Alternative derivation of relativistic three-particle quantization condition

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Based on work with Steve Sharpe

arXiv:2007.16188

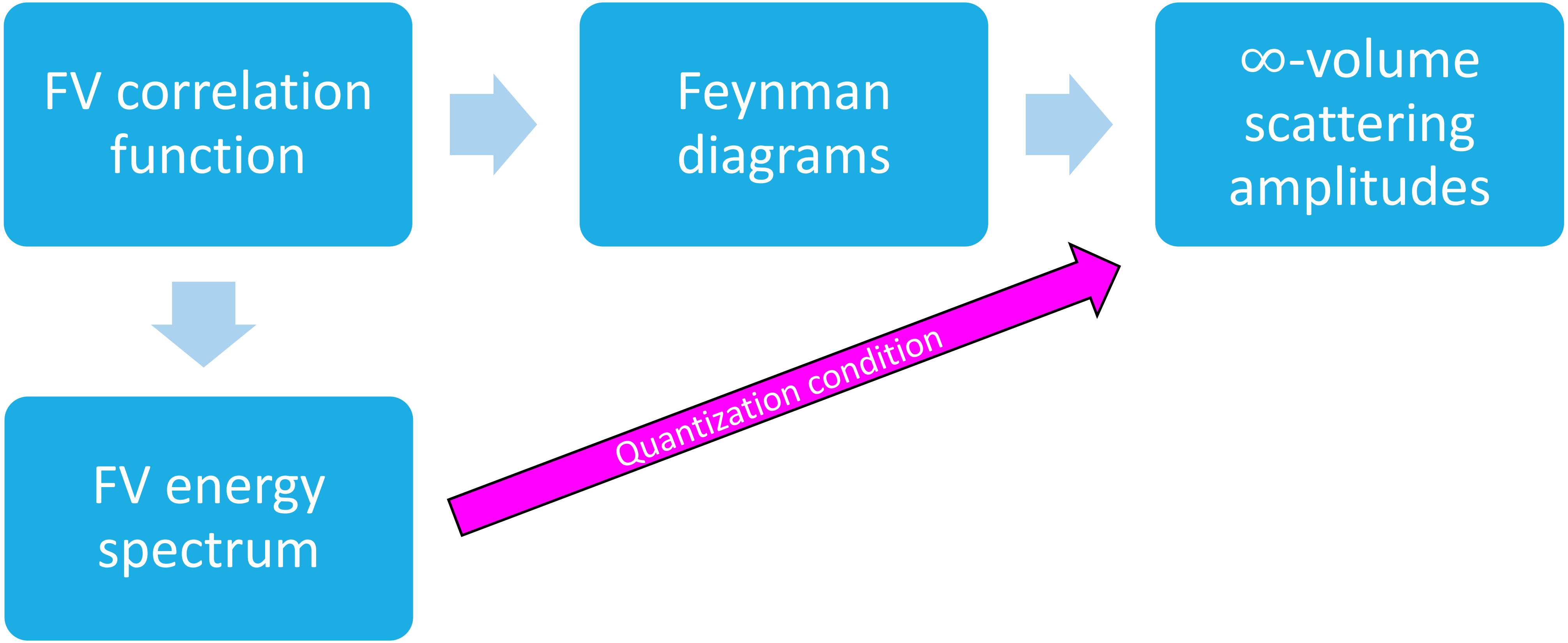
Status of 3→3 quantization conditions

Quantization condition (QC): Map from finite-volume (FV) energy spectrum to ∞-volume scattering amplitudes

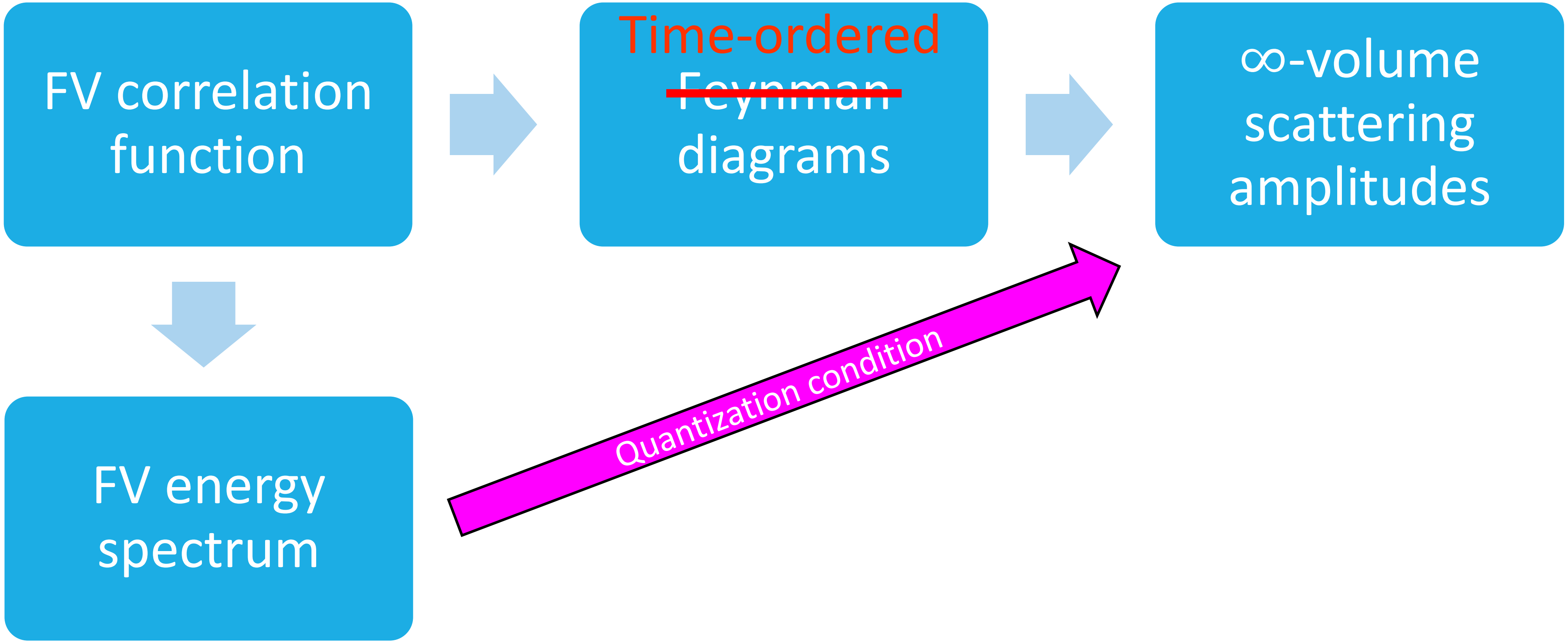
Hansen & Sharpe, 14	Relativistic EFT (RFT) approach	• Pros: Completely rigorous, most general • Cons: Complicated derivation, hard to generalize
Mai & Döring, 17	Finite-volume unitarity (FVU) approach	• Pros: Relativistic, simpler formalism • Cons: Only developed for s-wave dimers
Hammer et al., 17	Nonrelativistic EFT (NREFT) approach	• Pro: Simplest formalism • Con: NR, only developed s-wave dimers

Goal: Find simpler RFT derivation that's easier to generalize

Original RFT strategy

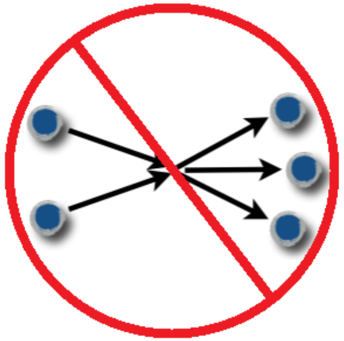


New RFT strategy



RFT framework

- Generic relativistic effective field theory
- Identical scalar particles with mass m
- Lagrangian has global $\mathbb{Z}_2$ symmetry (no odd-legged vertices)



(Side note: RFT generalizations to include $2 \to 3$ transitions and isospin have also been derived)

Briceño, Hansen, & Sharpe (2017)

Hansen, Romero-López, & Sharpe (2020)

Finite-volume (FV) correlation function

$$C_{3,L}(E,\vec{P}) = \int_L d^4x\ e^{i(Ex^0-\vec{P}\cdot x)}\ \langle 0|\mathrm{T}\sigma(x)\sigma^\dagger(0)|0\rangle$$

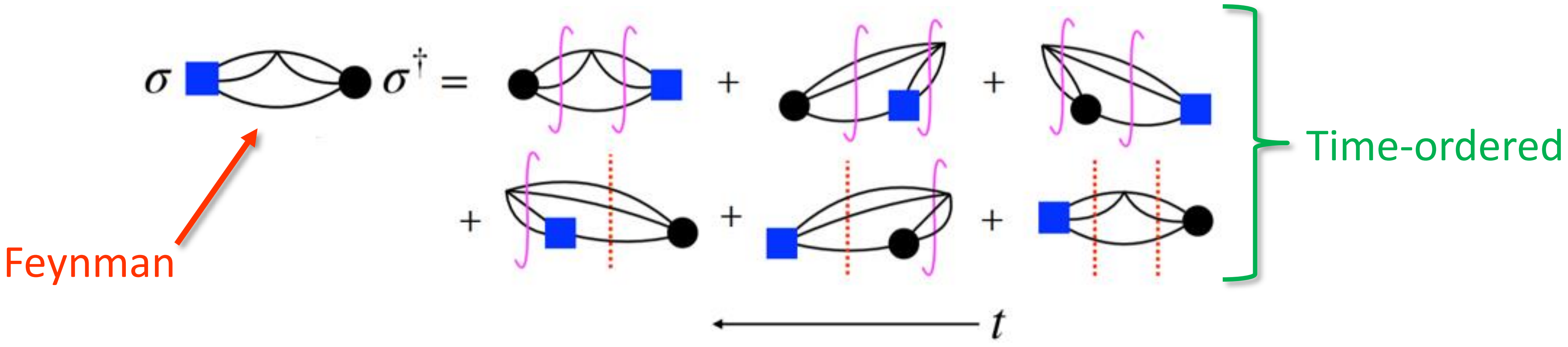
- Defined for cubic box with side length L and periodic boundary conditions
- $\sigma(x)$: operator coupling to 3-particle states
- $(E,\vec{P})$: fixed total 4-momentum with $\vec{P} \in \frac{2\pi}{L}\mathbb{Z}^3$
  - CM energy: $E^* = \sqrt{E^2 - \vec{P}^2}$

Assume $m < E^* < 5m$ so that only 3-particle states can go on shell

Key properties of $C_{3,L}$:
- Has poles at energies in the FV spectrum
- Can be expressed as an infinite sum of Feynman diagrams

Time-ordered perturbation theory (TOPT)

- Every Feynman diagram can be written as a sum of time orderings of the vertices
- In TOPT, all internal 4-momenta are on shell: $p^\mu = (+\omega_p, \vec{p}), \quad \omega_p = \sqrt{\vec{p}^2 + m^2}$
  - <u>Note</u> – total energy is generally **<u>NOT</u>** conserved: $\sum_{p \in \mathrm{cut}} \omega_p \neq E$



Evaluating time-ordered diagrams

- A propagator with momentum $\vec{p}$ gives a factor of $1/(2\omega_p)$
- A "cut" between sequential vertices gives a kinematic factor

$$iK_{\text{cut}} = \frac{i}{E_{\text{cut}} - \sum_{p\in\text{cut}} \omega_p + i\epsilon}, \qquad E_{\text{cut}} = \begin{cases} +E & \text{if } \; t_\sigma > t_{\text{cut}} > t_{\sigma^\dagger} \\ -E & \text{if } \; t_{\sigma^\dagger} > t_{\text{cut}} > t_\sigma \\ 0 & \text{otherwise} \end{cases}$$

Key insight: For $m < E^* < 5m$, K_{cut} can only be singular if:

1. The cut contains exactly 3 lines
2. $E_{\text{cut}} = +E \quad \Rightarrow \quad t_\sigma > t_{\text{cut}} > t_{\sigma^\dagger}$

} "relevant cut"

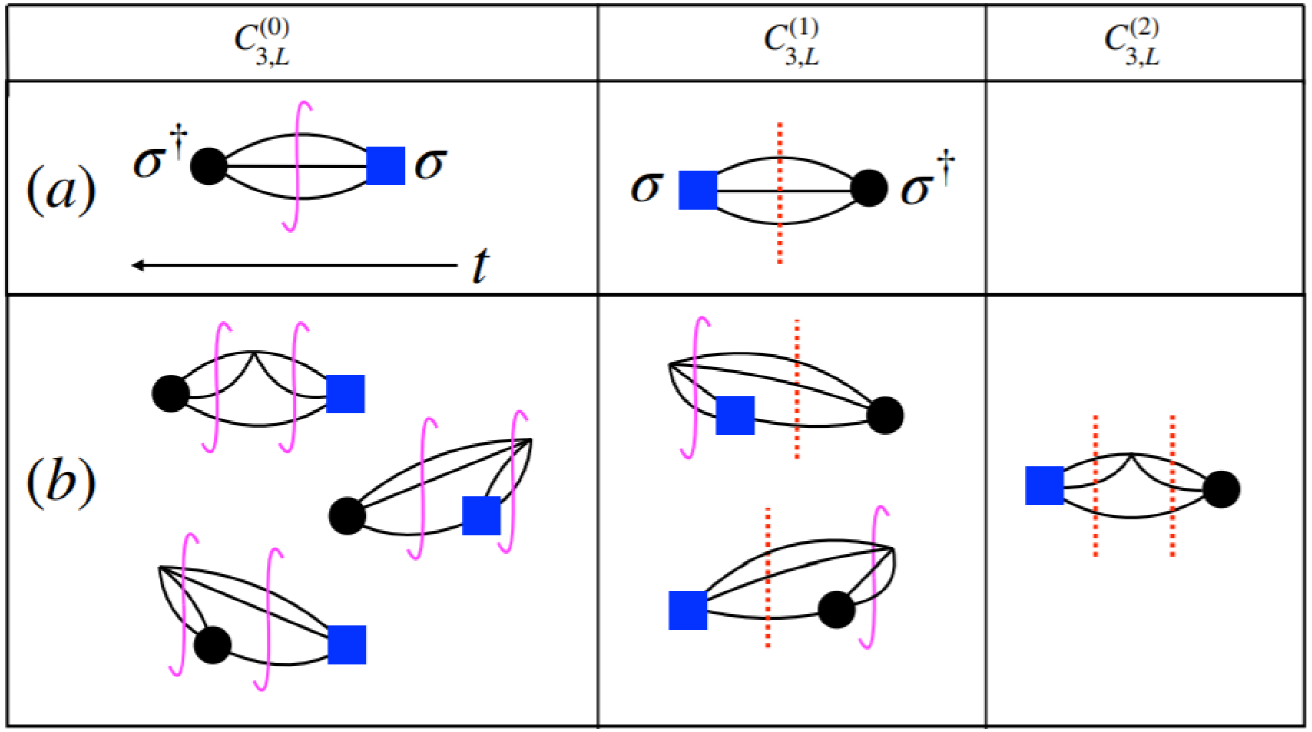
$\Rightarrow$ Can replace FV sums over irrelevant cuts with integrals up to $O(e^{-mL})$ corrections

Expanding $C_{3,L}$ in relevant cuts

- Strategy: Organize diagrams in $C_{3,L}(E,\vec{P})$ by number of relevant cuts n:

$$C_{3,L}(E,\vec{P}) = \sum_{n=0}^{\infty} C_{3,L}^{(n)}(E,\vec{P})$$

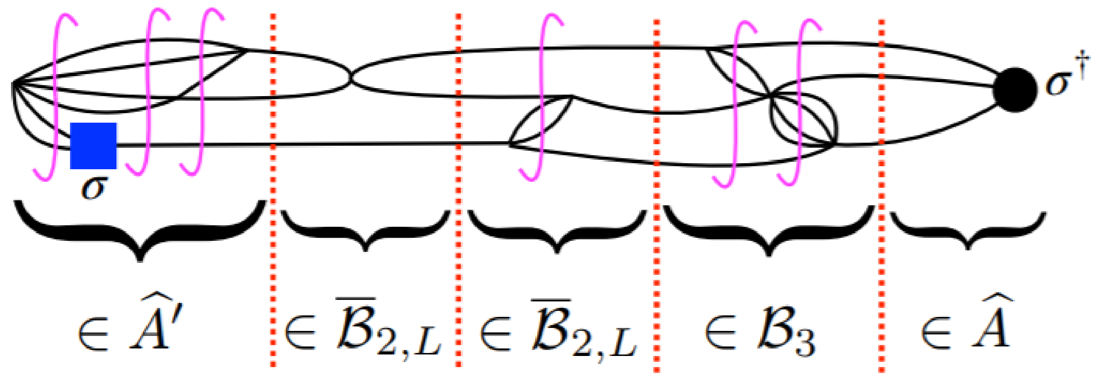
$$C_{3,L}^{(0)}(E,\vec{P}) = C_{3,\infty}^{(0)}(E,\vec{P}) + \mathcal{O}(e^{-mL})$$



3PIs building blocks

Step 1: Group irrelevant cuts into compact 3PIs quantities

- Sum of all 3PIs left (right) endcaps: $\widehat{A}' \ (\widehat{A})$
- Sum of all connected 3PIs $3 \to 3$ diagrams: $i\mathcal{B}_3$
- Sum of all disconnected (2+1) 3PIs $3 \to 3$ diagrams: $i\overline{\mathcal{B}}_{2,L} \equiv 2\omega L^3 i\mathcal{B}_2$
  - $i\mathcal{B}_2$: sum of all 2PIs $2 \to 2$ diagrams

$C^{(4)}_{3,L} \ni$ 

No relevant cuts ⇒ can treat these as ∞-volume quantities

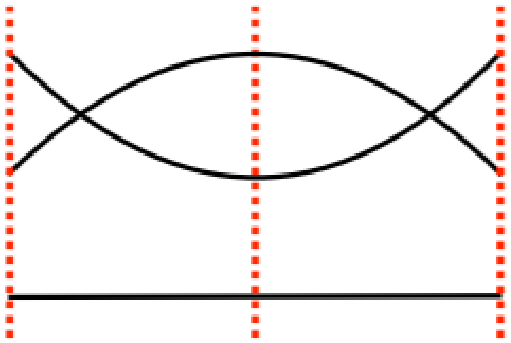
F and G cuts

<u>Step 2</u>: Define compact quantities for the relevant cuts

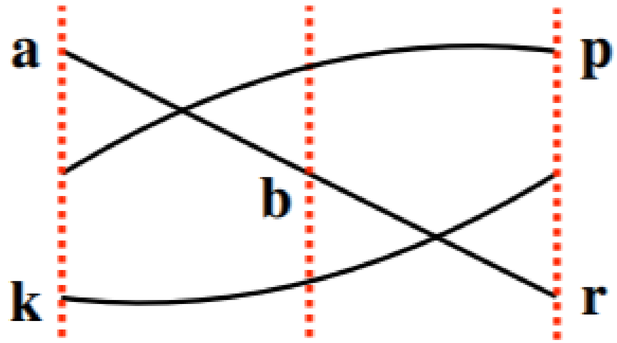
$$[iD_F]_{ka;pr} \equiv \delta_{kp}\delta_{ar}\frac{iD_{ka}}{2!}\,, \qquad [iD_G]_{ka;pr} \equiv \delta_{kr}\delta_{ap} iD_{kp}\,,$$

$$iD_{ka} \equiv \frac{1}{2\omega_k L^3}\frac{i}{2\omega_b(E-\omega_k-\omega_a-\omega_b)}\frac{1}{2\omega_a L^3}$$

$\overline{\mathcal{B}}_{2,L} D_F \overline{\mathcal{B}}_{2,L} \ni$



$\overline{\mathcal{B}}_{2,L} D_G \overline{\mathcal{B}}_{2,L} \ni$



Constructing $C_{3,L}$

$$C^{(1)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G)\widehat{A}$$

$$C^{(2)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A}$$

$$C^{(3)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G) i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\widehat{A}$$

$$\vdots$$

$$C^{(n)}_{3,L}(E,\vec{P}) = \widehat{A}' i(D_F + D_G) \left[i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)\right]^{n-1} \widehat{A}$$

$$\Rightarrow \boxed{C_{3,L}(E,\vec{P}) = C^{(0)}_{3,\infty}(E,\vec{P}) + \widehat{A}' i(D_F + D_G) \frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)} \widehat{A}}$$

This simple closed-form expression for $C_{3,L}$ is the main advantage of the new derivation

On-shell projection

$$C_{3,L}(E,\vec{P}) = C^{(0)}_{3,\infty}(E,\vec{P}) + \widehat{A}' i(D_F + D_G)\frac{1}{1 - i(\overline{\mathcal{B}}_{2,L} + \mathcal{B}_3) i(D_F + D_G)}\widehat{A}$$

- The kernels in this expression are generally evaluated off shell
  - Need to pick out contribution from on-shell amplitudes
  - Use same strategies as original derivation
- For F cuts: split up the nonspectator sum as $\sum = [\sum - \mathrm{PV}\int] + \mathrm{PV}\int$ with an appropriate principal-value (PV) prescription
- For G cuts:
  - Split up each kernel as on-shell part + remainder
  - Boost on-shell part to CM frame of nonspectator momentum pair and decompose in spherical harmonics
  - Introduce regulator function $H(\vec{k})$ to turn off 2-particle energies well below threshold

On-shell projection

$$X' D_F X = X'(\widetilde{F} + \widetilde{\mathcal{I}}_F) X \,, \qquad X' D_G X = X'(\widetilde{G} + \delta\widetilde{G}) X$$

$$\widetilde{F}_{k\ell m;p\ell' m'} \equiv \delta_{kp} H(\vec{k}) \left[\frac{1}{L^3} \sum_{\vec{a}}^{\mathrm{UV}} - \mathrm{PV} \int_{\vec{a}}^{\mathrm{UV}} \right] \frac{\mathcal{Y}_{\ell m}(\vec{a}_k^*)}{q_{2,k}^{*\ell}} \frac{L^3 D_{ka}}{2!} \frac{\mathcal{Y}_{\ell' m'}(\vec{a}_k^*)}{q_{2,k}^{*\ell'}}$$

$$\widetilde{G}_{k\ell m;p\ell' m'} \equiv \frac{1}{2\omega_k L^3} \frac{\mathcal{Y}_{\ell m}(\vec{p}_k^*)}{q_{2,k}^{*\ell}} \frac{H(\vec{k}) H(\vec{p})}{b^2 - m^2} \frac{\mathcal{Y}_{\ell' m'}(\vec{k}_p^*)}{q_{2,p}^{*\ell'}} \frac{1}{2\omega_p L^3}$$

- $\tilde{F}$ and $\tilde{G}$ project adjacent kernels on shell
- The $\widetilde{\mathcal{I}}_F$ and $\delta\widetilde{G}$ terms are off-shell remainders and can be treated as ∞-volume quantities

Final expression for $C_{3,L}$

$$\Rightarrow C_{3,L}(E,\vec{P}) = C^{(0)}_{3,\infty}(E,\vec{P}) + \widehat{A}' i(\widetilde{F}+\widetilde{G}+\widetilde{\mathcal{I}}_F+\delta\widetilde{G})\frac{1}{1-i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)i(\widetilde{F}+\widetilde{G}+\widetilde{\mathcal{I}}_F+\delta\widetilde{G})}\widehat{A}$$

$$= \widetilde{C}_{3,\infty}(E,\vec{P}) + \widetilde{A}'^{(u)} i(\widetilde{F}+\widetilde{G})\frac{1}{1-i\left(2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right)i(\widetilde{F}+\widetilde{G})}\widetilde{A}^{(u)}$$

$$i\left(2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}\right) = \frac{1}{1-i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)i(\widetilde{\mathcal{I}}_F+\delta\widetilde{G})}i(\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3)$$

- $\mathcal{K}_2$ and $\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}$ are on-shell ∞-volume amplitudes
- The combination $2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}^{(u,u)}_{\mathrm{df},3}$ appears naturally due to the $\overline{\mathcal{B}}_{2,L}+\mathcal{B}_3$ construction
- (u) labels indicate asymmetry under particle exchange due to the spectator-dependent $\overline{\mathcal{B}}_{2,L}$

New form of the quantization condition

$$C_{3,L} - \widetilde{C}_{3,\infty} = \widetilde{A}'^{(u)} i(\widetilde{F} + \widetilde{G}) \frac{1}{1 - i\left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right) i(\widetilde{F} + \widetilde{G})} \widetilde{A}^{(u)}$$

- Poles occur at energies in FV spectrum ⇒ when determinant of RHS is singular

- Asymmetric QC:

$$\boxed{\det\left[1 + \left(2\omega L^3 \mathcal{K}_2 + \widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right)\left(\widetilde{F} + \widetilde{G}\right)\right] = 0}$$

Understanding $\widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}$ and the new QC

$$\det\left[1+\left(2\omega L^3\mathcal{K}_2+\widetilde{\mathcal{K}}_{\mathrm{df},3}^{(u,u)}\right)\left(\widetilde{F}+\widetilde{G}\right)\right]=0$$

- Asymmetric divergence-free 3-particle K matrix with closed-form definition
  - In contrast to symmetric $\mathcal{K}_{\mathrm{df},3}$ in original RFT QC: $\det\left[1+\mathcal{K}_{\mathrm{df},3}F_3\right]=0$
- Can be symmetrized to give original QC
  - Similarly, can asymmetrize original QC into new form: $\det\left[1+\left(2\omega L^3\mathcal{K}_2+\mathcal{K}_{\mathrm{df},3}'^{(u,u)}\right)\left(\widetilde{F}+\widetilde{G}\right)\right]=0$
- Related to physical scattering amplitude $\mathcal{M}_3$ via integral equations
- Can be related to R matrix of FVU approach – topic of the next talk! arXiv:2007.16190

Conclusions

- We expect this new TOPT approach to organizing diagrams will greatly simplify the process of generalizing the RFT method

- Future goals:
  - 3-particle QC for nondegenerate particles
  - 4-particle QCs?

Thanks for listening! I’m happy to take any questions