Lattice QCD calculation of the pion charge radius using a model-independent method

Speaker: Yang Fu (Peking U.)

Co-authors: Xu Feng (Peking U.), Lu-Chang Jin (UConn/RIKEN BNL)

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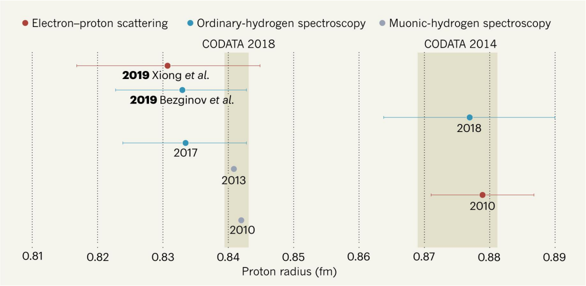
Reference: Phys. Rev. D 101, 051502 (2020)

Outline

1. Introduction and motivation
   - What is charge radius?
   - Traditional lattice approach: fitting the form factor
2. Model-independent method
   - Calculating the slope directly on lattice
   - Results: smaller statistical errors
3. Conclusions and outlook

Proton radius puzzle



JP Karr, Nature, 2019, 575, 61-62

- Over 5σ discrepancy between muon and electron based measurements;
- The debate over this puzzle mainly comes from experiments;
- Lattice simulations have not been able to give a comparable result.
  $\Longrightarrow$ Still affected by systematic uncertainties.

- Why pion charge radius?

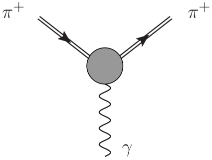
Pion structure is simple, no signal-to-noise problem, ...

$\Longrightarrow$ Well suited for a high precision benchmark of new methods.

- What is charge radius?

Defined as the derivative of the form factor $F(q^2)$ at zero momentum transfer

$$\langle r^2 \rangle = 6 \left. \frac{dF(q^2)/F(0)}{dq^2} \right|_{q^2=0}$$

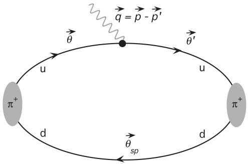


- Pion form factor $\leftrightarrow$ matrix element of the vector current

$$\langle \pi^+(p') | J_\mu | \pi^+(p) \rangle = F_\pi(q^2)(p + p')_\mu$$

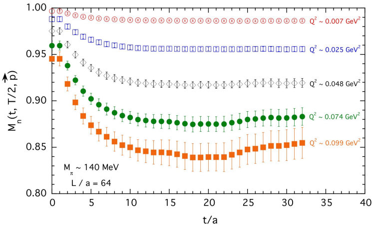
Traditional approach: form factor from lattice

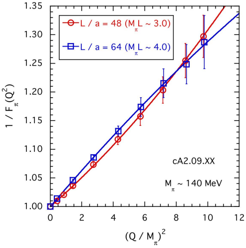
- Form factor under a certain momentum transfer



Construct ratio from two- and three-point functions

$$\frac{C^{(3)}(t, t', \vec{p}, -\vec{p})}{C^{(2)}(t', \vec{p})} \xrightarrow[(t'-t)\to\infty]{t\to\infty} \frac{\langle \pi^+(p')|J_4|\pi^+(p)\rangle}{2E_\pi(\vec{p})} = F_\pi(q^2)$$





C. Alexandrou, Phys. Rev. D 97, 014508 (2018)

- Fit $F(q^2)$ to get charge radius $\to$ model dependence.
- Widely used for over a decade!

Basic idea

- We start with this hadronic function

$$H(x) = \langle 0|O_\pi(t,\vec{x})J_\mu(0)|\pi(\vec{0})\rangle$$

In principle, this function contains all information about EM interactions.

- How to extract the part of interest?

By using an appropriate weight function $\omega(x)$

$$\langle A\rangle = \int \mathrm{d}^3\vec{x}\,\omega(x)H(x)$$

- E.g, $\omega(x) = e^{i\vec{p}\cdot\vec{x}}$ will extract the part under a certain momentum

$$\tilde{H}(t,\vec{p}) = \int \mathrm{d}^3\vec{x}e^{i\vec{p}\cdot\vec{x}}H(x) \sim F_\pi(q^2)$$

- Charge radius $\rightarrow$ derivative of the form factor $\rightarrow \dfrac{d}{d|\vec{p}|^2}\displaystyle\int \mathrm{d}^3\vec{x}e^{i\vec{p}\cdot\vec{x}}H(x)$

$\Longrightarrow$ $\omega(x) \sim |\vec{x}|^2$

Charge radius in the continuum theory

- In the infinite volume, following the above idea, one can construct

$$D(t) \equiv -\frac{m_\pi^2}{3!} \int \mathrm{d}^3\vec{x}|\vec{x}|^2 H(x), \quad \tilde{H}(t,\vec{0}) \equiv \int \mathrm{d}^3\vec{x} H(x)$$

- At large time t, the ratio of this two functions gives

$$R(t) \equiv \frac{D(t)}{\tilde{H}(t,\vec{0})} \to \frac{1}{4} - \frac{m_\pi t}{2} - c_1$$

where c_n is the expansion coefficients of the form factor

$$F_\pi(q^2) = \sum_n c_n \left(\frac{q^2}{m_\pi^2}\right)^n$$

with $c_0 = 1$ and $c_1 = \frac{m_\pi^2}{6}\langle r_\pi^2 \rangle$.

- One can determine c_1 directly using $H(x)$ as input.

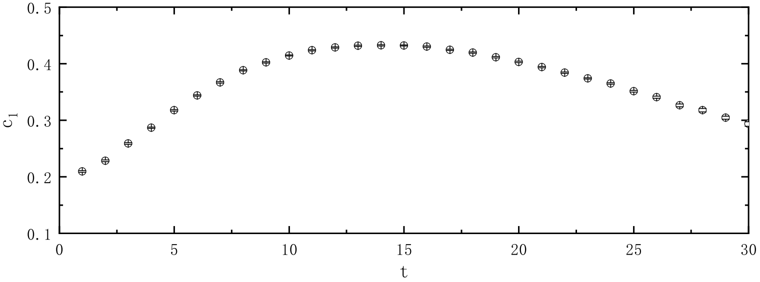


Figure: determine c_1 from the formula in continuum theory

- No plateau at all.
- Typical lattice size $m_\pi L \approx 4$, the integrand in $D(t)$ at the edge scales as $m_\pi^2 |\vec{x}|^2 \exp\left(-m_\pi \sqrt{|\vec{x}|^2 + t^2}\right) \approx 0.5 \not\ll 1$.

$\Longrightarrow$ Significant finite-volume effects!

Charge radius on the lattice

- On the lattice, one can still construct

$$D^{(L)}(t) \equiv -\frac{m_\pi^2}{3!} \sum_{x \in \mathbb{L}} |\vec{x}|^2 H(x), \quad \tilde{H}^{(L)}(t, \vec{0}) \equiv \sum_{x \in \mathbb{L}} H(x)$$

- In continuum theory, the ratio only contains contributions to the first order

$$R^{(\infty)}(t) \equiv \frac{D^{(\infty)}(t)}{\tilde{H}^{(\infty)}(t, \vec{0})} \to \beta_0^{(\infty)}(t) + \beta_1^{(\infty)}(t) c_1$$

- Now it contains contributions to all orders

$$R^{(L)}(t) \equiv \frac{D^{(L)}(t)}{\tilde{H}^{(L)}(t, \vec{0})} \to \sum_n \beta_n^{(L)}(t) c_n$$

with $\beta_n^{(L)}(t)$ known explicitly and can be calculated numerically

$$\beta_n^{(L)}(t) = -\frac{m_\pi^2}{3!} \sum_{\vec{x} \in \mathbb{L}^3} |\vec{x}|^2 I_n(x)$$

$$I_n(x) = \frac{1}{L^3} \sum_{\vec{p}} \frac{\hat{E}}{\tilde{E}} \frac{\tilde{m}}{\hat{m}} \frac{\hat{E} + \hat{m}}{2\hat{m}} \left(\frac{\hat{q}^2}{m_\pi^2} \right)^n e^{-(E - m_\pi)t} e^{-i\vec{p} \cdot \vec{x}}$$

- Naively, one can determine c_1 through

$$\frac{R^{(L)}(t)-\beta_0^{(L)}(t)}{\beta_1^{(L)}(t)} \to c_1\left(1+\sum_{n>1}\frac{\beta_n^{(L)}(t)c_n}{\beta_1^{(L)}(t)c_1}\right)$$

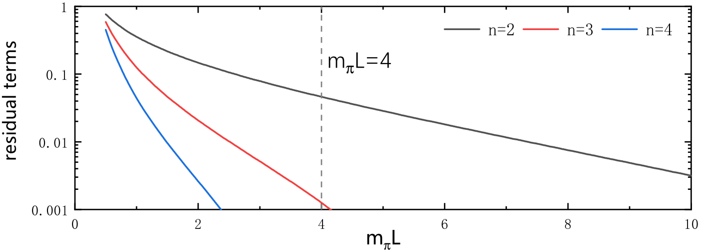


Figure: Residual terms as function of $m_\pi L$ with spacing $a=0$ at fixed $m_\pi t=1$

- Residual terms can be estimated by VMD model $F_\pi(q^2)=(1-q^2/m_\rho^2)^{-1}$ with $c_n=(m_\pi/m_\rho)^{2n}$.
- At $m_\pi L=4$, the lowest order $(n=2)$ of residual term $\sim 5\%$.

Error reduction: systematic effects

- higher-order terms are not well suppressed at $m_\pi L \approx 4$.

$\Longrightarrow$ determine $c_{n\geq 2}$ through weight function $\omega(x) = |\vec{x}|^4,\ |\vec{x}|^6,\ \cdots$

- one can generally construct

$$D_k^{(L)}(t) \sim \sum_{\vec{x}\in\mathbb{L}} |\vec{x}|^{2k} H(x), \qquad k = 1, 2, \ldots$$

with the ratio truncated to the m-th order

$$R^{(L)}(t) \equiv \frac{\sum_i^m f_i D_i^{(L)}(t)}{\tilde{H}^{(L)}(t, \vec{0})} + h \to c_1 + \mathcal{O}(c_{n>m} \text{ terms})$$

where parameters f_i and h are chosen to remove the c_0 and $c_{m\geq n\geq 2}$ terms.

- Our final choice is $m = 2$.
  - Contamination from $c_{n\geq 3}$ are negligibly small. ($\lesssim 0.1\%$)
  - The signal-to-noise ratio decreases as m increases.

Error reduction: statistical uncertainties

- Lattice data near the boundary of the box mainly contribute to the noise rather than signal since $H(x) \sim \exp\left(-m_\pi \sqrt{|\vec{x}|^2 + t^2}\right)$

$\Longrightarrow$ introduce an integral range ξL to reduce the statistical error.

$$D_k^{(L,\xi)}(t) \sim \sum_{|\vec{x}| \le \xi L} |\vec{x}|^{2k} H(x)$$

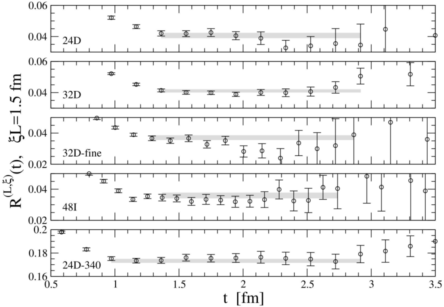
- The formula of the ratio is therefore changed to

$$R_k^{(L,\xi)}(t) \equiv \frac{D_k^{(L,\xi)}(t)}{\tilde{H}^{(L)}(t,\vec{0})} \to \sum_n \beta_{k,n}^{(L,\xi)}(t) c_n$$

with

$$\beta_{k,n}^{(L,\xi)}(t) \sim \sum_{|\vec{x}| \le \xi L} |\vec{x}|^{2k} I_n(x)$$

- with our final choice $\xi L = 1.5\text{fm}$, the statistical uncertainties are reduced by a factor of $1.3 - 1.8$.
- We expect the error reduction can be much more significant in the study of nucleon charge radius!



Ensemble	m_π[MeV]	L	T	a^{-1}[GeV]	$N_{\rm conf}$
24D	141.2(4)	24	64	1.015	47
32D	141.3(3)	32	64	1.015	47
32D-fine	143.2(3)	32	64	1.378	52
48I	139.1(3)	48	96	1.730	31
24D-340	340.9(4)	24	64	1.015	36

- We use 5 DWF ensembles from RBC/UKQCD. Phys. Rev. D 93, 074505 (2016)
- Relatively small statistics $n_{\rm conf} \approx 30 - 50$.
- Now we can observe clear plateau for each ensemble!

Ensemble	Parameters			New	Traditional
	m_π[MeV]	L	a^{-1}[GeV]	$\langle r_\pi^2 \rangle$ [fm^2]	$\langle r_\pi^2 \rangle$ [fm^2]
24D	141.2(4)	24	1.015	0.476(18)	0.466(30)
32D	141.3(3)	32	1.015	0.480(10)	0.479(15)
32D-fine	143.2(3)	32	1.378	0.423(15)	0.409(28)
48I	139.1(3)	48	1.730	0.434(20)	0.395(32)
24D-340	340.9(4)	24	1.015	0.3485(27)	0.3495(44)

- Traditional: fit $F_\pi(q^2) = 1 + \frac{1}{6}\langle r_\pi^2 \rangle q^2 + c_V(q^2)^2$. (To the same order)

Statistical errors are 1.5 − 1.9 times larger than the new method.

- 24D and 32D: finite volume effects are mild.
- Our final result

$$\langle r_\pi^2 \rangle = 0.434(20)(13)\ [\text{fm}^2]$$

is very consistent with the PDG value 0.434(5)fm^2.

Model-independent method

- Start with a basic idea: $\langle A \rangle = \sum_{\vec{x}} \omega(x) H(x)$, where the weight function $\omega(x)$ contains all the non-QCD information.
- Overcome the problem of finite-volume effects.
- It also has advantages in statistical uncertainties.

Next step: nucleon charge radius

- Finite-volume effects become insignificant since $m_N \gg m_\pi$.
- Error reduction techniques should be more effective.
- We aim at a result that can be compared with experiments.

Thank you!