# *Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field*

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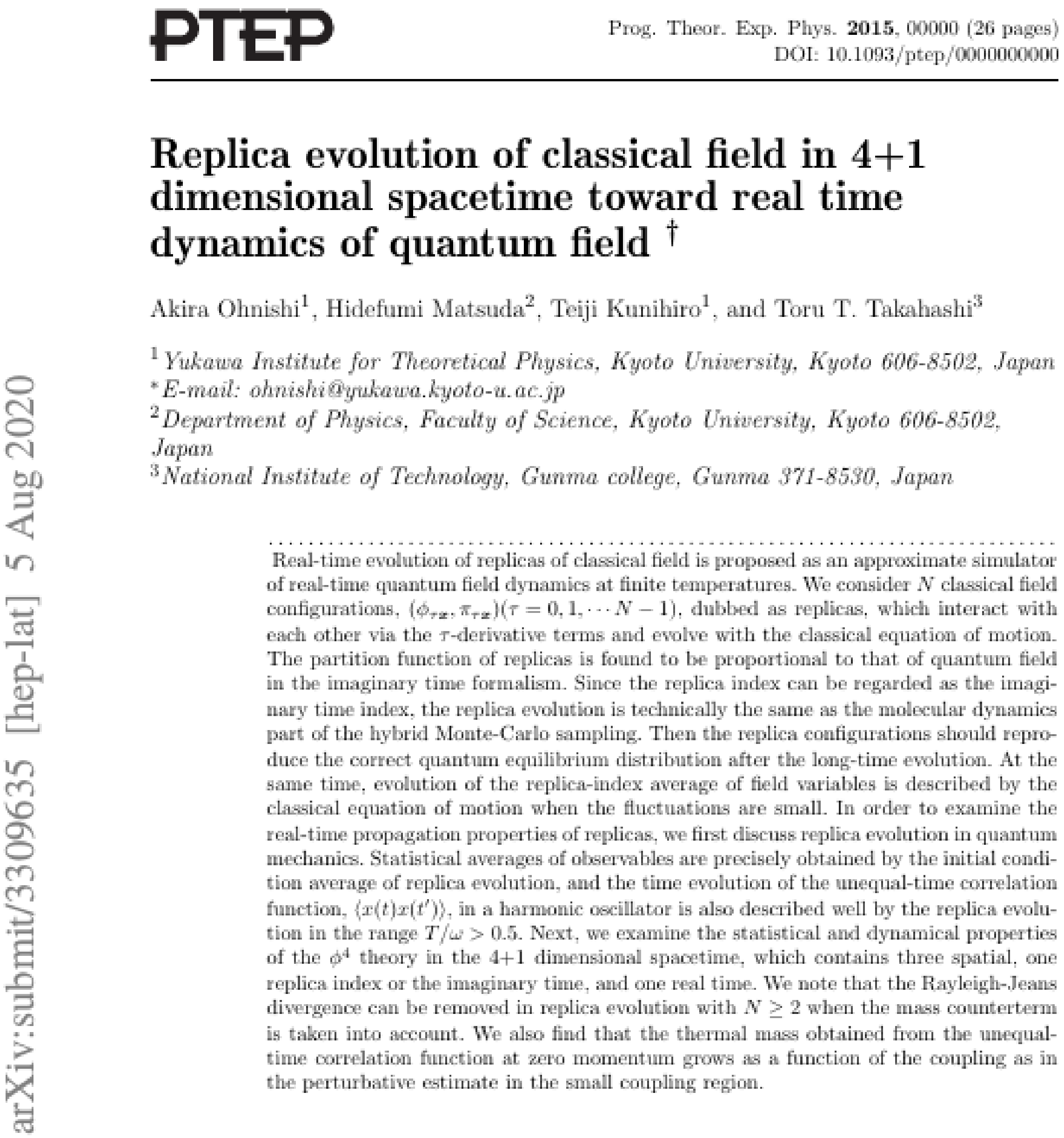
# *arXiv paper is still "on hold". Please be patient.*



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## Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field †

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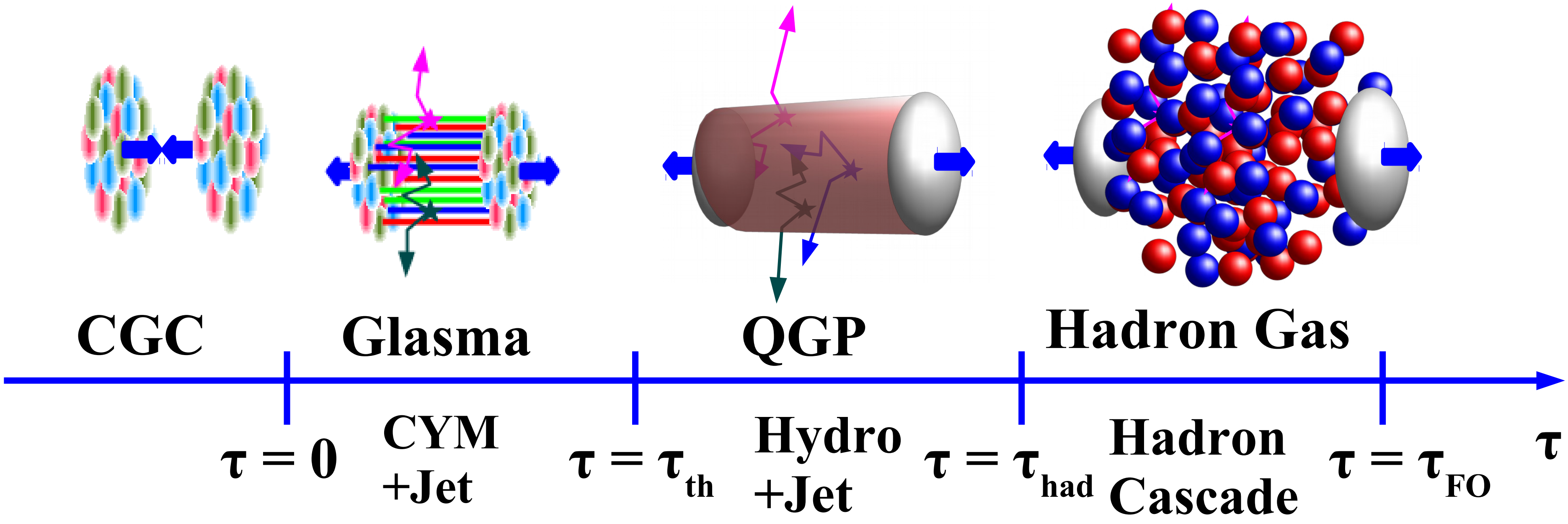
Real-time evolution of replicas of classical field is proposed as an approximate simulator of real-time quantum field dynamics at finite temperatures. We consider $N$ classical field configurations, $(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})(\tau = 0, 1, \cdots N-1)$, dubbed as replicas, which interact with each other via the $\tau$-derivative terms and evolve with the classical equation of motion. The partition function of replicas is found to be proportional to that of quantum field in the imaginary time formalism. Since the replica index can be regarded as the imaginary time index, the replica evolution is technically the same as the molecular dynamics part of the hybrid Monte-Carlo sampling. Then the replica configurations should reproduce the correct quantum equilibrium distribution after the long-time evolution. At the same time, evolution of the replica-index average of field variables is described by the classical equation of motion when the fluctuations are small. In order to examine the real-time propagation properties of replicas, we first discuss replica evolution in quantum mechanics. Statistical averages of observables are precisely obtained by the initial condition average of replica evolution, and the time evolution of the unequal-time correlation function, $\langle x(t)x(t')\rangle$, in a harmonic oscillator is also described well by the replica evolution in the range $T/\omega > 0.5$. Next, we examine the statistical and dynamical properties of the $\phi^4$ theory in the 4+1 dimensional spacetime, which contains three spatial, one replica index or the imaginary time, and one real time. We note that the Rayleigh-Jeans divergence can be removed in replica evolution with $N \geq 2$ when the mass counterterm is taken into account. We also find that the thermal mass obtained from the unequal-time correlation function at zero momentum grows as a function of the coupling as in the perturbative estimate in the small coupling region.

**http://www2.yukawa.kyoto-u.ac.jp/~akira.ohnishi/Src/Org/Replica-arXiv.pdf**

# Contents

# *Heavy-Ion Collisions on the Lattice ?*

- **Initial Condition: Color Glass Condensate (CGC)**
- **Early Stage: Glasma**
- **Main Stage: Quark Gluon Plasma (QGP)**
- **Final Stage: Hadron Gas**



**CGC** **Glasma** **QGP** **Hadron Gas**

$\tau = 0$ **CYM +Jet** $\tau = \tau_{th}$ **Hydro +Jet** $\tau = \tau_{had}$ **Hadron Cascade** $\tau = \tau_{FO}$ $\tau$

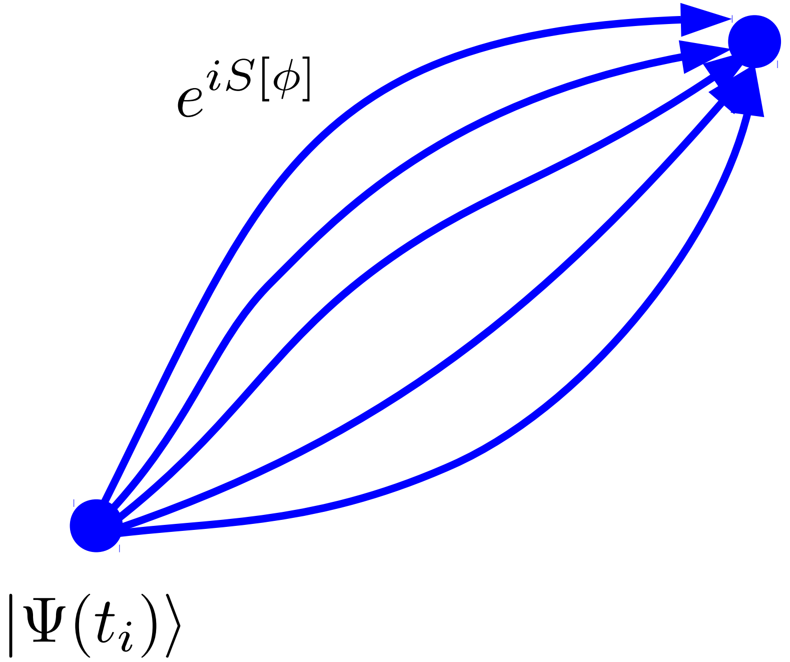
***Unreachable Dream ?***

# Classical Field Evolution

- **Path integral of real time evolution**

$$S_{fi} = \mathcal{N} \int \mathcal{D}\phi \, \langle \Psi(t_f) | \exp\left(iS[\phi]\right) | \Psi(t_i) \rangle$$

  - **Cancellation of amplitudes by exp(iS)**
    **→ Severe sign problem**



- **Classical Field Evolution**
  - **Euler-Lagrange equation choose path with δS=0 → No sign problem**
  - **Useful in discussing far-from-equilibrium phenomena**
    ***condensate evolution (Time dep. Gross-Pitaevski), nuclear excitation (TD Hartree-Fock), Inflation, high-energy heavy-ion collisions (classical Yang-Mills), …***
  - **But converges to incorrect equilibrium**

$$n_{\boldsymbol{k}} = T/\omega_{\boldsymbol{k}} (\text{Classical}), \quad n_{\boldsymbol{k}} = [\exp(\omega_{\boldsymbol{k}}) \mp 1]^{-1} \, (\text{Quantum})$$

***Is ther framework for far-from-equilibrium and around equilibrium ?***

# Replica evolution

# *Replica Partition Function (Quantum Mechanics)*

- **Part. func. in Classical Mechanics**

$$\mathcal{Z}_C(T) = \int \frac{dx\,dp}{2\pi} \exp\left[-\frac{H(x,p)}{T}\right] \qquad H = \frac{p^2}{2} + U(x)$$

- **Part. func. in Quantum Mechanics (imag. time formalism)**

$$\mathcal{Z}_Q(T) = \int \mathcal{D}x \exp\left(-S[x]\right) \qquad S = \frac{1}{\xi}\left[\mathcal{V} + \sum_{\tau=1}^{N} U(x_\tau)\right]$$

$$\xi = a/a_\tau,\, a^3 a_\tau = a^4/\xi,\, T = \xi/N,\, \mathcal{V} = \sum_{\tau=1}^{N} \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \simeq \xi \int_0^{1/T} d\bar{\tau} \frac{1}{2}\left[\frac{\partial x}{\partial \bar{\tau}}\right]^2$$

- **Part. func. in Replicas**
  **(N classical systems interacting with τ-derivative terms $\mathcal{V}$)**

$$\mathcal{Z}_R(\xi) = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \exp\left(-\frac{\mathcal{H}[x,p]}{\xi}\right) \qquad \mathcal{H} = \sum_{\tau=1}^{N} \underbrace{\left[\frac{p_\tau^2}{2} + U(x_\tau)\right]}_{H(x_\tau,\,p_\tau)} + \mathcal{V} \qquad \xi S[\phi]$$

$$= (2\pi\xi)^{NL^3/2} Z_Q(T)$$

***Part. fn. of N classical systems interacting via V at temp. ξ***
***∝ Part. fn. of quantum mech. at temp. T=ξ/N (MD in HMC)***

# *Evolution of Replica-Index Average*

- **Canonical equation of Motion for replica variables ($x_\tau$, $p_\tau$)**

$$\frac{dx_\tau}{dt} = \frac{\partial \mathcal{H}}{\partial p_\tau} = p_\tau$$

$$\frac{dp_\tau}{dt} = -\frac{\partial \mathcal{H}}{\partial x_\tau} = -\frac{\partial U(x_\tau)}{\partial x_\tau} + \underbrace{\xi^2(x_{\tau+1} + x_{\tau-1} - 2x_\tau)}_{\tau\text{-derivative terms}}$$

- **Replica index average**

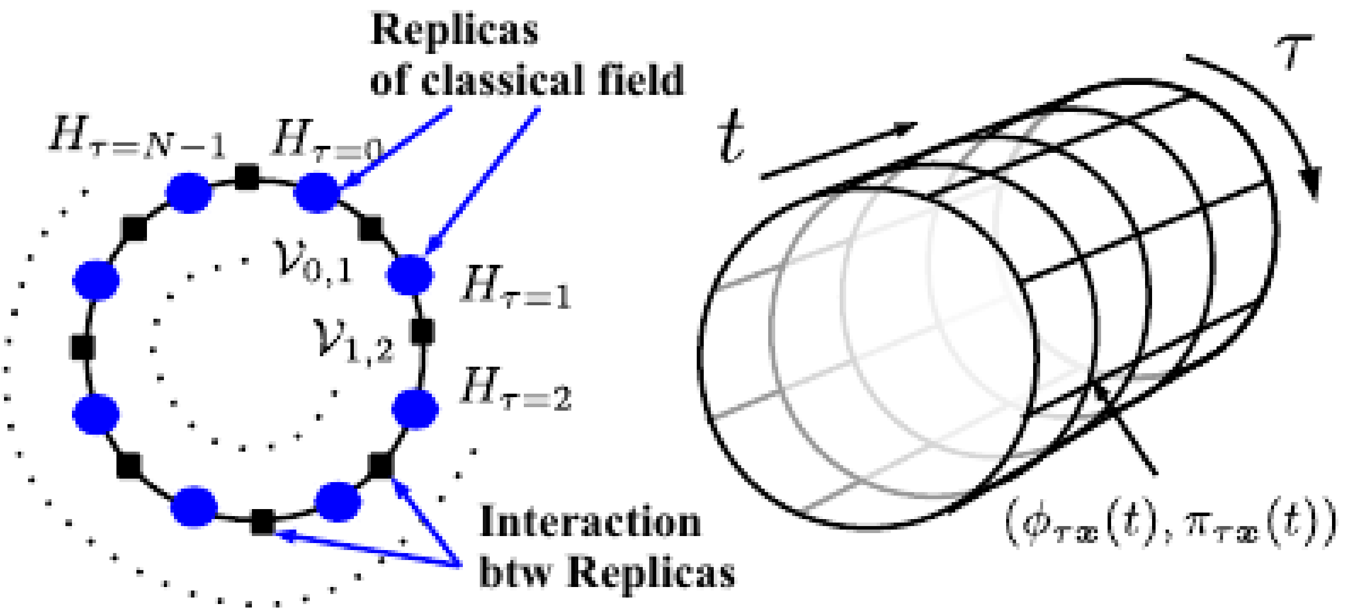
$$\frac{d\widetilde{x}}{dt} = \frac{1}{N}\sum_\tau \frac{dx_\tau}{dt} = \widetilde{p}$$

$$\frac{d\widetilde{p}}{dt} = \frac{1}{N}\sum_\tau \frac{dp_\tau}{dt} = -\frac{1}{N}\sum_\tau \frac{\partial U(x_\tau)}{\partial x_\tau} + 0 \quad \text{(Ehrenfest's theorem)}$$

$$= -\frac{\partial U(\widetilde{x})}{\partial \widetilde{x}} + \mathcal{O}((\delta x)^2)$$

***τ-averaged variables obey classical EOM approximately when fluc. among replicas are small.***

# Replica Evolution



$$\mathcal{H} = \sum_{\tau} H_\tau + \sum_{\tau} \mathcal{V}_{\tau,\tau+1} = \frac{1}{2}\sum_{\tau,\boldsymbol{x}} \pi^2_{\tau,\boldsymbol{x}} + \xi S[\phi]$$

$$\mathcal{Z}_R = \int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp(-\mathcal{H}/\xi) \propto \int \mathcal{D}\phi\,\exp(-S[\phi])$$

***Replica Evolution***
***= Classical Dynamics***
***with Quantum Statistics in Equilibrium***

# Replica Evolution

- Replicas = N classical systems interacting with τ-derivative terms (V)

$$\mathcal{V} = \sum_{\tau=1}^{N} \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \simeq \xi \int_0^{1/T} d\bar{\tau} \frac{1}{2} \left[\frac{\partial x}{\partial \bar{\tau}}\right]^2$$

  - Replica variables ($x_\tau$, $p_\tau$) are assumed to evolve with canonical EOM
  - Long (real) time evolution in 4+1D spacetime (space, τ, t) samples correct quantum statistical configurations of $x_\tau$.

    ~ MD part of HMC
  - Replica index (τ) average of ($x_\tau$, $p_\tau$) obeys the classical EOM.
- Replica evolution of field
  - Replace variables $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - Mass renormalization & Subtracting zero point contribution
- How about dynamical properties of replica evolution ?

*I will skip this page...*

# *Replica Evolution of a Single Harmonic Oscillator*

# *Replica Evolution of a Harmonic Oscillator*

- **Replica Hamiltonian = N free HO Hamiltonian**

$$\mathcal{H} = \sum_\tau \left[\frac{p_\tau^2}{2} + \frac{\omega^2 x_\tau^2}{2} + \underbrace{\frac{\xi^2}{2}(x_{\tau+1}-x_\tau)^2}_{\mathcal{V}}\right] = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2}\right]$$

$$M_n^2 = \omega^2 + 4\xi^2 \sin^2(\pi n/N)$$

**Fourier transf.**

- **Expectation value of x² in Replica**

$$\langle x^2\rangle = \frac{1}{N}\sum_\tau \langle x_\tau^2\rangle = \frac{1}{N}\sum_n \langle \bar{x}_n^2\rangle \overset{\exp(-\mathcal{H}/\xi)}{=} \frac{1}{N}\sum_n \frac{\xi}{M_n^2} \overset{\textbf{Matsubara freq. sum}}{=} \frac{\coth(\Omega/2T)}{2\omega\sqrt{1+\omega^2/4\xi^2}}$$

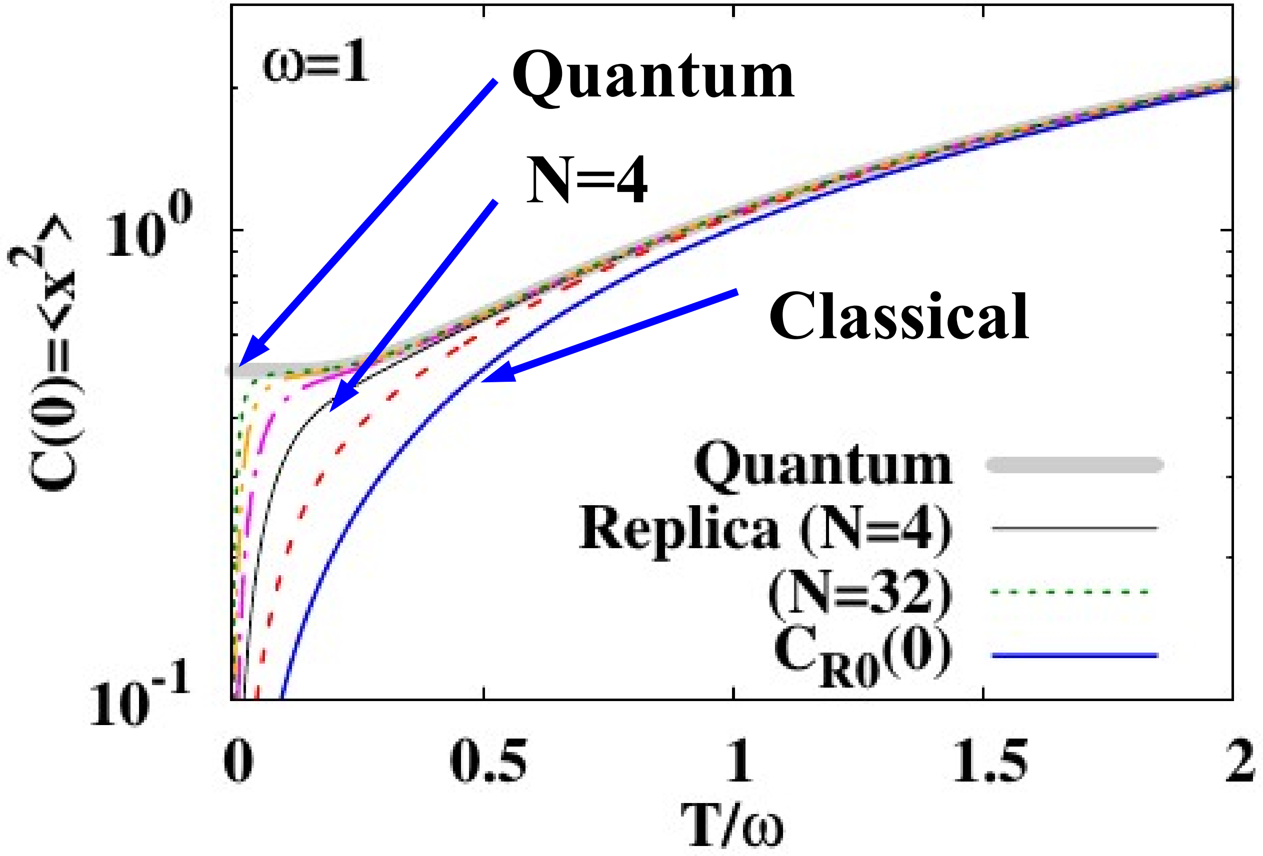
$$\Omega = 2\xi \operatorname{arcsinh}(\omega/2\xi)$$

$$\frac{T}{\omega^2}(N=1, \text{Classical})$$

$$\to \frac{\coth(\omega/2T)}{2\omega} = \frac{1}{\omega}\left[\overset{\textbf{zero point}}{\frac{1}{2}} + \overset{\textbf{thermal}}{\frac{1}{e^{\omega/T}-1}}\right](N \to \infty, \text{Quantum})$$

***Equal time observables of x are reproduced at $N \to \infty$***

# *Expectation value of $x^2$*



$$\frac{T}{\omega^2}(N=1,\mathrm{Classical}) \to \frac{\coth(\omega/2T)}{2\omega}(N\to\infty,\mathrm{Quantum})$$

***Equal time observables of x are reproduced at $N \to \infty$***

# *Time-Correlation Function*

- **Time-correlation function (Unequal-time two-point function)**

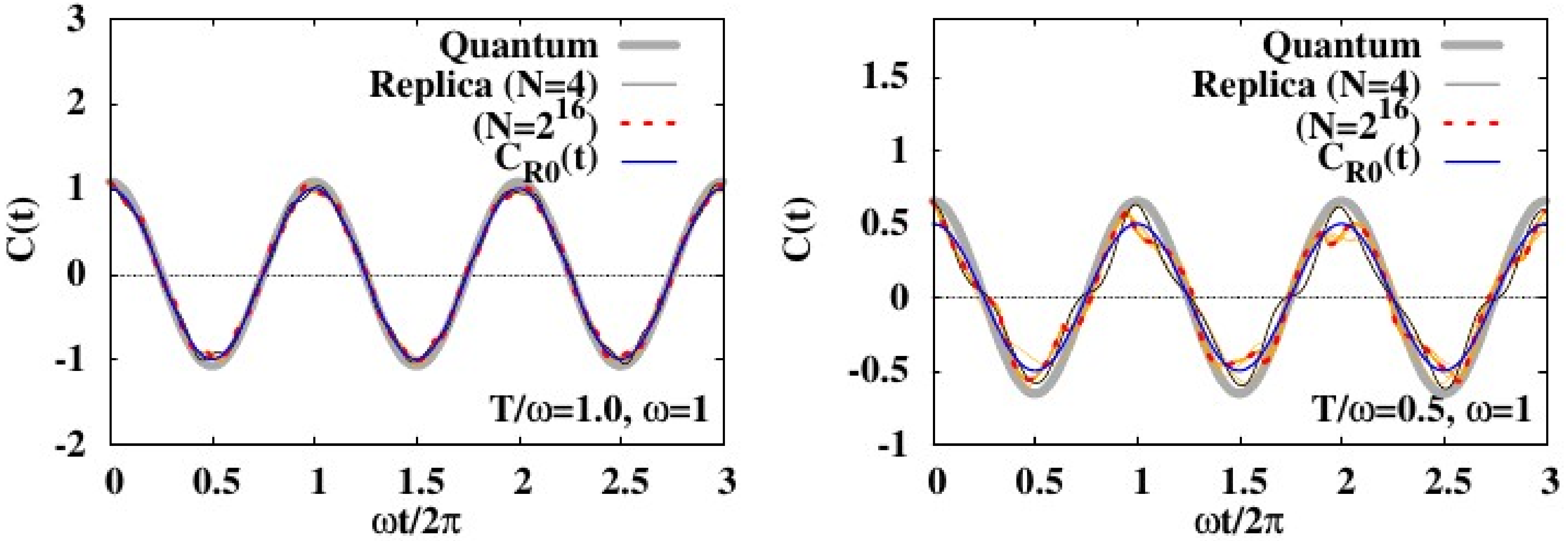
$$C(t) = \left\langle \frac{1}{2}\left[x(t)x(0) + x(0)x(t)\right] \right\rangle$$

**Quantum**

$$C_Q(t) = \frac{\coth(\omega/2T)}{2\omega}\cos\omega t$$

**Replica**

$$C_R(t) = \sum_n \frac{T}{M_n^2}\cos M_n t$$



***Not perfect, but $C_R(t)$ roughly explains $C_Q(t)$ at $T/\omega > 0.5$***

# Replica Evolution in Scalar Field Theory

# *Replica Evolution in Scalar Field Theory*

- **Replica evolution in field theory**
  - **Replace variables** $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - **Mass renormalization & Subtracting zero point contribution**
- **Example: $\phi^4$ theory**

$$\mathcal{H} = \sum_{\tau,\boldsymbol{x}} \left[ \frac{\pi^2_{\tau\boldsymbol{x}}}{2} + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2 + \frac{m^2}{2}\phi^2_{\tau\boldsymbol{x}} + \frac{\lambda}{24}\phi^4_{\tau\boldsymbol{x}} + \overset{\mathcal{V}}{\boxed{\frac{\xi^2}{2}(\phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}})^2}} \right]$$

$H(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$

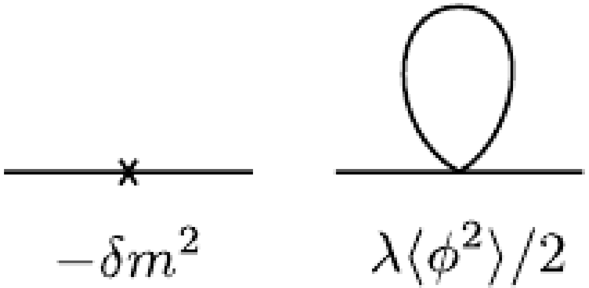
$\xi S[\phi]$

$$-\frac{\delta m^2}{2}\phi^2_{\tau\boldsymbol{x}}$$

- **Counterterm (one loop)**
  *Aarts, Smit ('97), Kapusta, Gale (textbook)*

$$\delta m^2 = \frac{\lambda}{2}\langle\phi^2\rangle_{\rm div}$$

$$\langle\phi^2\rangle_{\rm div} = \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}$$



# Momentum Distribution

- **Momentum distribution in replica**

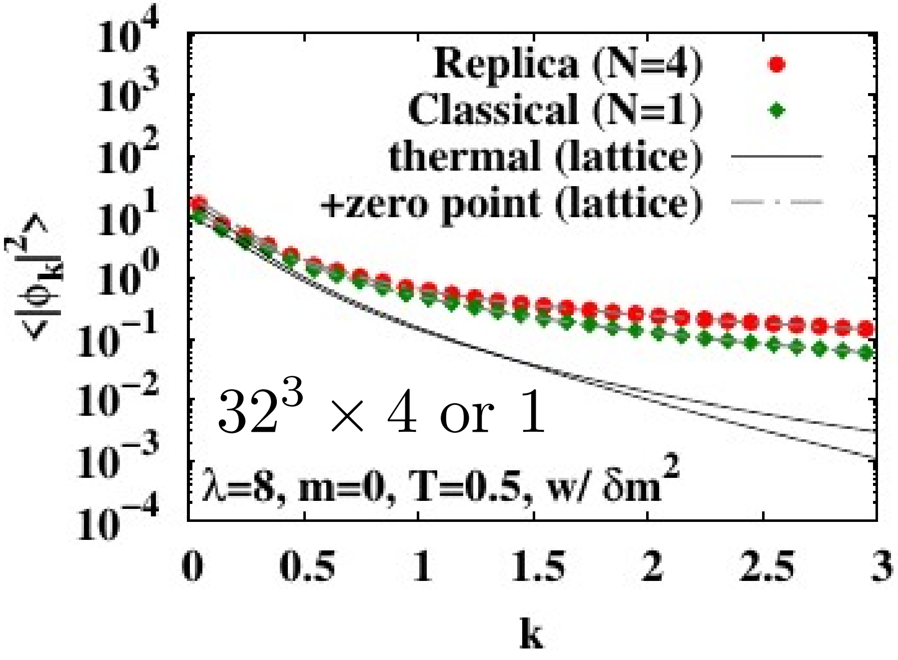
$$\langle|\phi_{\boldsymbol{k}}|^2\rangle = \frac{1}{N}\sum_n \langle \phi_{n\boldsymbol{k}}\phi^*_{n\boldsymbol{k}}\rangle \overset{\uparrow}{=} \frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\underline{\frac{1}{2}} + \underline{\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}}\right]$$

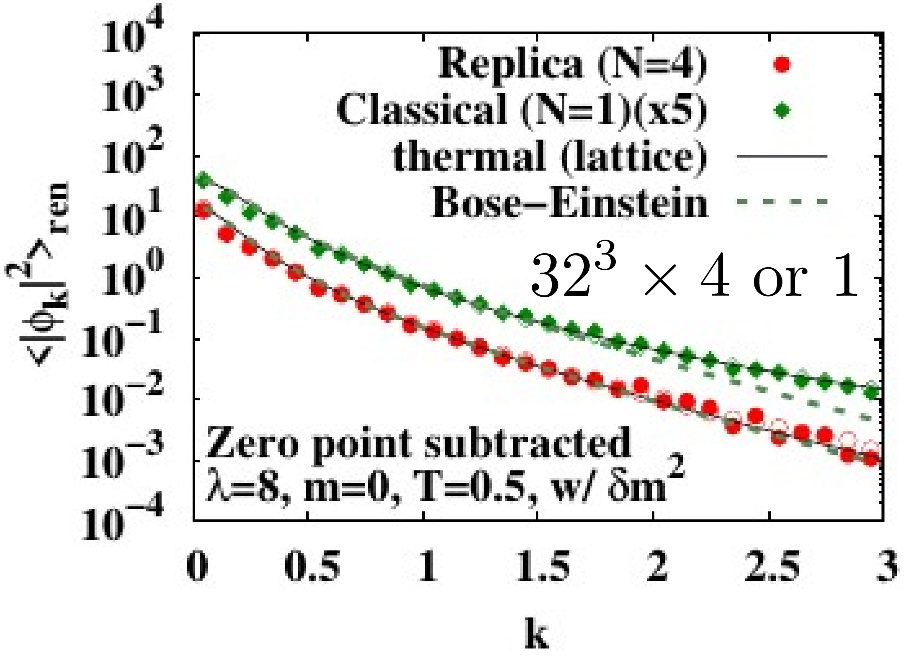
**Free field, Matsubara sum**

**Zero point**

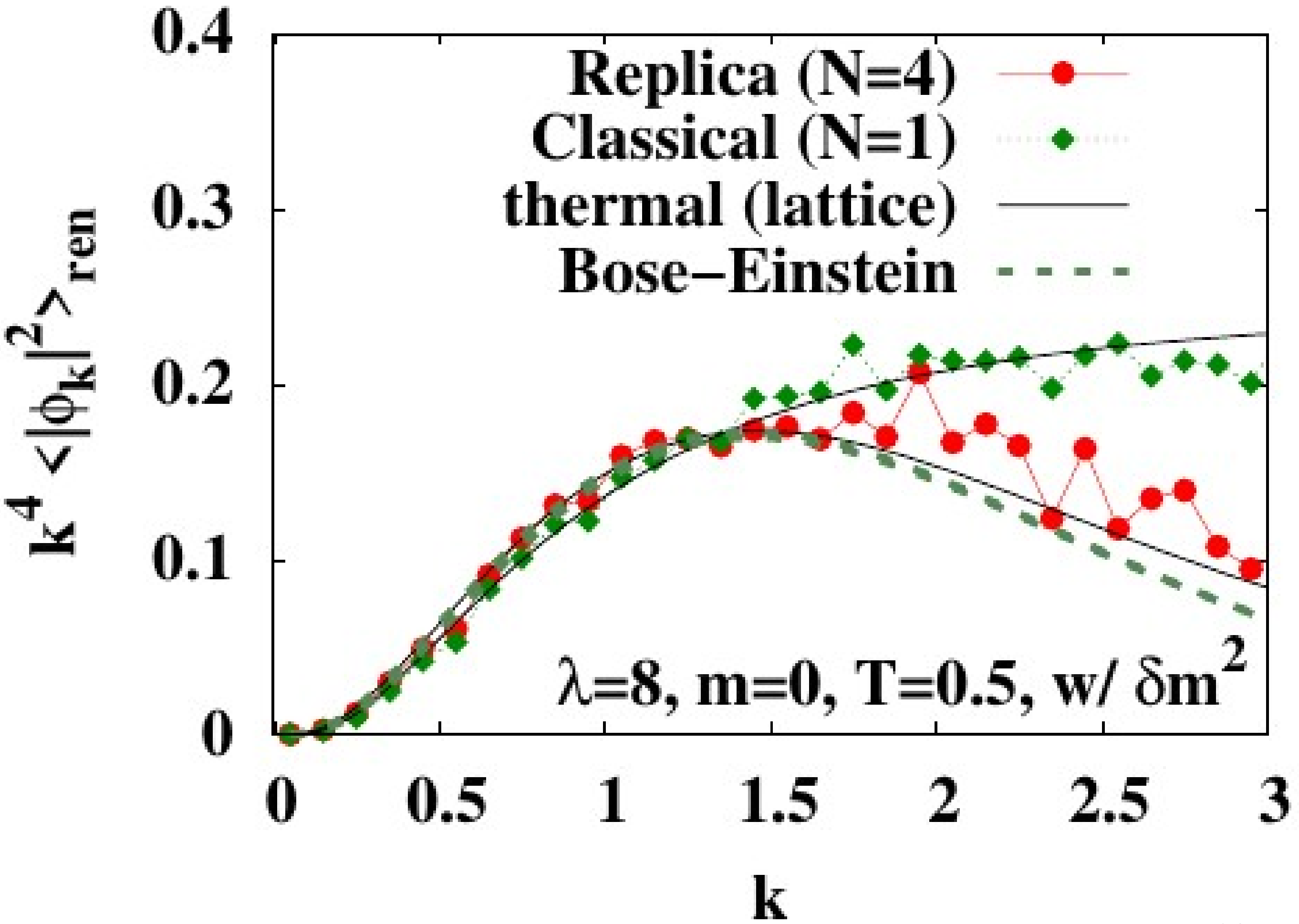
**Thermal**
**→ Bose-Einstein**

  - **By subtracting the zero point part, we can avoid equipartition & Rayleigh-Jeans divergence.**





# *Momentum Distribution*



$$\frac{1}{N}\sum_{\tau\boldsymbol{x}}\frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2=\frac{1}{N}\sum_{n\boldsymbol{k}}\frac{1}{2}\boldsymbol{k}^2|\phi_{n\boldsymbol{k}}|^2\to L^3\int\frac{d\boldsymbol{k}}{(2\pi)^3}\boldsymbol{k}^2|\phi_{n\boldsymbol{k}}|^2$$

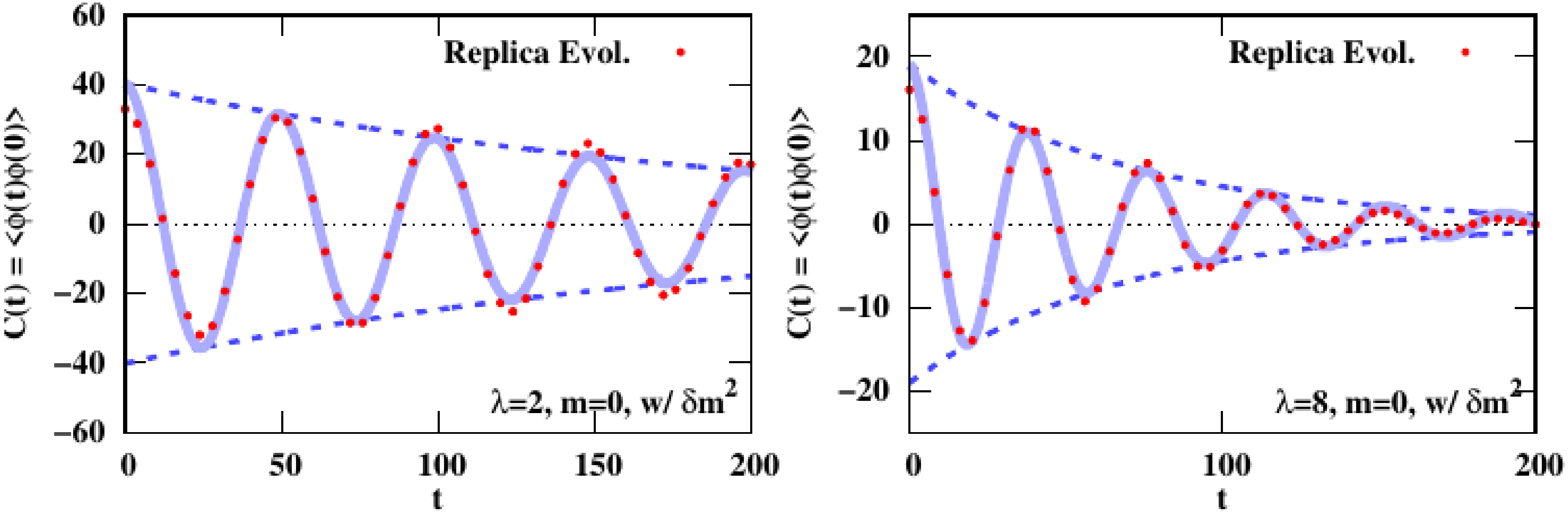
***By subtracting the zero point part,***
***we can avoid the Rayleigh-Jeans divergence of energy.***

# *Time-Correlation Function*

- **Time-correlation function (unequal-time two-point function at zero momentum)**

$$C(t)=\frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}}\langle\phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle \xrightarrow[\text{free}]{} \sum_n \frac{T}{M_n^2}\cos M_n t$$

- **With interaction (non-zero λ), C(t) shows damped oscillatory behavior. → Thermal mass & damping rate**

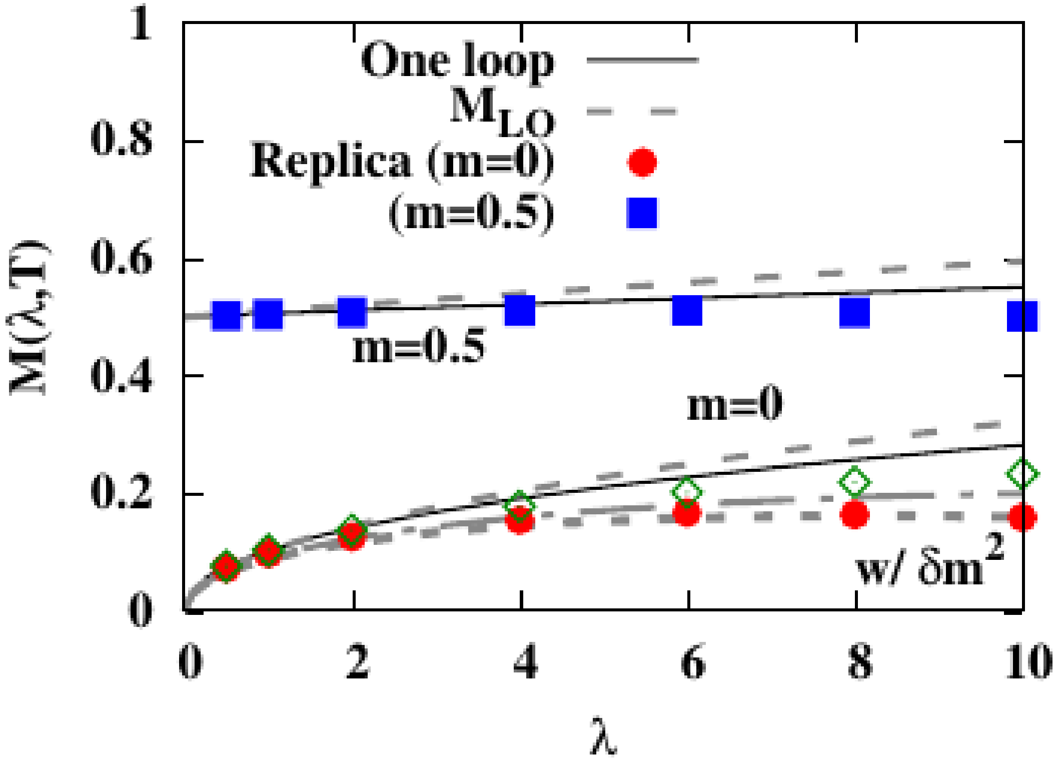


# *Thermal Mass*

- **Thermal Mass**
  - **Leading Order (one-loop)** $M_{\rm LO}^2 = m^2 + \lambda T^2/24.$
  - **Resummed One-Loop** $M_{\rm resum}^2 = \frac{\lambda T^2}{24}\left[1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}}\right]$
  - **Two-Loop**

$$M_{2-\rm loop}^2 = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\},$$

*Kapusta, Gale (textbook)*
*Parwani ('92, '93)*



***Thermal mass in Replica Evolution ~ Two-loop results***

# *Summary*

- **Replica evolution is proposed as a quantum-statistics-improved classical field framework.**
  - **N classical field configurations evolves with the τ-derivative terms.**
  - **4+1D classical evolution → quantum stat. ensemble (HMC).**
  - **Replica-index (=imag. time) average provides classical field.**
  - **Subtracting zero point part from $< \phi^2 >$**
    **→ mass renormalization and removing Rayleigh-Jeans divergence**
  - **Thermal mass ~ 2-loop perturbation ressults.**
- **To be investigated**
  - **Comparison with previously proposed frameworks.**
    ***Hard mode effects [Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97)], Field-particle sim. [Dumitru, Nara ('05), Dumitru, Nara, Strickland ('07)], 2PI [Aarts, Berges ('02), Hatta, Nishiyama ('12)], …***
  - **Formal discussions, e.g. relation to Boltzmann Eq., *A.Muller, Son ('04).***
  - **Shear viscosity *[Matsuda, 6-1C]*, Thermalization, ...**

## *Thank you for your attention !*

**AO** **Hidefumi Matsuda**

**Toru T. Takahashi**

**Teiji Kunihiro**

**http://www2.yukawa.kyoto-u.ac.jp/~akira.ohnishi/Src/Org/Replica-arXiv.pdf**

# *Lattice Setup*

- **Lattice size: $32^3$ x 4 (L=32, N=4)**
- **Temperature: T=0.5, Coupling: λ=0.5-10, bare mass:m=0, 0.5**
- **Average over replica index (τ) and replica ensemble ($N_{conf}$=1000)**
- **Thermal ensemble is prepared by solving the Langevin equation at temperature ξ=NT=2.**
- **EOM is solved in the leap-frog method (reversible !) with the time step of Δt=0.025 until t=500 after equilibration.**
- **A few hours for each (λ, m) on iCore7 PC (w/o MP).**