APLAT 2020

Finite-volume effects in $(g-2)_\mu^{\textbf{HVP,LO}}$

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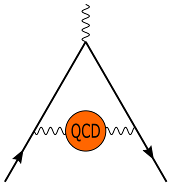
in collaboration with **Maxwell T. Hansen**

MH and AP, Phys. Rev. Lett. 123 (2019) 172001, arXiv:1904.10010 [hep-lat].
MH and AP, accepted on JHEP, arXiv:2004.03935 [hep-lat].

Introduction

$$a^{\mathrm{HVP}}(T,L) = -\frac{1}{3}\int_0^{T/2} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_k(x_0,\mathbf{x}) j_k(0)\rangle_{T,L}$$



D. Bernecker, H. B. Meyer, *Vector Correlators in Lattice QCD: methods and applications*, Eur. Phys. J. A **47** (2011), 148.

Setup:

- Euclidean $T \times L^3$ torus with (anti)periodic boundary conditions.
- QCD in isosymmetric limit.
- Continuum limit has already been taken.

Goal: Understand the finite-volume corrections

$$\Delta a^{\mathrm{HVP}}(T,L) = a^{\mathrm{HVP}}(T,L) - a^{\mathrm{HVP}}(\infty)\ ,$$

possibly in a model-independent way.

Structure of finite-volume corrections

$$\Delta a^{\mathrm{HVP}}(T,L) = \mathcal{O}\left(e^{-m_\pi L}\right) + \mathcal{O}\left(e^{-\sqrt{2}m_\pi L}\right) + \mathcal{O}\left(e^{-\sqrt{3}m_\pi L}\right) + \mathcal{O}\left(e^{-\sqrt{2+\sqrt{3}}m_\pi L}\right)$$

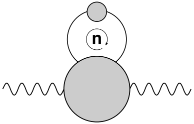
$$+ \mathcal{O}\left(e^{-m_\pi T}\right) + \mathcal{O}\left(e^{-\frac{3}{2}m_\pi T}\right) + \mathcal{O}\left(e^{-m_\pi\sqrt{T^2+L^2}}\right) + \mathcal{O}\left(e^{-m_K L}\right) + \mathcal{O}\left(e^{-m_K T}\right) + \dots$$

Loops can wrap around the spatial torus (in coordinate space!)

M. Lüscher, *Volume Dependence [...] Stable Particle States*, Comm.Math.Phys.104 (1986) 177.

A complete set of independent loops exist such that:

- one pion loop has wrapping number $\mathbf{n} = (n_x, n_y, n_z) \neq \mathbf{0}$;
- all other loops have zero wrapping number.



$$= O\left(e^{-m_\pi L|\mathbf{n}|}\right)$$

$$\Delta a^{\mathrm{HVP}}(T, L) = \mathcal{O}\left(e^{-m_\pi L}\right) + \mathcal{O}\left(e^{-\sqrt{2} m_\pi L}\right) + \mathcal{O}\left(e^{-\sqrt{3} m_\pi L}\right)$$

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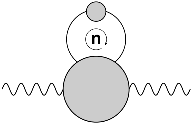
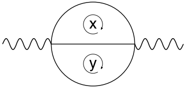


$$= O\left(e^{-m_\pi L|\mathbf{n}|}\right)$$

At least two loops wrap around the spatial torus, e.g.



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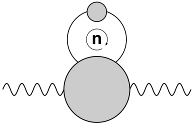

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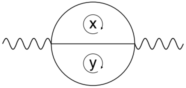

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Leading finite-temperature corrections. For full analysis see *Hansen, AP, arXiv:1904.10010*. Subleading (with respect to first line) if $T = 2L$.

Subleading finite-temperature corrections, and mixed corrections. Subleading (with respect to first line) if $T = 2L$.

Contributions from kaon loops wrapping around the torus. Subleading (with respect to first line) at the physical point $m_K \simeq 3.5 m_\pi$.

Finite-L corrections, $T = \infty$

$$\Delta a^{\mathrm{HVP}}(L) = -\sum_{\mathbf{n}\neq\mathbf{0}} \int \frac{dp}{2\pi} \frac{e^{-|\mathbf{n}|L\sqrt{m_\pi^2+p^2}}}{24\pi|\mathbf{n}|L} \int_0^\infty dx_0\, K(x_0)\times$$
$$\times \int \frac{dk}{2\pi} \cos(x_0 k)\, \mathrm{Re}\, \boxed{T(-k^2, -kp)} + \mathcal{O}(e^{-1.93 m_\pi L})$$

Forward Compton scattering amplitude $\pi\gamma^* \to \pi\gamma^*$

$$\boxed{T(k^2, kp)} = i \lim_{\mathbf{p}'\to\mathbf{p}} \sum_{q=0,\pm1} \int d^4x e^{ikx} \langle \mathbf{p}', q | \mathrm{T} \mathcal{J}_\rho(x) \mathcal{J}^\rho(0) | \mathbf{p}, q \rangle \; .$$

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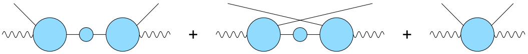
- Reference to the effective field theory has disappeared, everything is now defined in terms of the fundamental theory (QCD).
- We can use this formula for a numerical estimate of finite-L effects!
  Use a model for the Compton scattering amplitude, calculate integrals and sum numerically.

Estimates

Compton scattering amplitude in the space-like region

$$\boxed{T(-\mathbf{k}^2, -\mathbf{kp})} = \frac{2(4m_\pi^2 + \mathbf{k}^2)\,\boxed{F_\pi^2(\mathbf{k}^2)}}{\mathbf{k}^2 + 2\mathbf{pk} - i\epsilon} + \frac{2(4m_\pi^2 + \mathbf{k}^2)\,\boxed{F_\pi^2(\mathbf{k}^2)}}{\mathbf{k}^2 - 2\mathbf{pk} - i\epsilon} + \boxed{T^{\text{reg}}(-\mathbf{k}^2, -\mathbf{kp})}$$



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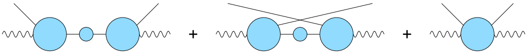

Pole contribution. Space-like electromagnetic form factor of the pion, phenomenological fit
D. Brömmel *et al.* [QCDSF/UKQCD Collaboration], *The Pion form-factor...*, Eur. Phys. J. C **51** (2007) 335.

$$\boxed{F_\pi(\mathbf{k}^2)} = \left(1+\frac{\mathbf{k}^2}{M^2}\right)^{-1}, \qquad M = 727\ \mathrm{MeV}$$

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Compton scattering amplitude in the space-like region

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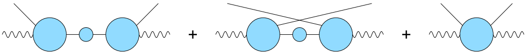

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Regular contribution. NLO chiral perturbation theory

$$T^{\mathrm{reg}}(-\mathbf{k}^2, -\mathbf{kp}) = c_0 + c_1\mathbf{k}^2 + \frac{7m_\pi^2 + 4\mathbf{k}^2}{6\pi^2 f_\pi^2}\sqrt{1 + \frac{4m_\pi^2}{\mathbf{k}^2}}\,\mathrm{cotg}^{-1}\sqrt{1 + \frac{4m_\pi^2}{\mathbf{k}^2}}$$

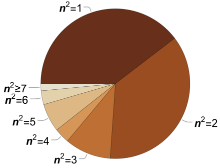
Estimates

$m_\pi = 137$ Mev $\quad m_\mu = 106$ MeV $\quad M = 727$ MeV $\quad a = 700 \times 10^{-10}$

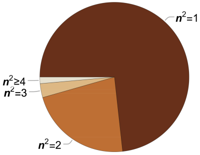
$m_\pi L$	$-10^2 \frac{\Delta a(L)}{a}$	pole (form factor)	regular
4	**3.19**	3.168	0.024
5	**1.43**	1.424	0.006
6	**0.631**	0.6300	0.0015
7	**0.275**	0.2744	0.0004
8	**0.118**	0.1178	0.0001

Contribution of various $e^{-m_\pi L|\mathbf{n}|}$ to the pole term (i.e. form-factor).

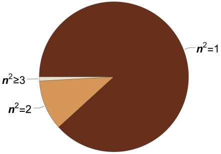
$m_\pi L = 4$



$m_\pi L = 6$



$m_\pi L = 8$



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Contribution of various terms of $F_\pi(\mathbf{k}^2) = 1 - \mathbf{k}^2/M^2 + \Delta F_\pi(\mathbf{k}^2)$.

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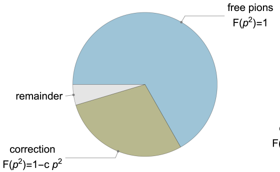

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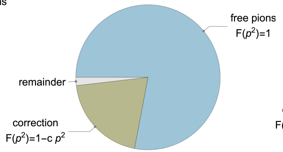

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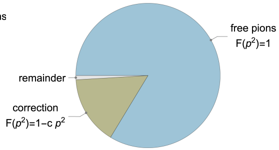

Some comments – Conclusions

- We studied the finite-volume effects of a particular estimator of a^{HVP}, i.e.

  $$a^{\mathrm{HVP}}(T,L) \to \int_0^{T/2} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0)\rangle_{T,L}$$

  Different estimators will have different finite-volume effects.

- Finite-L corrections are exponentially suppressed and the leading exponential accounts only for $\sim 1/3$ of the effect at $m_\pi L = 4$.

- The space-like form factor is the natural quantity that enters in the finite-L effects, not the time-like one (but one can always use dispersion relation).

- The pion-structure contribution amounts to $\sim 1/3$ of the finite-L correction at $m_\pi L = 4$.

- Because of the noise at large distance, one may need to cut the integral over x_0, i.e.

  $$a^{\mathrm{HVP}}(T,L) \to \int_0^{\bar{x}_0} dx_0\, K(x_0) \int_0^L d^3\mathbf{x} \sum_{\mu=1}^{3} \langle j_\mu(x_0,\mathbf{x}) j_\mu(0)\rangle_{T,L}$$

  The contribution to the finite-volume effects of each time-slice is analyzed in `arXiv:2004.03935`.

- Finite-T corrections are analyzed in `arXiv:2004.03935`, and they are negligible in standard setups $T \geq 2L$.