The effects of fermions in the complex Langevin simulation of the Lorentzian type IIB matrix model

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work in progress

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1. Introduction 1

◆The type IIB matrix model [Ishibashi-Kawai-Kitazawa-Tsuchiya ('96)]

Promising candidate for nonperturbative formulation of superstring theory

Symmetries: SU(N), SO(9,1), $\mathcal{N}$ =2 SUSY

$$S = -\frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{4}[A^{\mu}, A^{\nu}][A_{\mu}, A_{\nu}] + \frac{1}{2}\bar{\Psi}\Gamma^{\mu}[A_{\mu}, \Psi]\right) \qquad (\mu = 0, 1, \ldots, 9)$$

$N \times N$ Hermitian matrices

A_{μ} : 10d Lorentz vector
Ψ : 10d Majorana-Weyl spinor

under SO(9,1) transformation

In this model, space-time does not exist a priori
but emerges dynamically from the degrees of freedom of matrices.

1. Introduction 2

This model has been studied by numerical methods.

◆Lorentzian version of the type IIB matrix model

SSB of SO(9) to SO(3) and existence of (3+1)-dim. expanding universe

[Kim-Nishimura-Tsuchiya ('12), Aoki-Hirasawa-Ito-Nishimura-Tsuchiya ('19), etc]

In previous works, approximations were used to avoid the sign problem, which arise from the inability to consider e^{-S} as the probability distribution. It is hard to perform Monte Carlo simulations due to this problem.

Examples of systems where the sign problem occurs:

- Theta term [Yosprakob-san's talk, Matsumoto-san's, Honda-san's]
- Real-time dynamics
- Finite density QCD
- Other systems with fermions

1. Introduction 3

◆Methods to overcome the sign problem

- Lefschetz thimble method

  In this method, the sign problem is minimized by the deformation of the integral path.

- Complex Langevin method (CLM)

  In this method, expectation values are calculated
  by the stochastic process for complexified variables.

◆Studies of the type IIB model by the CLM

- Euclidean version of the type IIB matrix model

  [Azuma-san's talk; Anagnostopoulos-Azuma-Ito-Nishimura-Okubo-Papadoudis ('20)]

- Lorentzian version of the type IIB matrix model

  [Nishimura-Tsuchiya ('19), this talk, Hirasawa-san's talk]

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2. The type IIB matrix model

◆ Action

$$S = -\frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{4}[A^\mu, A^\nu][A_\mu, A_\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \Psi] + m_{\mathrm{f}}\,\mathrm{Tr}\left(\bar{\Psi} i \Gamma_0 \Psi\right)\right)$$

$$= S_{\mathrm{b}} + S_{\mathrm{f}}$$

SO(5) symmetry is preserved.

$$S_{\mathrm{b}} = -\frac{1}{4g^2}\,\mathrm{Tr}\left([A^\mu, A^\nu][A_\mu, A_\nu]\right), \quad S_{\mathrm{f}} = -\frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \Psi] + m_{\mathrm{f}}\bar{\Psi} i \Gamma_0 \Psi\right)$$

A_μ, Ψ : $N \times N$ Hermitian matrices

m_{f} : deformation parameter $\begin{cases} m_{\mathrm{f}} \to \infty & \text{: bosonic} \\ m_{\mathrm{f}} \to 0 & \text{: fermionic} \end{cases}$

Partition function

$$Z = \int dA d\Psi e^{iS}$$

$$= \int dA \, e^{iS_{\rm b}} \, {\rm Pf}\mathcal{M}(A)$$

phase factor → sign problem!

- IR cutoffs to make this model well-defined

$$\frac{1}{N}{\rm Tr}(A_0)^2 = \kappa \ , \ \frac{1}{N}{\rm Tr}(A_i)^2 = 1 \quad (i = 1, \ldots, 9)$$

There is a nice way to treat these cutoffs in the CLM.

Wick rotation

$$S_{\rm b} = -\frac{1}{4g^2}\,{\rm Tr}\left([A^\mu, A^\nu][A_\mu, A_\nu]\right)$$

$$= N\beta\left[-\frac{1}{2}\,{\rm Tr}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2\right]$$

$$\beta = 1/(g^2 N),\;\; F_{\mu\nu} = i[A_\mu, A_\nu]$$

$$Z = \int dA e^{iS_{\rm b}}{\rm Pf}\mathcal{M}(A)$$

Wick rotation on the world-sheet: multiplying by $e^{is\pi/2}$

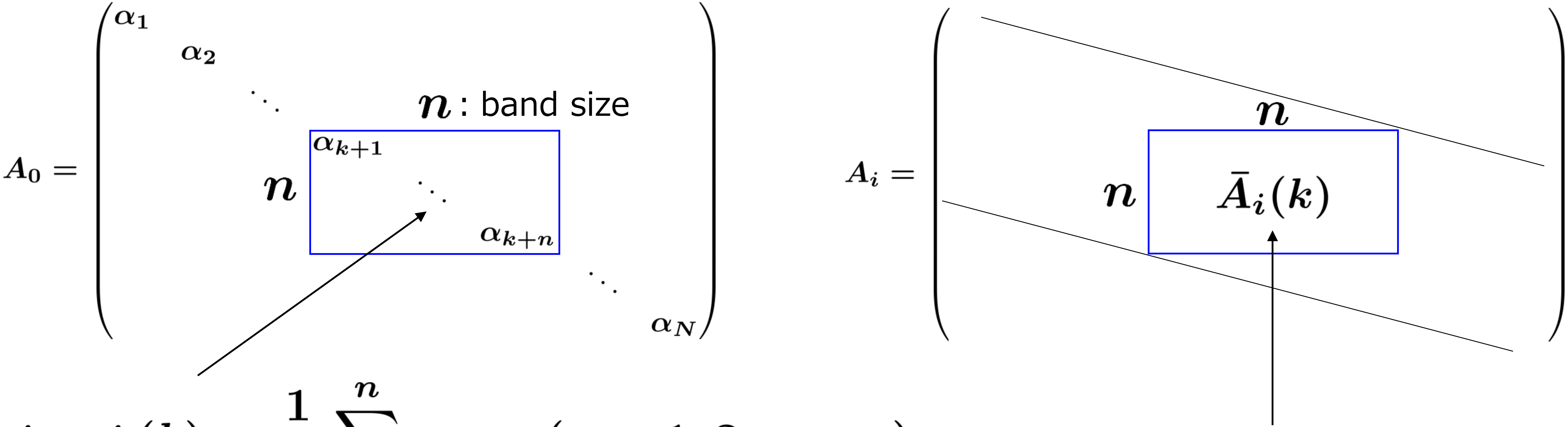
$$\tilde{S}_{\rm b} = -iN\beta e^{is\pi/2}\left[-\frac{1}{2}\,{\rm Tr}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2\right]$$

$$Z = \int dA e^{-\tilde{S}_{\rm b}}{\rm Pf}\mathcal{M}(A)$$

$s = 0$: Lorentzian type IIB matrix model

How to extract the time evolution

$A_\mu \to U A_\mu U^\dagger$, U diagonalizes A_0.



$$t = t\,(k) = \frac{1}{n}\sum_{p=1}^{n} \alpha_{k+p} \;\; (p = 1, 2, \ldots, n)$$

$$(k = 1, 2, \ldots, N - n + 1)$$

represent space at $t\,(k)$

3. Complex Langevin method

Complex Langevin equation

Complex-valued function

$$Z = \int dx\, w(x), \quad x \in \mathbb{R}$$

[Parisi ('83), Klauder ('84)]

↓ complexify variable

$$z \in \mathbb{C}$$

- Complex Langevin equation (t: Langevin time)

$$\frac{dz_k}{dt} = \frac{1}{w}\frac{\partial w}{\partial z_k} + \eta_k(t)$$

drift term

Gaussian noise, real $P(\eta_k(t)) \propto \exp\left(-\frac{1}{4}\int dt \sum_k [\eta_k(t)]^2\right)$

- Necessary and sufficient condition to justify the CLM

[Nagata-Nishimura-Shimasaki ('16)]

The probability distribution of the drift term should be exponentially suppressed for large values.

Application to the Lorentzian type IIB matrix model

◆Order of eigenvalues of A_0 [Nishimura-Tsuchiya ('19)]

Change of variables

$$\alpha_1 < \alpha_2 < \cdots < \alpha_N \Rightarrow \alpha_1 = 0,\ \alpha_2 = e^{\tau_1},\ \alpha_3 = e^{\tau_1} + e^{\tau_2},\ \cdots,\ \alpha_N = \sum_{k=1}^{N-1} e^{\tau_k}$$

◆Complexification of τ_k

$\alpha_k \in \mathbb{R}$ Hermitian matrices: $A_i \in \mathrm{SU}(N)$	➡	$\alpha_k \in \mathbb{C}$ General complex matrices: $A_i \in \mathrm{SL}(N,\mathbb{C})$

- Complex Langevin equation

$$S_{\text{eff}} = -iN\beta e^{is\pi/2}\left[-\frac{e^{-ik\pi}}{2}\frac{\mathrm{Tr}(F_{0i})^2}{(\mathrm{Tr}\,A_0^2/N)(\mathrm{Tr}\,A_i^2/N)} + \frac{1}{4}\frac{\mathrm{Tr}(F_{ij})^2}{(\mathrm{Tr}\,A_i^2/N)^2}\right] + \frac{N}{2}(\mathrm{Tr}\,A_0^2 + \mathrm{Tr}\,A_i^2)$$

$$\frac{d\tau_k}{dt} = -\frac{\partial S_{\text{eff}}}{\partial \tau_k} + \eta_k(t) \qquad P(\eta_k(t)) \propto \exp\left(-\frac{1}{4}\int dt \sum_k [\eta_k(t)]^2\right)$$

$$\frac{d(A_i)_{kl}}{dt} = -\frac{\partial S_{\text{eff}}}{\partial (A_i)_{lk}} + (\eta_i)_{kl}(t) \qquad P(\eta_i(t)) \propto \exp\left(-\frac{1}{4}\int dt \sum_i \mathrm{Tr}[\eta_i(t)]^2\right)$$

4. Results

Parameter setting

$$S = -\frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{4}[A^\mu, A^\nu][A_\mu, A_\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[A_\mu, \Psi] + m_{\mathrm{f}}\,\mathrm{Tr}\left(\bar{\Psi} i \Gamma_0 \Psi\right)\right)$$

IR cutoffs: $\frac{1}{N}\mathrm{Tr}(A_0)^2 = \kappa\ ,\ \frac{1}{N}\mathrm{Tr}(A_i)^2 = 1 \quad (i = 1, \ldots, 9)$

$$\tilde{S}_{\mathrm{b}} = -iN\beta e^{is\pi/2}\left[-\frac{1}{2}\mathrm{Tr}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2\right]$$

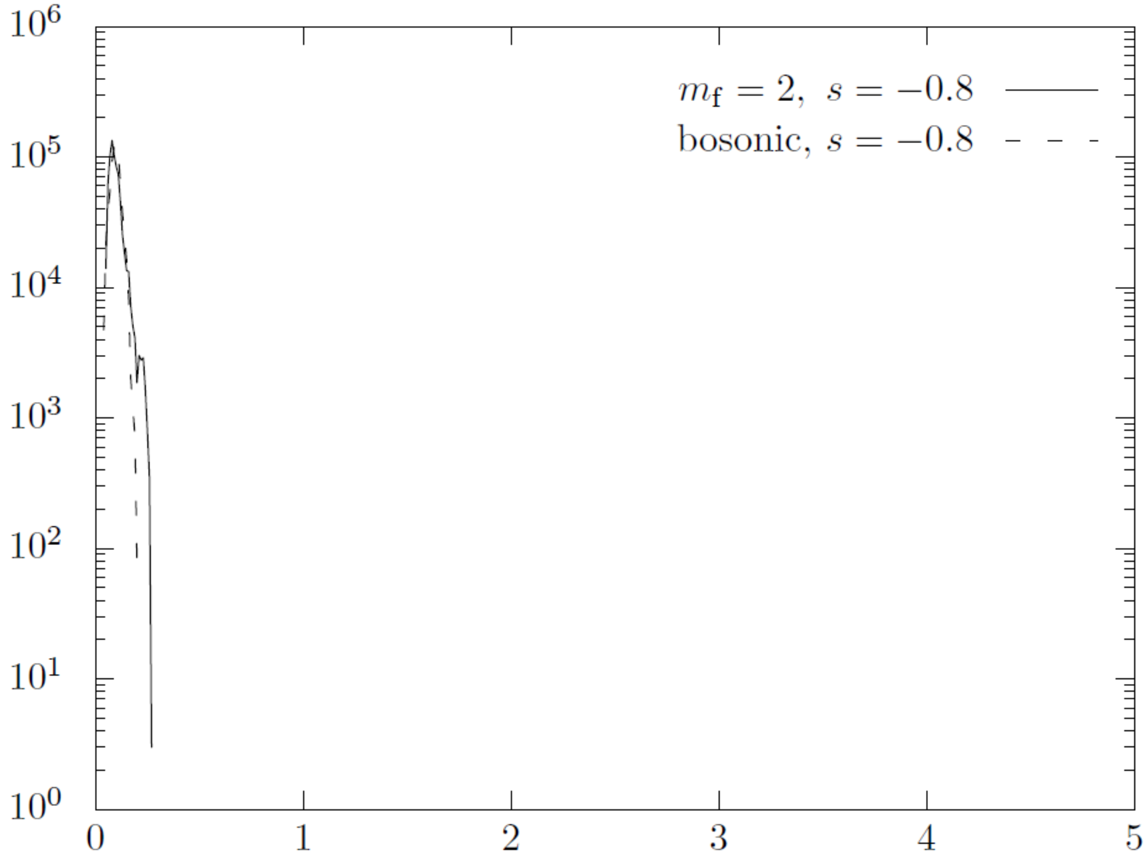
- Matrix size: $N = 32$
- Band size: $n = 8$
- Kappa: $\kappa = 0.01$
- Beta: $\beta = 1/(g^2 N) = 32$
- Deformation parameter: $m_{\mathrm{f}} = 2$
- Wick rotation parameter: $s = -0.8$

n : band size

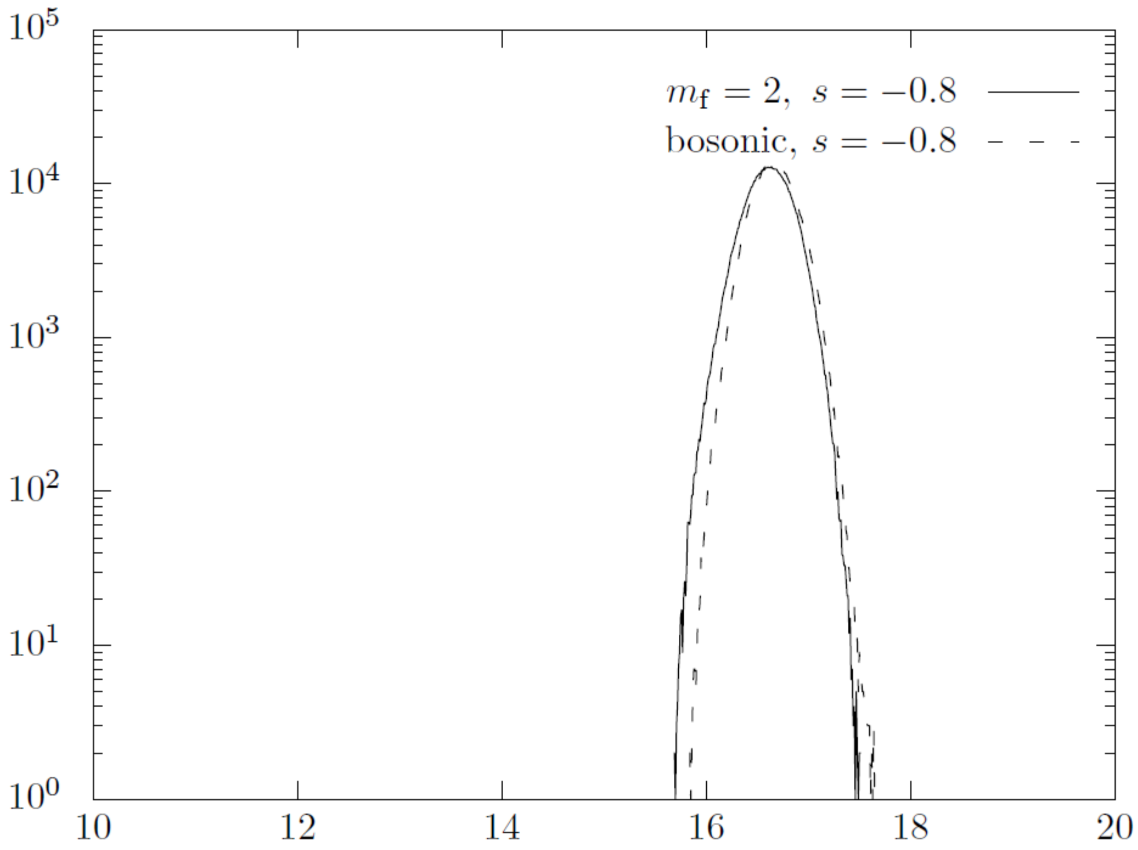
$$A_0 = \begin{pmatrix} \alpha_1 & & & & & & \\ & \alpha_2 & & & & & \\ & & \ddots & & & & \\ & & & \alpha_{k+1} & & & \\ & & & & \ddots & & \\ & & & & & \alpha_{k+n} & \\ & & & & & & \ddots \\ & & & & & & & \alpha_N \end{pmatrix}$$

Histograms of drift term

Drift term of $\boldsymbol{\tau_k}$

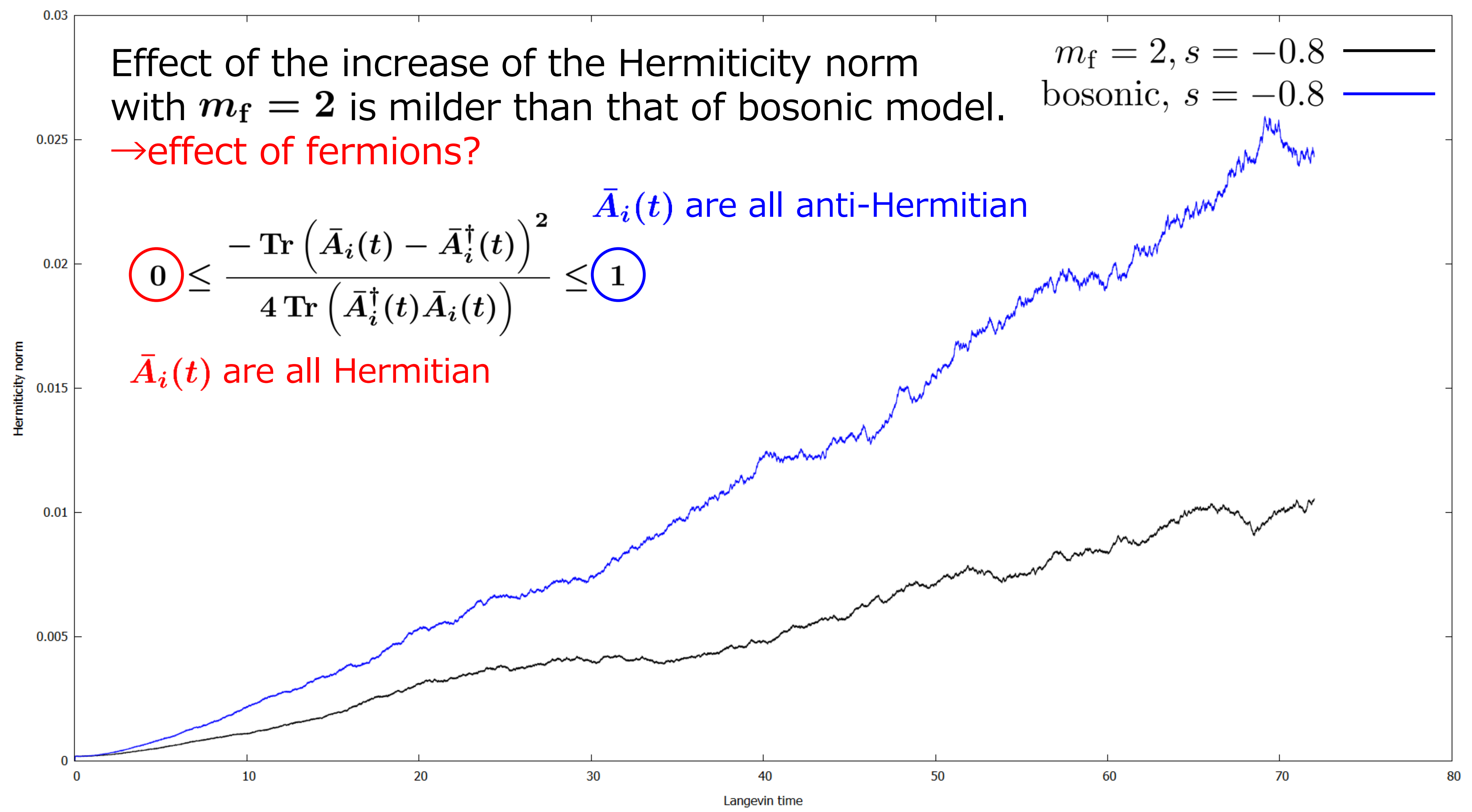


Drift term of $\boldsymbol{A_i}$



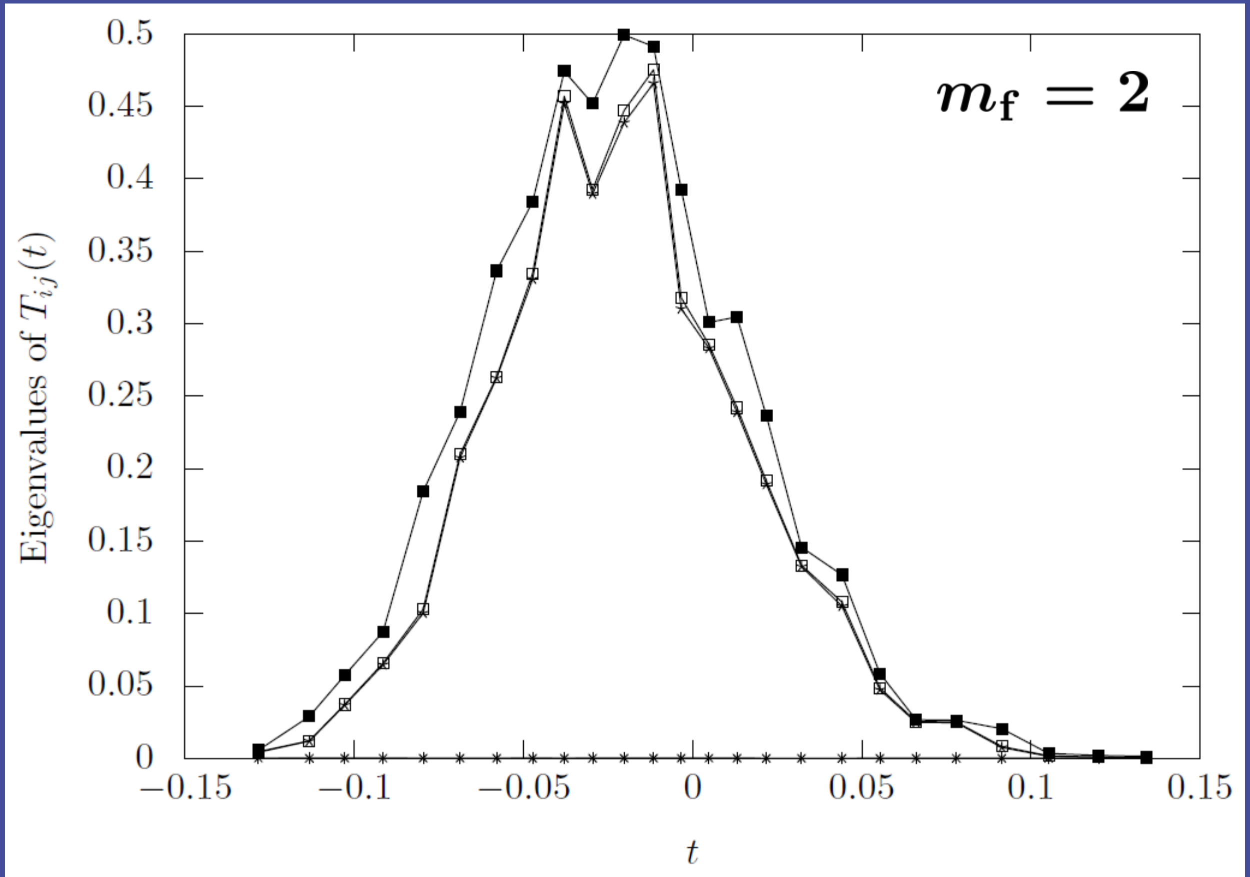
The CLM does not fail in both of $\boldsymbol{m_f = 2}$ and bosonic cases.
For smaller $\boldsymbol{m_f}$, we expect that the CLM will fail.

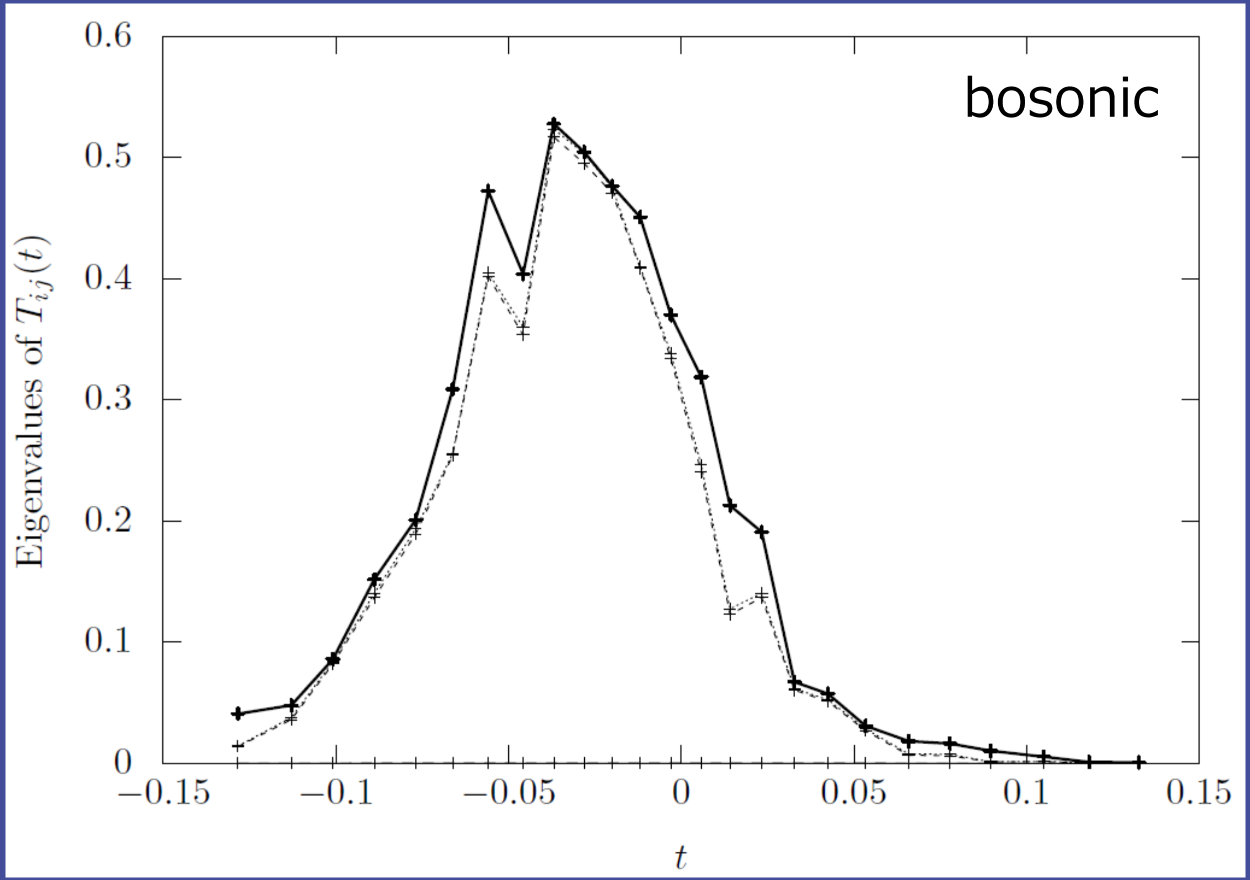
Hermiticity norm



Eigenvalues of $T_{ij}(t) = \frac{1}{n}\mathrm{tr}\left(\bar{A}_i(t)\,\bar{A}_j(t)\right)$

This quantity represents how the space extends in 5 dimensions.
If SO(5) symmetric, 5 eigenvalues are degenerate.

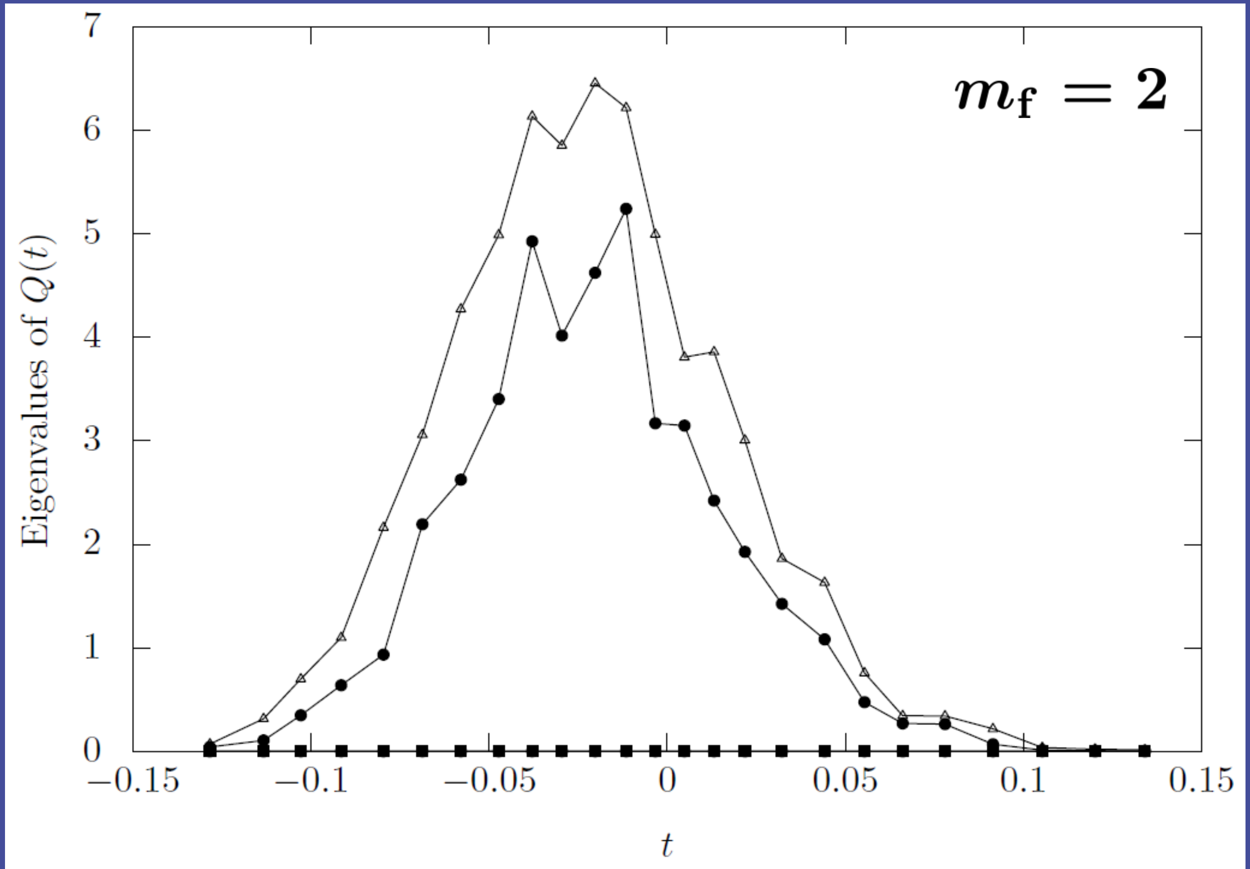


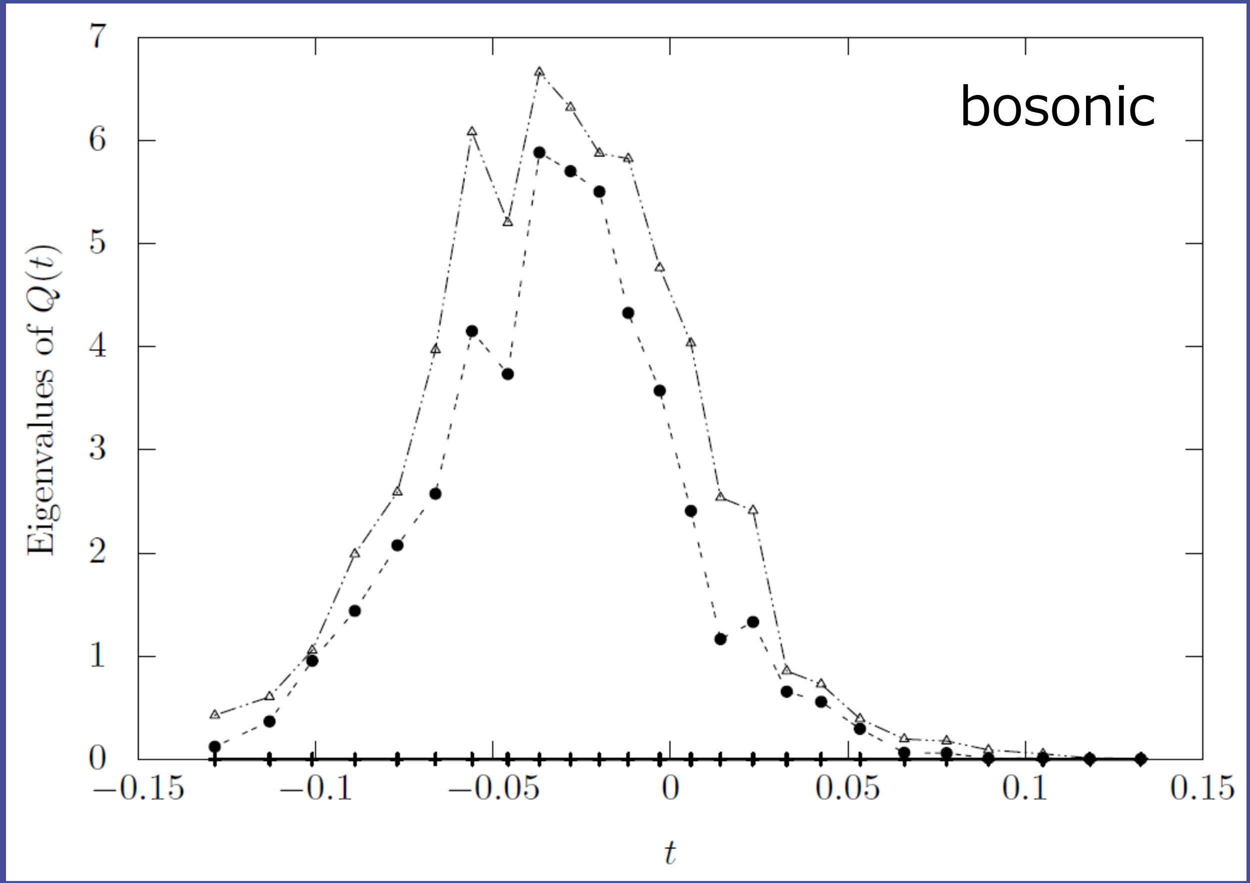


Only 3 eigenvalues distribute, which implies SSB of SO(5) to SO(3).

Eigenvalues of $Q(t) = \sum_{i=1}^{5} \bar{A}_i(t)\bar{A}_i(t)$

This quantity represents how the space extends in the radial direction.





Only 2 eigenvalues grow.

At $s = -1$, configurations which minimize the action are dominating.
The following matrices maximize the noncommutativity between the spatial matrices under the constraints (IR cutoffs) in the classical analysis:

$$\bar{A}_i = C\sigma_i \oplus 1_{n-2} \ (i = 1, 2, 3), \quad 0_n \ (i \geq 4) \quad \Rightarrow \quad Q = (C^2 1_2) \oplus 0_{n-2}$$

Pauli matrices

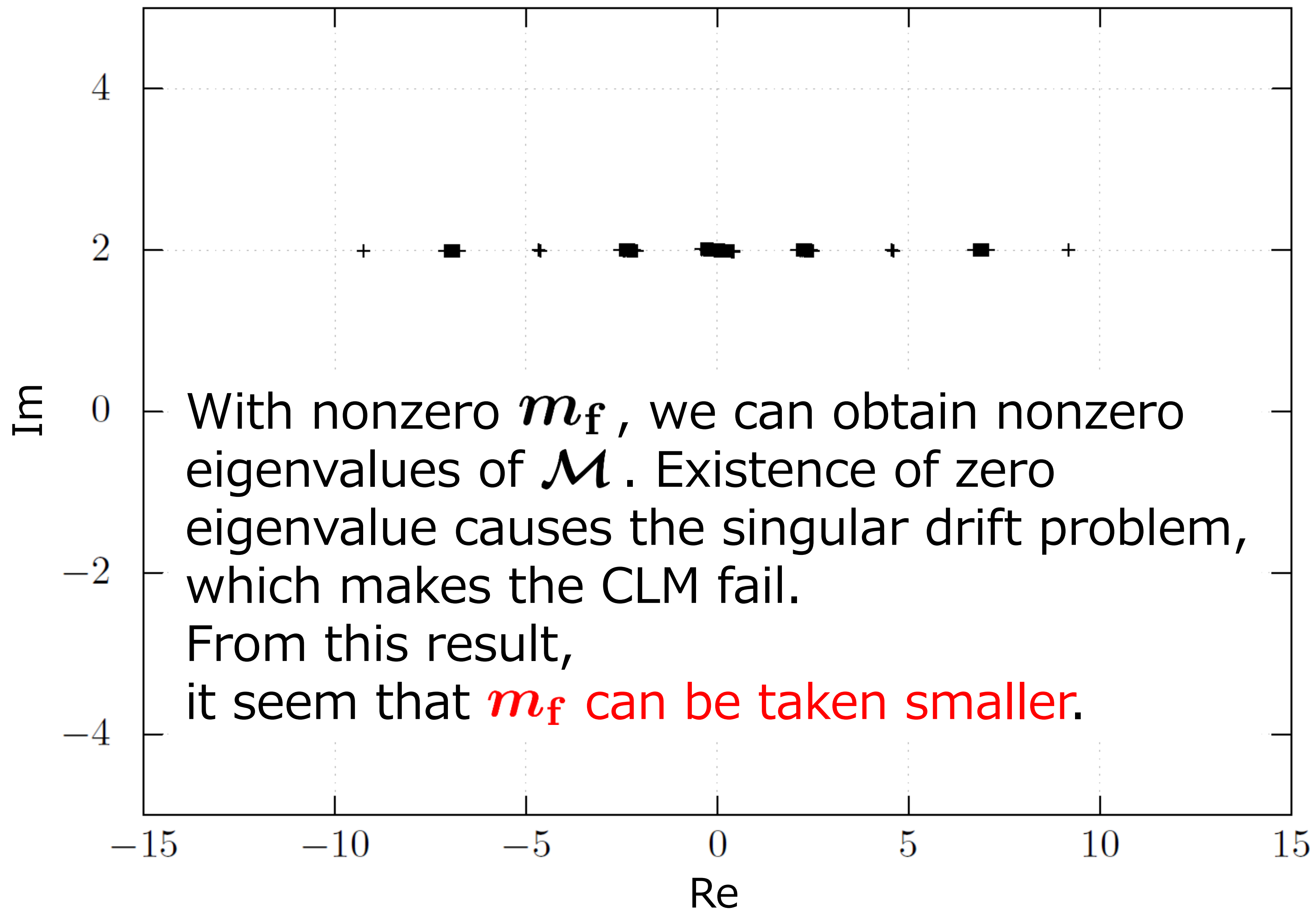
Fuzzy sphere-like structure

$$S = -\frac{1}{g^2} \mathrm{Tr} \left(\frac{1}{4} [A^\mu, A^\nu][A_\mu, A_\nu] + \frac{1}{2} \bar{\Psi} \Gamma^\mu [A_\mu, \Psi] \right)$$

In the $s \to 0$ regime,
we expect that eigenvalues of $Q(t)$ have smooth distribution.

Cf) [Hatakeyama-Matsumoto-Nishimura-Tsuchiya-Yosprakob ('19)]

Eigenvalues of $\mathcal{M}$

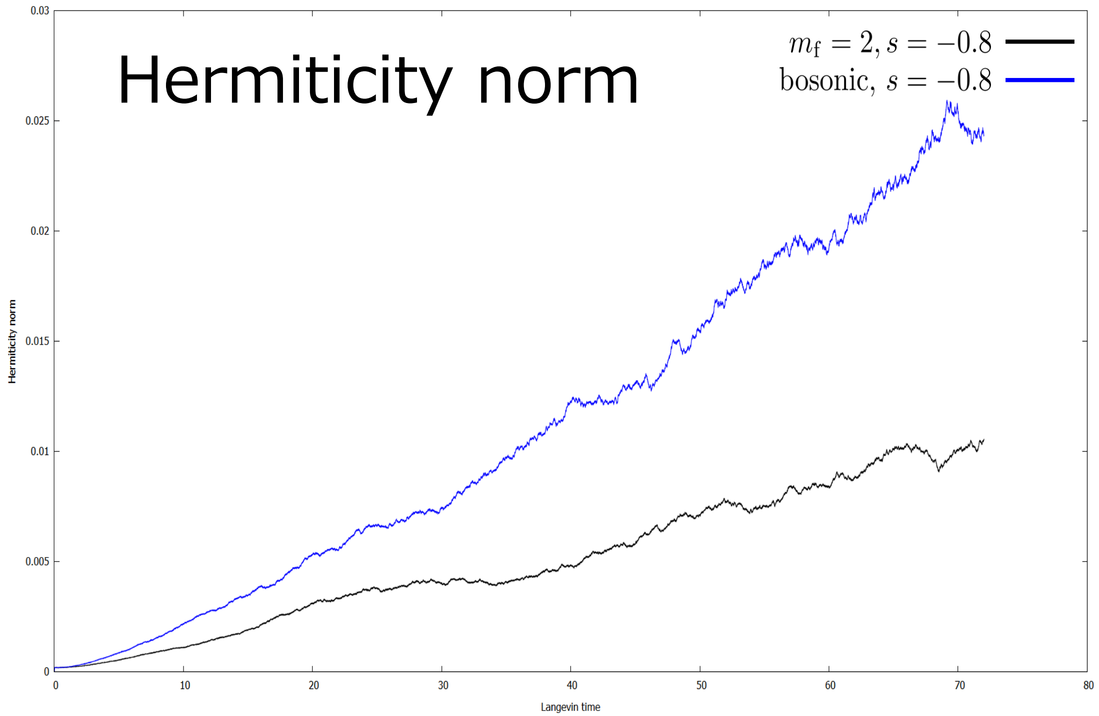


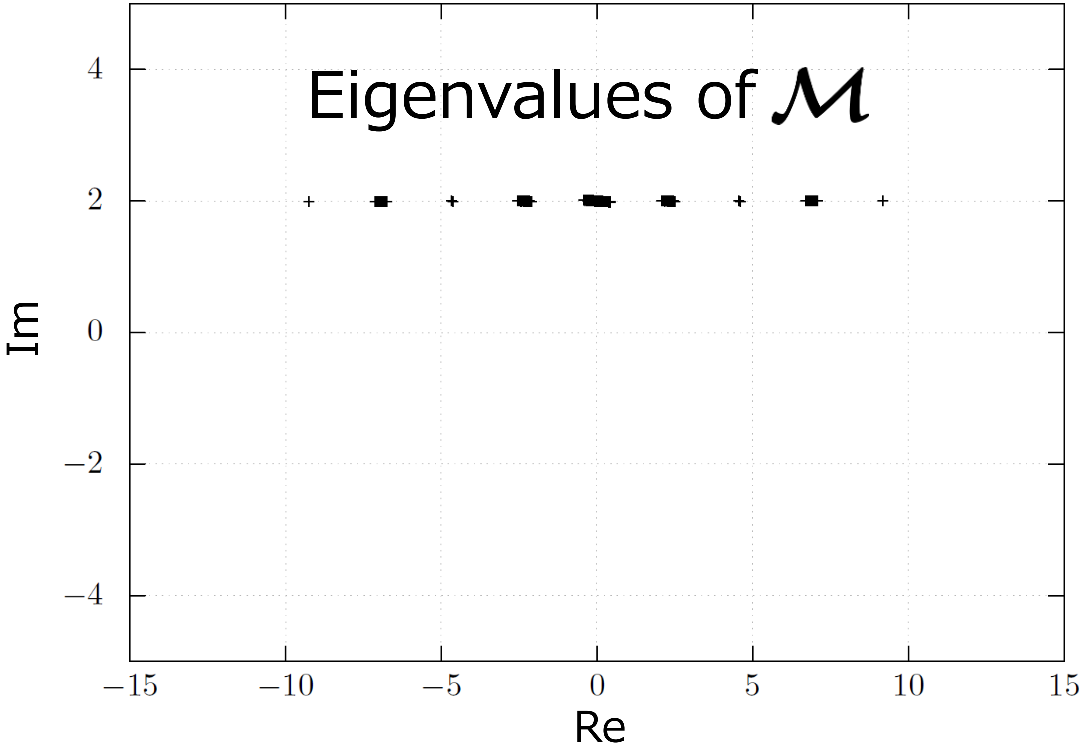
With nonzero $m_{\mathbf{f}}$, we can obtain nonzero eigenvalues of $\mathcal{M}$. Existence of zero eigenvalue causes the singular drift problem, which makes the CLM fail.

From this result,

it seem that $m_{\mathbf{f}}$ can be taken smaller.

5. Conclusion





From these results, it seems that complex Langevin simulations with fermions are more stable than those of the bosonic case.
At Hirasawa-san's talk, the effect of the matrix size will be discussed.

Back up

$$Z = \int dA e^{-\tilde{S}_{\rm b}(A)} \delta\left(\frac{1}{N} \mathrm{Tr}\,(A_0)^2 - \kappa \right) \delta\left(\frac{1}{N} \mathrm{Tr}\,(A_i)^2 - 1 \right)$$

↓ Introduce auxiliary field

Constraints on matrices (IR cutoff)

$$= \int_0^\infty du dv \int dA u^p v^q e^{-N^2(u+v)/2} e^{-\tilde{S}_{\rm b}(A)} \delta\left(\frac{1}{N} \mathrm{Tr}\, A_0^2 - \kappa \right) \delta\left(\frac{1}{N} \mathrm{Tr}\, A_i^2 - 1 \right)$$

↓ Change variables: $A_0 \to A_0 \sqrt{\kappa/u},\ A_i \to A_i/\sqrt{v}$

$$Z = \int dA e^{-S_{\rm eff}}$$

$$S_{\rm eff} = -iN\beta e^{is\pi/2} \left[-\frac{1}{2} e^{-ik\pi} \frac{\mathrm{Tr}(F_{0i})^2}{\frac{1}{N}\mathrm{Tr}\,(A_0)^2 \frac{1}{N}\mathrm{Tr}\,(A_i)^2} + \frac{1}{4} \frac{\mathrm{Tr}(F_{ij})^2}{\left(\frac{1}{N}\mathrm{Tr}\,(A_i)^2\right)^2} \right] + \frac{N}{2}\left(\mathrm{Tr}\,(A_0)^2 + \mathrm{Tr}\,(A_i)^2 \right)$$