Cosmological Constant Problem and Scale Invariance

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1 Cosmological Constant Problem

Dark Clouds hanging over the two well-established theories

Quantum Field Theory $\iff$ Einstein Gravity Theory

I first explain my view point on what is actually the problem.
Recently observed Dark Energy $\Lambda_0$, looks like a small Cosmological Constant (CC):

Present observed CC $10^{-29}\text{gr/cm}^3 \sim 10^{-47}\text{GeV}^4 \equiv \Lambda_0$ (1)

We do not mind this tiny CC now, which will be explained after our CC problem is solved. However, we use it as the scale unit $\Lambda_0$ of our discussion in the Introduction.
What is the true problem?

→ Essential point: multiple mass scales are involved!

There are several dynamical symmetry breakings and they are necessarily accompanied by Vacuum Condensation Energy (potential energy):

In particular, from the success of the Standard Model, we are confident of the existence of at least TWO symmetry breakings:

\[
\begin{align*}
\text{Higgs Condensation} & \sim (200 \text{ GeV})^4 \sim 10^9 \text{GeV}^4 \sim 10^{56} \Lambda_0 \\
\text{QCD Chiral Condensation} & \langle \bar{q}q \rangle^{4/3} \sim (200 \text{ MeV})^4 \sim 10^{-3} \text{GeV}^4 \sim 10^{44} \Lambda_0
\end{align*}
\]

Nevertheless, these seem not contributing to the Cosmological Constant!

It is a Super fine tuning problem:

\[
\begin{align*}
c & : \text{initially prepared CC} (> 0) \\
\left( c - 10^{56} \Lambda_0 \right) & : \text{should cancell, but leaving 1 part per } 10^{12}; \text{ i.e., } \sim 10^{44} \Lambda_0 \\
\left( c - 10^{56} \Lambda_0 - 10^{44} \Lambda_0 \right) & : \text{should cancell, but leaving 1 part per } 10^{44}; \text{ i.e., } \sim \Lambda_0 \\
\left( c - 10^{56} \Lambda_0 - 10^{44} \Lambda_0 \right) & \sim \Lambda_0 : \text{present Dark Energy}
\end{align*}
\]
\[ c = \text{initially prepared CC} \]
\[ 654321, 0987654321, 0987654321, 0987654321, 0987654321, 0987654321 \times \Lambda_0 \sim 10^{56} \Lambda_0 \]
\[ c + V_{\text{Higgs}} = \]
\[ 4321, 0987654321, 0987654321, 0987654321, 0987654321 \times \Lambda_0 \sim 10^{44} \Lambda_0 \]
\[ c + V_{\text{Higgs}} + V_{\text{QCD}} = \text{present Dark Energy} \]
\[ 1 \times \Lambda_0 \sim \Lambda_0 \]

Note that the vacuum energy is almost totally cancelled \textit{at each stage of spontaneous breaking} as far as the the relevant energy scale order.
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In this context, the use of quantum scale-invariant prescription was proposed by M. Shaposhnikov and D. Zenhausern, Phys. Lett. B 671 (2009)
162 This is actually a very good paper. I introduce you their scenario and point the problem.
Part I: SI is Necessary

2 Vacuum Energy $\simeq$ vacuum condensation energy

People may suspect that there are “Two” origins of Cosmological Constant

(Quantum) Vacuum Energy

$$\sum_{k,s} \frac{1}{2} \hbar \omega_k - \sum_{k,s} \hbar E_k$$ (2)

Infinite, No controle, simply discarded

$\Downarrow$

(Classical) Potential Energy

$$V(\phi_c) : \text{potential}$$ (3)

Finite, vacuum condensation energy

They are separately stored in our (or my, at least) memory, but actually, almost the same object, as we see now.

We now show for the vacuum energies in the SM that

quantum Vacuum Energy = Higgs Potential Energy (4)
Let us see this more explicitly. For that purpose, consider

**Simplified SM:**

\[
\mathcal{L}_r = \bar{\psi} \left( i \gamma^\mu \partial_\mu - y \phi(x) \right) \psi(x) + \frac{1}{2} \left( \partial^\mu \phi(x) \partial_\mu \phi(x) - m^2 \phi^2(x) \right) - \frac{\lambda}{4!} \phi^4(x) - \hbar m^4. \quad (5)
\]

Effective Action (Effective Potential) is calculated prior to the vacuum choice. (i.e., calculable independently of the choice of the vacuum)

**Calculating Formula:**

\[
\mathcal{L}(\Phi + \phi) = \mathcal{L}(\phi) + \frac{\partial \mathcal{L}(\phi)}{\partial \phi} \Phi + \frac{1}{2} \Phi \left( iD_F^{-1}(\phi) \right) \Phi + \mathcal{L}_{\text{int.}}(\Phi; \phi) \quad (6)
\]

\[
\Gamma[\phi] = \int d^4x \mathcal{L}(\phi) + \frac{i}{2} \hbar \ln \det \left[ iD_F^{-1}(\phi) \right] - \hbar \left\langle \exp \left( \frac{i}{\hbar} \int d^4x \mathcal{L}_{\text{int}}(\Phi; \phi) \right) \right\rangle_{1\text{PI}} \quad (7)
\]

\[
V[\phi] = V_0(\phi) + \frac{1}{2} \hbar \int \frac{d^4k}{i(2\pi)^4} \ln \det \left[ iD_F^{-1}(k; \phi) \right] + \hbar \left\langle \exp \left( \frac{i}{\hbar} \int d^4x \mathcal{L}_{\text{int}}(\Phi; \phi) \right) \right\rangle_{1\text{PI}} \quad (8)
\]
1-loop effective potential in the Simplified SM

Use dimensional regularization for doing Mass-Independent (MI) renormalization

\[ V(\phi, m^2) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + hm^4 + V_{\text{1-loop}} + \delta V^{(1)}_{\text{counterterms}} \]

\[ V_{\text{1-loop}} = \frac{1}{2} \int \frac{d^4 k}{i(2\pi)^4} \ln(-k^2 + m^2 + \frac{1}{2} \lambda \phi^2) - 2 \int \frac{d^4 p}{i(2\pi)^4} \ln(-p^2 + y^2 \phi^2) \]

\[ \delta V^{(1)}_{\text{counterterms}} = \frac{D^{(1)}}{4!} \lambda \phi^4 + \frac{1}{2} (E^{(1)} m^2 + (\delta m^2)^{(1)}) \phi^2 + (F^{(1)} m^4 + G^{(1)} m^2 + H^{(1)}) \]

dropping the $1/\bar{\varepsilon}$ parts in $\overline{\text{MS}}$ renormalization scheme, we find

\[ \Gamma^{(4,0)}_{\phi^4} : \quad D^{(1)} \lambda = \frac{3}{16\pi^2} \frac{\lambda^2}{2 \bar{\varepsilon}} - \frac{4!}{16\pi^2} \frac{1}{\bar{\varepsilon}} \]

\[ \Gamma^{(2,0)}_{\phi^2} : \quad E^{(1)} m^2 = \frac{\lambda}{32\pi^2} \frac{1}{\bar{\varepsilon}} m^2, \quad (\delta m^2)^{(1)} = 0 \] (9)

\[ \Gamma^{(0,0)} : \quad F^{(1)} m^4 = \frac{1}{32\pi^2} \frac{1}{\bar{\varepsilon}} m^4, \quad G^{(1)} m^2 = 0, \quad H^{(1)} = 0 \] (10)

since

\[ \frac{1}{2} \int \frac{d^4 k}{i(2\pi)^4} \ln(-k^2 + M^2) = \frac{M^4}{64\pi^2} \left( -\frac{1}{\bar{\varepsilon}} + \ln \frac{M^2}{\mu^2} - \frac{3}{2} \right). \] (11)

Coleman-Weinberg potential
So we get finite well-known renormalized 1-loop effective potential:

\[
V(\phi, m^2) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + h m^4 \\
+ \frac{(m^2 + \frac{1}{2} \lambda \phi^2)^2}{64 \pi^2} \left( \ln \frac{m^2 + \frac{1}{2} \lambda \phi^2}{\mu^2} - \frac{3}{2} \right) - 4 \frac{(y \phi)^4}{64 \pi^2} \left( \ln \frac{y^2 \phi^2}{\mu^2} - \frac{3}{2} \right)
\]  

(12)

Note that the general 1-loop contributions are given by

\[
V_{1\text{-loop}}(\phi) = \sum_i \pm n_i F_{\ln}(M_i^2(\phi)), \quad F_{\ln}(M^2) \equiv \frac{1}{2} \int \frac{d^4k}{i(2\pi)^4} \ln(-k^2 + M^2)
\]  

(13)

But, this shows it’s nothing but (quantum) Vacuum Energies: Zero-point osc. for boson and Dirac’s sea negative energies.

Indeed, we can evaluate the LHS as follows:

\[
F_{\ln}(M^2) - F_{\ln}(0) = \frac{1}{2} \int_0^{M^2} dm^2 \frac{\partial}{\partial m^2} \int \frac{d^4k}{i(2\pi)^4} \ln(-k^2 + m^2 - i\varepsilon)
\]

\[
= \frac{1}{2} \int_0^{M^2} dm^2 \int \frac{d^4k}{i(2\pi)^4} \frac{1}{-k^2 + m^2 - i\varepsilon}
\]

\[
= \frac{1}{2} \int_0^{M^2} dm^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{k^2 + m^2}}
\]

\[
= \int \frac{d^3k}{(2\pi \hbar)^3} \left( \frac{\hbar}{2} \sqrt{k^2 + M^2} - \frac{\hbar}{2} \sqrt{k^2} \right)
\]  

(14)
Note also that $F_{ln}(0)$ in the massless case vanishes in the dimensional regularization. If you apply the dimensional formula to the last expression, you can also recover the original RHS result.

3 Conclusions from these simple observation

We have shown that the equivalence between the (quantum) vacuum energies and (‘classical’) Higgs potential energy. From this simple observation, we can draw very interesting and important conclusions:

As far as the **matter fields** and **gauge fields** are concerned, whose mass comes solely from the Higgs condensation $\langle \phi \rangle$,

Their vacuum energies are calculable and finite quantities in terms of the renormalized $\lambda$ and $m^2$ parameters!

Note that this is because that their divergences are proportional to $\phi^4$ and $m^2\phi^2$. (At 1-loop, only $\phi^4$ divergences appear.)

However, the **Higgs itself is an exception**! The divergences of the Higgs vacuum energy are not only $m^2\phi^2$ and $\phi^4$ but also the zero-point function proportional to $m^4$. In order to
cancel that part, we have to prepare the counterterm:

\[ h_0 m_0^4 = Z_h Z_m^2 h m^4 = (1 + F) h m^4 \]
\[ F^{(1)} h = \frac{1}{64\pi^2} \frac{1}{\bar{\varepsilon}}. \]  \hspace{1cm} (15)

And the renormalized CC term \( h m^4 \) is a Free Parameter. Then, there is no chance to explain CC.

Thus, for the calculability of CC, we need \( m^2 = 0 \), or

No dimensionful parameters in the theory \( \Rightarrow \) (Classical) Scale-Invariance
Part II: Scale Invariance is a Sufficient Condition?

4 Scale Invariance may solve the problem

Our world is almost scale invariant: that is, the standard model Lagrangian is scale invariant except for the Higgs mass term!

If the Higgs mass term comes from the spontaneous breaking of scale invariance at higher energy scale physics, the total system can be really be scale invariant:

$$\lambda(h^\dagger h - m^2)^2 \rightarrow (h^\dagger h - \varepsilon \Phi^2)^2.$$  \hspace{1cm} (16)

where $\Phi$ may be a field which appear also in front of Einstein-Hilbert term:

$$\int d^4x \sqrt{-g} \Phi^2 R$$  \hspace{1cm} (17)
4.1 Classical Scale Invariance

Suppose that our world has no dimensionful parameters.
Suppose that the effective potential $V$ of the total system looks like

$$V(\phi) = V_0(\Phi) + V_1(\Phi, h) + V_2(\Phi, h, \varphi)$$

and it is scale invariant. Then, classically, it satisfies the scale invariance relation:

$$\sum_i \phi^i \frac{\partial}{\partial \phi^i} V(\phi) = 4V(\phi),$$

so that the vacuum energy vanishes at any stationary point $\langle \phi^i \rangle = \phi^i_0$:

$$V(\phi_0) = 0.$$ 

Important point is that this holds at every stages of spontaneous symmetry breaking.

This miracle is realized since the scale invariance holds at each energy scale of spontaneous symmetry breaking.
For example, we can write a toy model of potentials.

\[
V_0(\Phi) = \frac{1}{2} \lambda_0 (\Phi_1^2 - \varepsilon_0 \Phi_0^2)^2,
\]

in terms of two real scalars \(\Phi_0, \Phi_1\), to realize a VEV

\[
\langle \Phi_0 \rangle = M \quad \text{and} \quad \langle \Phi_1 \rangle = \sqrt{\varepsilon_0} M \equiv M_1.
\]

This \(M\) is totally spontaneous and we suppose it be Planck mass giving the Newton coupling constant via the scale invariant Einstein-Hilbert term

\[
S_{\text{eff}} = \int d^4x \sqrt{-g} \left\{ c_1 \Phi_0^2 R + c_2 R^2 + c_3 R_{\mu\nu}R^{\mu\nu} + \cdots \right\}
\]

If GUT stage exists, \(\varepsilon_0\) may be a constant as small as \(10^{-4}\) and then \(\Phi_1\) gives the scalar field breaking GUT symmetry; e.g., \(\Phi_1 : 24\) causing \(SU(5) \to SU(3) \times SU(2) \times U(1)\).

\(V_1(\Phi, h)\) part causes the electroweak symmetry breaking:

\[
V_1(\Phi, h) = \frac{1}{2} \lambda_1 \left( h^\dagger h - \varepsilon_1 \Phi_1^2 \right)^2,
\]

with very small parameter \(\varepsilon_1 \simeq 10^{-28}\). This reproduces the Higgs potential when \(h\) is the Higgs doublet field and \(\varepsilon_1 \Phi_1^2\) term is replaced by the VEV \(\varepsilon_1 M_1^2 = \mu^2/\lambda_1\).

\(V_2(\Phi, h, \varphi)\) part causes the chiral symmetry breaking, e.g., \(SU(2)_L \times SU(2)_R \to SU(2)_V\). Using the \(2 \times 2\) matrix scalar field \(\varphi = \sigma + i \tau \cdot \pi\) (chiral sigma-model field), we may
similarly write the potential

\[
V_2(\Phi, h, \varphi) = \frac{1}{4} \lambda_2 \left( \text{tr}(\varphi^\dagger \varphi) - \varepsilon_2 \Phi_1^2 \right)^2 + V_{\text{break}}(\Phi, h, \varphi)
\]

with another small parameter \( \varepsilon_2 \simeq 10^{-34} \). The first term reproduces the linear \( \sigma \)-model potential invariant under the chiral \( \text{SU}(2)_L \times \text{SU}(2)_R \) transformation \( \varphi \rightarrow g_L \varphi g_R \) when \( \varepsilon_2 \Phi_1^2 \) is replaced by the VEV \( \varepsilon_2 M_1^2 = m^2 / \lambda_2 \). The last term \( V_{\text{break}} \) stands for the chiral symmetry breaking term which is caused by the explicit quark mass terms appearing as the result of tiny Yukawa couplings of \( u, d \) quarks, \( y_u, y_d \), to the Higgs doublet \( h \); e.g.,

\[
V_{\text{break}}(\Phi, h, \varphi) = \frac{1}{2} \varepsilon_2 \Phi_1^2 \text{tr} \left( \varphi^\dagger \left( y_u \epsilon h^* \ y_d h \right) + \text{h.c.} \right)
\]
4.2 Quantum Mechanically

Is there Anomaly for the Scale Invariance?

Usual answer is YES in quantum field theory. If we take account of the renormalization point \( \mu \), so that we have dimension counting identity

\[
\left( \mu \frac{\partial}{\partial \mu} + \sum_i \phi_i \frac{\partial}{\partial \phi_i} \right) V(\phi) = 4V(\phi).
\]

and, also have renormalization group equation (RGE):

\[
\left( \mu \frac{\partial}{\partial \mu} + \sum_a \beta_a(\lambda) \frac{\partial}{\partial \lambda_a} + \sum_i \gamma_i(\lambda) \phi_i \frac{\partial}{\partial \phi_i} \right) V(\phi) = 0
\]

From these we obtain

\[
\left( \sum_i (1 - \gamma_i(\lambda)) \phi_i \frac{\partial}{\partial \phi_i} - \sum_a \beta_a(\lambda) \frac{\partial}{\partial \lambda_a} \right) V(\phi) = 4V(\phi)
\]

which replaces the above naive one:

\[
\sum_i \phi_i \frac{\partial}{\partial \phi_i} V(\phi) = 4V(\phi)
\]
This shows the anomalous dimension $\gamma_i(\lambda)$ is not the problem, but $\beta_a(\lambda)$ terms may be problematic.

Still, if I assume the existence of Infrared Fixed Points: $\beta_a(\lambda_{\text{IR}}) = 0$, then, I can prove that the potential value $V(\phi_0)$ at the stationary point $\phi = \phi_0$ is zero at any $\mu$. The vanishing property of the stationary potential value $V(\phi)$ is not injured by the scale-inv anomaly.

Probably, however, it will not be sufficient to guarantee the vanishing CC.
Stationary point $\phi_0$ may be the trivial point $\phi_0 = 0$.
Non-trivial is the existence of the flat direction even after the quantum corrections are included.

Shaposhnikov-Zenhausern’s New Idea is: \textbf{SI exists even quantum mechanically.}

\textbf{Quantum Scale Invariance}


Extension to $n$-dimension keeping S.I. is possible by introducing \textit{dilaton field} $\Phi \rightarrow \text{NO ANOMALY.}$
1. Usual dimensional regularization

\[
\lambda (h \dagger(x) h(x))^2 \rightarrow \lambda \mu^{4-n} (h \dagger(x) h(x))^2 \quad [h] = \frac{n-2}{2}
\]

\[
y \bar{\psi}(x) \psi(x) h(x) \rightarrow y \mu^{\frac{4-n}{2}} \bar{\psi}(x) \psi(x) h(x) \quad [\psi] = \frac{n-1}{2}
\]

(20)

2. SI prescription Using ‘dilaton’ field \(\Phi(x)\),

\[
\lambda (h \dagger(x) h(x))^2 \rightarrow \lambda \left(\Phi(x)^2\right)^{\frac{4-n}{n-2}} (h \dagger(x) h(x))^2
\]

\[
y \bar{\psi}(x) \psi(x) h(x) \rightarrow y \left[\Phi(x)^{\frac{4-n}{n-2}} \bar{\psi}(x) \psi(x)\right] h(x)
\]

(21)

This introduces

FAINT but Non-Polynomial “evanescent” interactions \(\propto 2\epsilon = 4 - n\)

\[
\Phi = M e^{\phi/M}, \quad \langle \Phi \rangle \equiv M \rightarrow \left[\Phi(x)^{\frac{4-n}{n-2}}\right] = M^{\frac{\epsilon}{1-\epsilon}} \left(1 + \frac{\epsilon}{1-\epsilon} \frac{\phi}{M} + \frac{1}{2} \left(\frac{\epsilon}{1-\epsilon}\right)^2 \frac{\phi^2}{M^2} + \cdots\right)
\]

(22)

This scenario would give quantum scale invariant theory, which may realize the vanishing CC.
5 Quantum scale-invariant renormalization

Explicit calculations were performed by


in a simple scalar model: (h → φ, Φ → σ)

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{1}{2} \partial_\mu \sigma \cdot \partial^\mu \sigma - V(\phi, \sigma) \]  (23)

with scale-invariant potential in n dimension:

\[ V(h, \Phi) = \mu(\sigma)^{4-n} \left( \frac{\lambda_\phi}{4} \phi^4 - \frac{\lambda_m}{2} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4} \sigma^4 \right) \]  (24)

with

\[ \mu(\sigma) = z \sigma^{\frac{2}{n-2}} \quad (z : \text{renormalization point parameter}) \]  (25)

At tree level, \( \lambda_m^2 = \lambda_\phi \lambda_\sigma \) is assumed so that

\[ V(\phi, \sigma) = \mu(\sigma)^{4-n} \frac{\lambda_\phi}{4} (\phi^2 - \varepsilon \sigma^2)^2 \]

\[ \lambda_m = \varepsilon \lambda_\phi, \quad \lambda_\sigma = \varepsilon^2 \lambda_\phi \]  (26)
Ghilencea has shown:

1. **Non-renormalizability**: higher and higher order non-polynomial interaction terms of the form

\[
\frac{\phi^{4+2p}}{\sigma^{2p}} \quad (p = 1, 2, 3, \cdots)
\]

are induced by the evanescent interactions at higher loop level, and they must also be included as counterterms, can be neglected in the low-energy region \( E < \langle \sigma \rangle \sim M_{\text{Pl}} \).

2. **Mass hierarchy is stable**: If we put

\[
\lambda_\phi = \bar{\lambda}_\phi, \quad \lambda_m = \varepsilon \bar{\lambda}_m, \quad \lambda_\sigma = \varepsilon^2 \bar{\lambda}_\sigma
\]

with \( \bar{\lambda}_i \)'s \((i = \phi, m, \sigma)\): \(O(1)\) and very tiny \( \varepsilon = (\frac{100\text{GeV}}{10^{18}\text{GeV}})^2 = 10^{-32} \), then, \( \bar{\lambda}_i \)'s remain \(O(1)\) stably against radiative corrections. This is essentially because \( \sigma^2 \phi^2 \) term comes only through the \( \lambda_m \phi^2 \sigma^2 \) interaction.

One-loop potential at \( n = 4 \): **scale Invariant!**

\[
V(\phi, \sigma) = \frac{\lambda_\phi}{4} \phi^4 - \frac{\lambda_m}{2} \phi^2 \sigma^2 + \frac{\lambda_\sigma}{4} \sigma^4 + \frac{\hbar}{64\pi^2} \left\{ M_1^4 \left( \ln \frac{M_1^2}{z^2 \sigma^2} - \frac{3}{2} \right) + M_2^4 \left( \ln \frac{M_2^2}{z^2 \sigma^2} - \frac{3}{2} \right) + \Delta V \right\}
\]

\[
\Delta V = -\lambda_\phi \lambda_m \frac{\phi^6}{\sigma^2} + (16\lambda_\phi \lambda_m - 6\lambda_m^2 + 3\lambda_\phi \lambda_\sigma) \phi^4 + (-16\lambda_m + 25\lambda_\sigma) \lambda_m \phi^2 \sigma^2 - 21\lambda_\sigma^2 \sigma^4
\]
However, the problem is that

3. **Vanishing CC again requires fine tuning!** owing to quantum corrections.

\[ V(\phi, \sigma) = \sigma^4 W(x) \text{ with } x \equiv \frac{\phi^2}{\sigma^2}. \]

Since the stationarity
\[
\begin{cases}
\frac{\partial V}{\partial \phi} V = \sigma^4 W'(x) \cdot 2x = 0 \\
\frac{\partial V}{\partial \sigma} V = \sigma^4 \left(4W(x) + W'(x) \cdot (-2x)\right) = 0
\end{cases}
\]

requires
\[ W'(x) = 0 \text{ and } W(x) = 0 \text{ are satisfied.} \]

Let us examine these conditions with the above 1-loop potential
\[
W(x) = \frac{\lambda_\phi}{4} x^2 - \frac{\lambda_m}{2} x + \frac{\lambda_\sigma}{4}
\]
\[
+ \frac{\hbar}{64\pi^2} \left\{ \frac{M_1^4}{\sigma^4} \left( \ln \frac{M_1^2}{z^2 \sigma^2} - \frac{3}{2} \right) + \frac{M_2^4}{\sigma^4} \left( \ln \frac{M_2^2}{z^2 \sigma^2} - \frac{3}{2} \right) \right. \\
- \lambda_\phi \lambda_m x^3 + (16\lambda_\phi \lambda_m - 6\lambda_m^2 + 3\lambda_\phi \lambda_\sigma) x^2 + (-16\lambda_m + 25\lambda_\sigma) \lambda_m x - 21\lambda_\sigma^2 \right\}
\]
At tree level, the stationary point $x = x_0$ is

\[
\begin{aligned}
W'(x_0) = 0 & \rightarrow \frac{\lambda_{\phi}}{2} x_0 - \frac{\lambda_m}{2} = 0 \rightarrow x_0 = \frac{\lambda_m}{\lambda_{\phi}} \\
W(x_0) = 0 & \rightarrow \frac{\lambda_{\phi}}{4} x_0^2 - \frac{\lambda_m}{2} x_0 + \frac{\lambda_\sigma}{4} = 0 \rightarrow \lambda_\sigma = \frac{\lambda_m^2}{\lambda_{\phi}}
\end{aligned}
\]  

(33)

At one-loop level, the stationary point may be shifted and the coupling constants may be adjusted:

\[
x = x_0 + \hbar x_1, \quad \lambda_i \Rightarrow \lambda_i + \hbar \delta \lambda_i \quad (i = \phi, m, \sigma)
\]

(34)

$W'(x) = 0$ requires

\[
W'(x) \bigg|_{O(\hbar)} = \frac{\lambda_{\phi}}{2} x_1 + \frac{\delta \lambda_{\phi}}{2} x_0 + \frac{\delta \lambda_m}{2} + \frac{\hbar}{64\pi^2} \left( 12 \lambda_m (\ln 2\lambda_{\phi} - 1) \right) + O(\lambda_m^2)
\]

(35)

→ consistent with the VEV (mass) hierarchy

\[
x = \frac{\langle \phi \rangle^2}{\langle \sigma \rangle^2} = O(\varepsilon) \quad \text{since} \quad \lambda_m, \delta \lambda_m \sim O(\varepsilon), \lambda_{\phi}, \delta \lambda_{\phi} \sim O(1), \rightarrow x_{0,1} \sim O(\varepsilon).
\]

(36)
Next,

\[
W(x)\bigg|_{O(h)} = \frac{\lambda_\phi}{2}(x_0 + \frac{\lambda_m}{\lambda_\phi})x_1 + \frac{\delta\lambda_\phi}{4}x_0^2 + \frac{\delta\lambda_m}{2}x_0 + \frac{\delta\lambda_\sigma}{4}
+ \frac{\hbar}{64\pi^2}\left(2\lambda_m(1 + \frac{\lambda_m}{\lambda_\phi})\right)^2 \left(\ln 2(\lambda_\phi - \lambda_m) - \frac{3}{2}\right) + O(\lambda_m^3) \quad (37)
\]

All the terms are consistently of \(O(\varepsilon^2)\), so that \(W(x) = 0\) is realized by \(O(1)\) tuning of \(\tilde{\lambda}_\phi, \tilde{\lambda}_m, \tilde{\lambda}_\sigma\). However, Note: the Vacuum Energy \(\sigma^4W(x)\) at the stationary point vanishes only in the sense of \(O(\varepsilon^2) \times \sigma^4 = O((100\text{GeV})^4)\).

If we require the vanishingness up to the order of \(\Omega_0 \sim (1\text{meV})^4 \sim 10^{-56} \times (100\text{GeV})^4\), then, we have still to tune \(\tilde{\lambda}_\phi, \tilde{\lambda}_m, \tilde{\lambda}_\sigma\) in 56 digits!

We still need Superfine Tuning even in quantum Scale-Invariant theory \(\text{(38)}\)

This is the original CC problem!

Quantum SI is not enough to solve the CC problem.

Note also, however, that this is also the problem beyond the perturbation theory. We are discussing the Vacuum energy in much much finer precision than the perturbation expansion parameter \(O(\hbar/16\pi^2)\).
What happens?

If the theory is quantum scale-invariant, then

\[ \sum_{i} \phi_i \frac{\partial}{\partial \phi_i} V(\phi) = 4V(\phi) \]  (39)

implying \( V(\phi_i^0) = 0 \) at any stationary point \( \phi_i^0 \), and any point in that direction, \( \rho \phi_i^0 \) with \( \forall \rho \in \mathbb{R} \) also realizes the vanishing energy \( V(\rho \phi_i^0) = \rho^4 V(\phi_i^0) = 0 \). (flat direction)

If \( V(\phi) \neq 0 \) at \( \exists \phi \), then the potential is not stationary at \( \phi \).

In the above: \( V(\phi, \sigma) = \sigma^4 W(x) \), is flat in the direction \( x_0 \) at the tree level, but does not satisfy \( W(x_0 + \hbar x_1) = 0 \) exactly for the ‘stationary point’ realizing \( W'(x_0 + \hbar x_1) = 0 \) exactly.

This means from the above Eq. (31) that the point \( x_0 + \hbar x_1 \) realizes the stationarity with respect to \( \phi \) but not necessarily to \( \sigma \). If \( W(x_0 + \hbar x_1) = 0 \) is not exactly satisfied by superfine tuning of couplings, then the potential has a small gradient \( \sigma (\partial/\partial \sigma) V = \sigma^4 W(x) = \sigma^4 O(\varepsilon^2) \neq 0 \) in the \( \sigma \)-direction, implying that the potential is stationary only at the origin \( \sigma = 0 \! \)!

The flat direction is lifted by the radiative correction

\[ \text{(40)} \]

Quantum scale invariance alone does not protect the flat direction, automatically.
6 Discussion

We may need still other symmetry?

I have no definite idea now, but I would like to examine the dynamical symmetry breaking in this quantum scale-invariant theory.

Then, the running coupling becomes stronger, starting from the energy scale $\langle \sigma \rangle$, and reaches the critical value at a scale $\Lambda_{QCD}$ to break the chiral symmetry. So some connection appear between the scales $\langle \sigma \rangle$ and $\Lambda_{QCD}$. I will calculate the effective potential for the SD self-energy and examine whether the vacuum energy is lifted or not.

In this connection, C. Tamarit, JHEP12(2013)098 has derived the usual form of RGE equation

$$\left( \frac{\partial}{\partial \ln z} + \sum_a \beta_a(\lambda) \frac{\partial}{\partial \lambda_a} + \sum_i \gamma_i(\lambda) \phi_i \frac{\partial}{\partial \phi_i} \right) V(\phi; \lambda) = 0$$

by introducing an renormalization point parameter $z$

$$\mu(\sigma) = z \sigma^{\frac{2}{n-2}}$$

and argued that the coupling constant actually runs as the energy scale $z \langle \sigma \rangle$ changes after the SI is spontaneously broken by $\langle \sigma \rangle \neq 0$. 
THANK YOU
1. **Running coupling explain the hierarchy** after VEV $\langle \sigma \rangle \neq 0$ appears.

   e.g., Chiral symmetry breaking scale in QCD:

   Usually the coupling $\alpha_3 \equiv g_3^2/4\pi$ runs according to

   \[
   \mu \frac{d}{d\mu} \alpha_3(\mu) = 2b_3 \alpha_3^2(\mu) \quad \rightarrow \quad \frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(M)} - b_3 \ln \frac{\mu^2}{M^2}
   \]

   \[
   \rightarrow \quad \frac{1}{\alpha_3^{\text{cr}}} = \frac{1}{\alpha_3(M)} - b_3 \ln \frac{\Lambda_{\text{QCD}}^2}{M^2}
   \]

   where $\alpha_3^{\text{cr}} = O(1)$ quantity like $\pi/3$, so explains the huge hierarchy:

   \[
   \varepsilon = \frac{\Lambda_{\text{QCD}}^2}{M^2} = \exp \frac{1}{b_3} \left( \frac{1}{\alpha_3(M)} - \frac{1}{\alpha_3^{\text{cr}}} \right). \tag{43}
   \]

   This is the usual explanation.

   The following is still a handwaving argument to be confirmed.

   In quantum SI theory, $\alpha_3(M)$ here, probably, should be replaced by $M$-independent initial gauge coupling $\alpha_3^{\text{init}}$, while the initial scale $M^2$ should be replaced by the dilaton field VEV $\langle \sigma \rangle^2$. Then

   \[
   \frac{1}{\alpha_3^{\text{cr}}} - \frac{1}{\alpha_3^{\text{init}}} = -b_3 \ln \frac{\Lambda_{\text{QCD}}^2}{\langle \sigma \rangle^2} \tag{44}
   \]

   so that the QCD scale $\Lambda_{\text{QCD}}$ is always scaled with the dilaton VEV $\langle \sigma \rangle$.
2. Hierarchy and Effective Potential

This hierarchy should show up in the effective potential. And the effective potential should be calculable prior to the spontaneous breaking.

Since $\Lambda_{\text{QCD}}^2$ should stand for the VEV $\phi^\dagger \phi$ of the chiral sigma model scalar field $\phi$, we suspect that we should be able to derive the effective potential of the Coleman-Weinberg type like

$$\frac{(\phi^\dagger \phi)^2}{64\pi^2} \left( -b_3 \ln \frac{\phi^\dagger \phi}{\sigma^2} + \frac{1}{\alpha_3^{\text{init}}} - \frac{1}{\alpha_3^{\text{cr}}} \right)^2$$

(45)
7 Other Problems

1. More sound proof, for the claim that
   Quantum scale invariance persists by the SI prescription.

2. Gauge hierarchies; how do those potentials appear possessing tiny $\varepsilon_i$’s?

3. Global or Local scale invariance?

4. If global, What is $\exists$Dilaton? $\rightarrow$ Higgs?

5. The fate of dilaton? $\rightarrow$ does it remain massless?

6. How is the present CC value $\Lambda_0$ explained?

7. How does the inflation occur in this scale invariant scenario?

8. Thermal effects.


10. (Super)Gravity theory with (local or global) scale invariance.