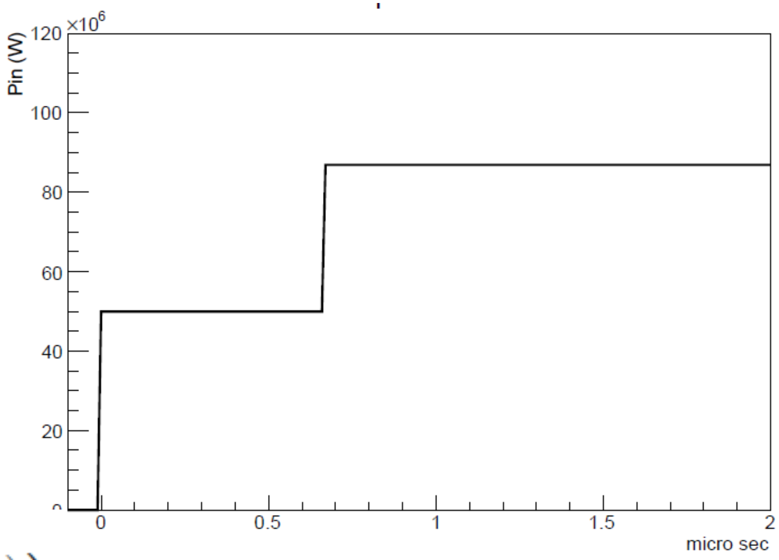
- Acceleration field is varied when we start the beam acceleration.
- We consider its compensation for uniform acceleration.



Amplitude Modulation

Amplitude modulation on E(t) for compensation,

$$E(t) = E_0 u(t) + E_1 u(t - t_f)$$

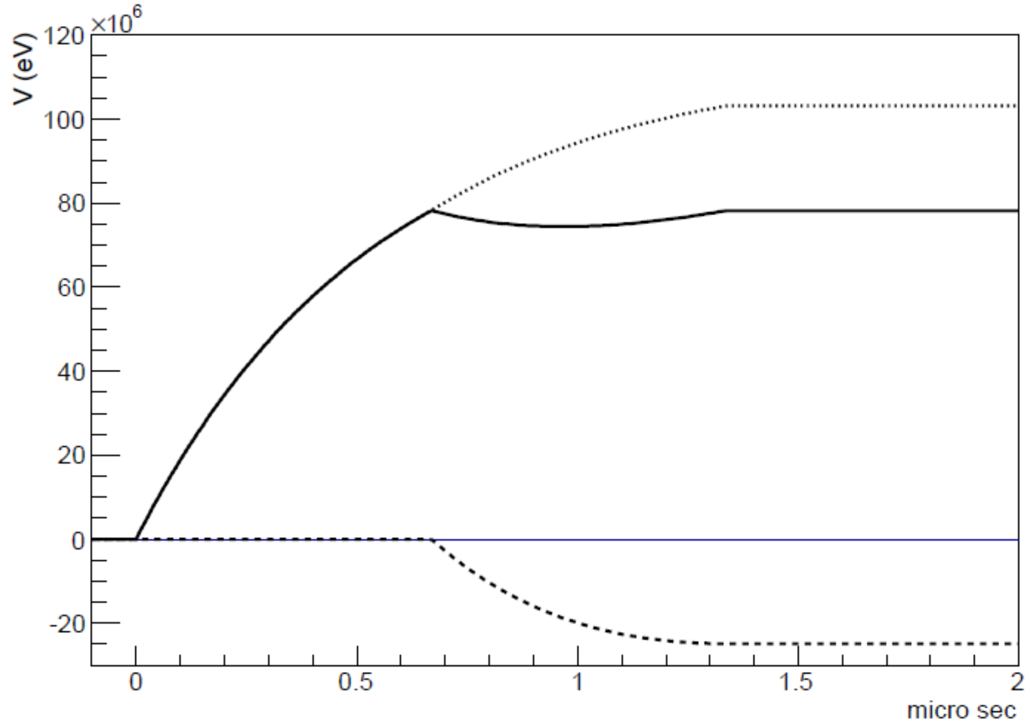


$$E(s) = \frac{E_0}{s} + \frac{E_1}{s} e^{-st_f}$$

$$V(t) = E_0 L + \frac{LE_1}{1 - e^{-2\tau}} \left(1 - e^{-\frac{\omega}{Q}(t - t_f)}\right)$$

$$- \frac{r_0 L I_0}{2(1 - e^{-2\tau})} \left[-\frac{\omega}{Q} e^{-2\tau}(t - t_f) + 1 - e^{-\frac{\omega}{Q}(t - t_f)} \right],$$

$$E_1 = \frac{r_0 I_0}{2} \left(\frac{2\tau e^{-2\tau}}{1 - e^{-2\tau}} - 1 \right),$$



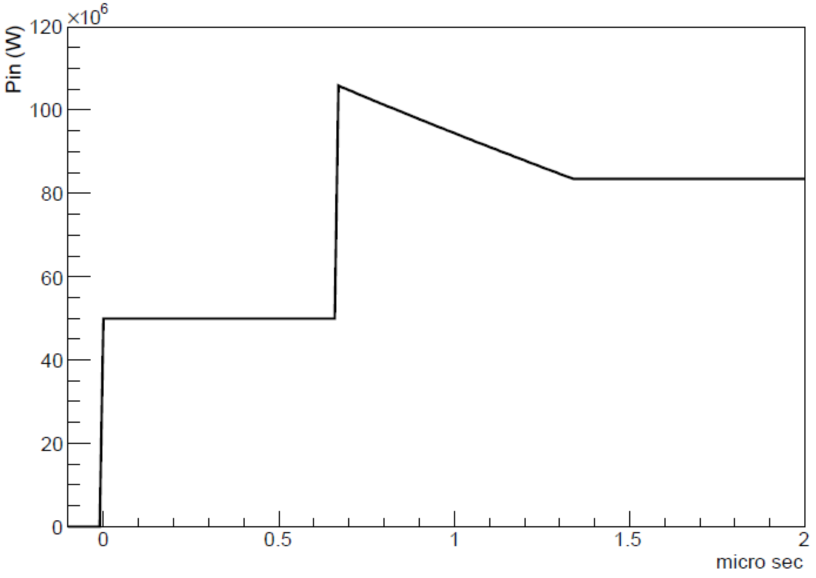
The variation is much compensated, but not perfect.

Two components AM

$$V(t) = E_0 L - \frac{r_0 L I_0}{2(1-e^{-2\tau})}\left[-\frac{\omega}{Q} e^{-2\tau}(t-t_f) + 1 - e^{2\tau - \frac{\omega}{Q}t}\right].$$

The beam loading term has two components.
AM should have also two components for a perfect compensation.

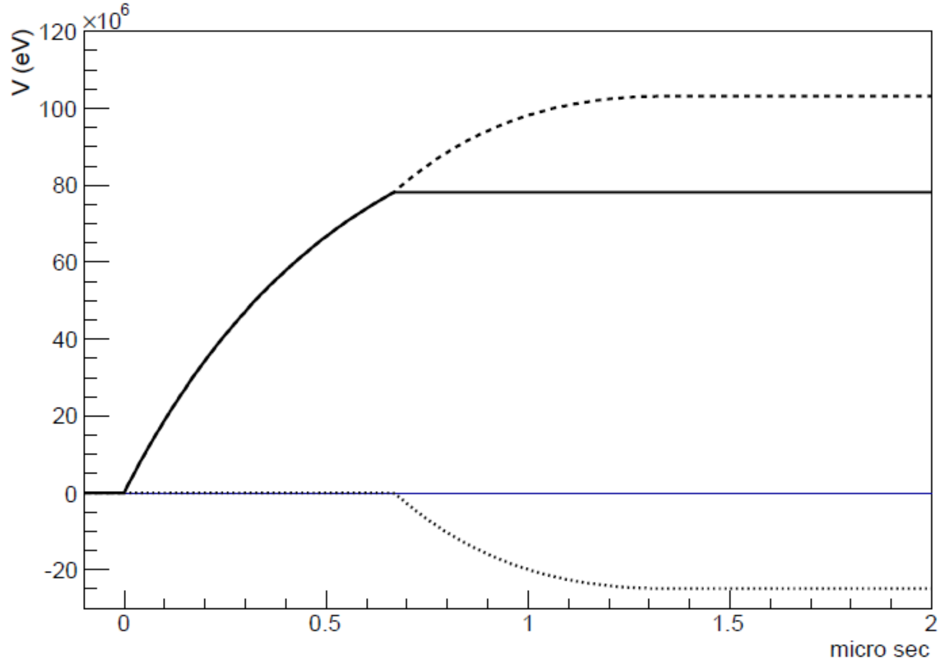
$$E(t) = E_0 u(t) + E_1 u(t-t_f) + E_2(t-t_f)u(t-t_f) - E_2(t-t_f)u(t-2t_f)$$



$$V(t) = E_0 L + \frac{L}{1-e^{-2\tau}}\left(E_1 - \frac{Q}{\omega}E_2\right)\left(1-e^{-\frac{\omega}{Q}(t-t_f)}\right) + \frac{Le^{-2\tau}}{1-e^{-2\tau}}E_2(t-t_f)$$
$$-\frac{r_0 L I_0}{2(1-e^{-2\tau})}\left[-\frac{\omega}{Q}e^{-2\tau}(t-t_f) + 1 - e^{-\frac{\omega}{Q}(t-t_f)}\right],$$

$$V(t) = E_0 L + \frac{L}{1-e^{-2\tau}}\left(E_1 - \frac{Q}{\omega}E_2\right) - \frac{r_0 L I_0}{2(1-e^{-2\tau})} = E_0 L,$$

$$E_1 = \frac{r_0 I_0}{2}(1-e^{-2\tau}),$$
$$E_2 = -\frac{r_0 I_0}{2}\frac{\omega}{Q}e^{-2\tau},$$



Summary

- RF acceleration is a great invention realizing high energy acceleration beyond the vacuum discharge.
- For acceleration, appropriate boundary condition and synchronization are essential.
- Two types of accelerator: Standing wave and Travelling wave.
- Beam loading compensation is mandatory for multi-bunch acceleration.

Stored Energy

$$
\begin{aligned}
W &= \frac{\varepsilon_0}{2}\int_V E_z^2 dV \\
&= \frac{\varepsilon_0}{2}\int_V [E_0 J_0(kr)]^2 dV \\
&= \frac{\varepsilon_0 E_0^2}{2}\int_z \int_r J_0^2(kr) 2\pi r dr dz \\
&= \frac{\varepsilon_0 E_0^2 L\pi}{2}\int_r J_0^2(kr) 2r dr \\
&= \frac{\varepsilon_0 E_0^2 L\pi}{2}\left[r^2[J_1^2(kr) - J_0^2(kr)]\right]_0^b \\
&= \frac{\varepsilon_0 E_0^2}{2}\pi b^2 L J_1^2(p_{01}),
\end{aligned}
$$

Dissipated Power

$$
\begin{aligned}
P &= \frac{R_s}{2}\int_A H_\phi^2 dA \\
&= \frac{R_s E_0^2}{2Z_0^2}\int_A J_1^2(kr)dA \\
&= \frac{R_s E_0^2}{2Z_0^2}\left[\int J_1^2(kr)2\pi r dr + 2\pi b\int J_1^2(p_{01})dz\right] \\
&= \frac{R_s E_0^2}{2Z_0^2}\left(2\pi\left[r^2[J_1^2(kr) - J_0(kr)J_2(kr)]\right]_0^b + 2\pi b L J_1^2(p_{01})\right) \\
&= \frac{R_s E_0^2 \pi b}{Z_0^2}(b+L)J_1^2(p_{01})
\end{aligned}
$$