# Relativistic $N$ particle energy shifts in finite volume

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**Nikolas M. Schlage**
in cooperation with
Fernando Romero-López, Akaki Rusetsky, Carsten Urbach

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UNIVERSITÄT BONN

# State of the Art

- **Lattice QCD**
  - ★ Multi hadron states
  - ★ 2 and 3 particle scattering parameters
- $N$ **particles**
  - ★ Non-rel. threshold expansion
    [Beane, Detmold, Savage, 2007]
  - ★ Rel. threshold expansion
    [Romero-López, Rusetsky, Schlage, Urbach, in preparation]

    $\hookrightarrow$ Tests in complex scalar $\varphi^4$-theory

# Motivation – Finite Volume Quantities

## 1) **Lüscher threshold expansion** [Huang, Yang, 1957, Lüscher, 1986]

- Two identical particles
- Relates $\Delta E_2$ due to two particle scatt. in finite volume $L^3$ and $a_0$

$$\Delta E_2 = -\frac{4\pi a_0}{M L^3}\left[1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2}\right]$$

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## 2) Expansion in $L$ to $\mathcal{O}(L^{-6})$

[Beane, Detmold, Savage, 2007, Hansen, Sharpe, 2017, Pang *et al.*, 2019]

**Two-body sector** effective range

**Three-body sector** effective range and three-body contact interaction

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    $\rightarrow$ technical challenge: signal $\leftrightarrow$ thermal pollution
  - ★ **2 and 3 particle scattering parameters** can be identified **consistently**
  - ★ Extracted three body coupling constant is significantly **different from zero**

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    - → Influence of corrections on extracted parameters
  - ★ Results in agreement with predictions from **perturbation theory**?
  - ★ Impact of **exponentially suppressed corrections** on extracted scattering parameters?

# Theoretical Foundations – $N$ Particle Energy Shift

$\Delta E_N(L) \rightarrow S$**-wave scattering parameters**

**2007:**
**Non-relativistic threshold expansion, $N$ particle ground state**

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**2020:**
**Relativistic threshold expansion, $N$ particle ground state**

[Romero-López, Rusetsky, Schlage, Urbach, in preparation]

# Theoretical Foundations – $N$ Particle Energy Shift

## Threshold expansion for $N$ particle ground state

$$\Delta E_N(L) = \binom{N}{2} \frac{4\pi a_0}{M_\varphi L^3} \left[ 1 + c_1 \left(\frac{a_0}{\pi L}\right) + c_2(N) \left(\frac{a_0}{\pi L}\right)^2 + c_3(N) \left(\frac{a_0}{\pi L}\right)^3 + c_4(N) \frac{\pi a_0}{M_\varphi^2 L^3} + c_5(N) \frac{\pi a_0^2 r_0}{L^3} \right] + \binom{N}{3} \left[ \frac{32\pi a_0^4}{M_\varphi L^6} \left(3\sqrt{3} - 4\pi\right) \left(\ln(M_\varphi L)^2 - \Gamma'(1) - \ln(4\pi)\right) - \frac{\bar{\mathcal{T}}}{6L^6} \right] + \mathcal{O}(L^{-7})$$

[Romero-López, Rusetsky, Schlage, Urbach, in preparation]

| | | | |
|---|---|---|---|
| $N$ | $\hat{=}$ Number of particles | $a_0$ | $\hat{=}$ Scattering length |
| $E_N$ | $\hat{=}$ Interaction $N$ particle energy | $r_0$ | $\hat{=}$ Effective range |
| $M_\varphi$ | $\hat{=}$ Large vol. particle mass | $\bar{\mathcal{T}}$ | $\hat{=}$ Rel. threshold amplitude |

# Simulation – Parameter extraction

- **Complex $\varphi^4$-Theory**

$$\mathcal{S}[\varphi] = \int \mathrm{d}^4x \left[\partial_\mu \varphi^* \, \partial^\mu \varphi - m^2 \, |\varphi|^2 - \lambda \, |\varphi|^4\right]$$

$$\hookrightarrow \mathcal{S} = \sum_x \left[ \left( -\kappa \sum_\mu \varphi_x^* \, \varphi_{x+\mu} + c.c. \right) + \lambda' \left( |\varphi_x|^2 - 1 \right)^2 + |\varphi_x|^2 \right]$$

$a \mathrel{\hat{=}}$ lattice spacing $\kappa \mathrel{\hat{=}}$ hopping parameter $\lambda' \equiv \kappa^2 \lambda$

- **Parameter choice:**

$$m^2 = -4.9, \quad \lambda = 10.0 \qquad \Rightarrow \qquad \lambda' = 0.253308, \quad \kappa = 0.159156$$

- **Field configurations** (Metropolis) $\rightarrow$ 1 to 5 particle correlation function

$\Rightarrow$ Extract $a_0, r_0, \bar{\mathcal{T}}$

# Technical Details

- **Periodic** boundary conditions
- **Interpolation operator** for $N$-particle state

$$\hat{\mathcal{O}}_{N\varphi}(x) = (\varphi(x))^N$$

- **Correlation functions**

**$N$-particle correlator**

$$C_N(t) = \left\langle \hat{\mathcal{O}}'_{N\varphi}(t)\, \hat{\mathcal{O}}^\dagger_{N\varphi}(0) \right\rangle$$

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## $N$-particle correlator

$$C_N(t) = \left\langle \hat{\mathcal{O}}'_{N\varphi}(t)\, \hat{\mathcal{O}}^{\dagger}_{N\varphi}(0) \right\rangle$$

## Expand sum in limit $T \to \infty$

$$C_N(t) = A_N^2 \cdot \frac{1}{2}\left(e^{-E_N t} - e^{-E_N (T-t)}\right)$$

$\hookrightarrow$ pollution terms vanish in this limit

# Technical Details – Thermal Pollution Terms

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- Expansion of correlation functions in an infinite sum of eigenstates:
  - $C_1(t)$: Unpolluted signal
  - $C_2(t)$: Signal + constant pollution term
  - $C_3(t)$:

$$C_3(t) = \left|A^{(3)}_{3\leftrightarrow 0}\right|^2 \exp\left(-E_3 \tfrac{T}{2}\right) \cosh\left(E_3(t - \tfrac{T}{2})\right) + \left|A^{(3)}_{2\leftrightarrow 1}\right|^2 \exp\left(-(E_2 + M_\varphi)\tfrac{T}{2}\right) \cosh\left((E_2 - M_\varphi)(t - \tfrac{T}{2})\right)$$

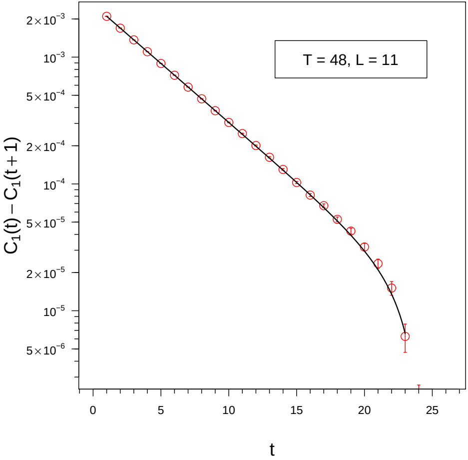
  - $C_4(t), C_5(t)$:

$t$-dependent + const. pollution, $t$-dependent pollutions

# Numerical Results – Correlator Example

**One particle correlator**

$C_1(t)$-fit $\rightarrow$ $M_\varphi \equiv E_1$



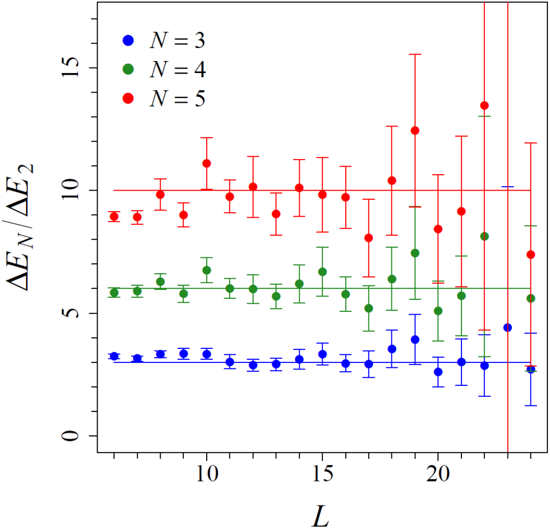
# Numerical Results – Energy Shift Ratios

**Threshold expansion:**

expect for shift ratio

$$\frac{\Delta E_N}{\Delta E_2} = \binom{N}{2} + \mathcal{O}(L^{-5})$$

$\hookrightarrow$ **Valid as a first test**, not enough for parameter extraction



# Numerical Results – $M_\varphi$ at Infinite $L$

**$L$ dependence of single particle mass**

$$M_\varphi(L) = M_\infty + c_M \cdot \frac{K_1(M_\infty L)}{M_\infty L}$$

$K_\nu(z) \hat{=}$ mod. Bessel func. of 2nd kind [Gasser, Leutwyler, 1987]

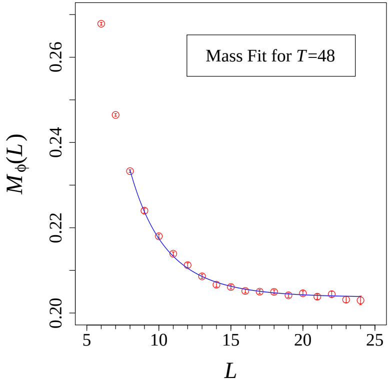
- **Result from fit:**

$$M_\infty = 0.2037(2)$$

- **Result from publication***:

$$M_\infty = 0.2027(2)$$

*[Romero-López, Rusetsky, Urbach, 2018]

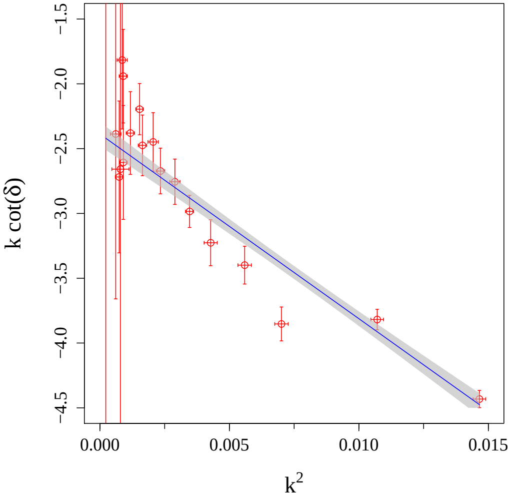


# Numerical Results – 2 Particle Sector: Phase Shift Fit

**$S$-wave phase shift**
[Lüscher, 1991]

$$\cot\delta = \frac{Z_{00}(1,q^2)}{\pi^{3/2}\,q}, \qquad q = \frac{L\,k}{2\pi}$$

with $E_2 = 2\sqrt{k^2 + M(L)^2}$



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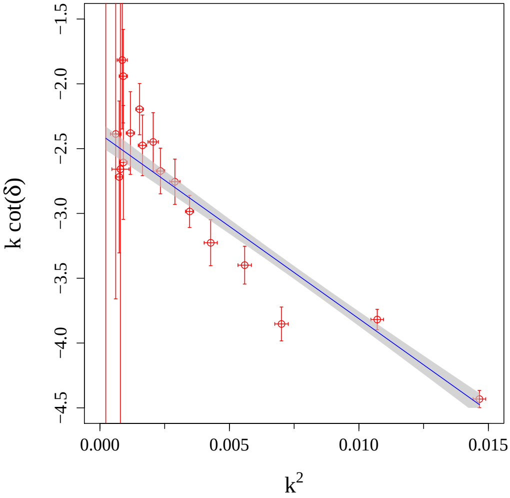
**Effective range expansion**

$$[k\,\cot\delta]\,(k^2) = -\frac{1}{a_0} + \frac{r_0\,k^2}{2} + \mathcal{O}(k^4)$$

Results ($p$-value = 0.62):

$a_0 = 0.407(17)$, $r_0 = -274(23)$



# Numerical Results – Global Energy Shift Fit

**Global fit model**

$$\chi^2_{\text{global}}(\vec{p}) = \sum_{N=2}^{5} \sum_{i \in D_N} \frac{\left[\Delta E_{N,\text{data}}(L_i) - \Delta E_N(L, \vec{p})\right]^2}{\left[\Delta\left(\Delta E_{N,\text{data}}(L_i)\right)\right]^2}$$

$L \mathrel{\hat{=}}$ independent variable
$\vec{p} \mathrel{\hat{=}}$ vector containing the fit parameter

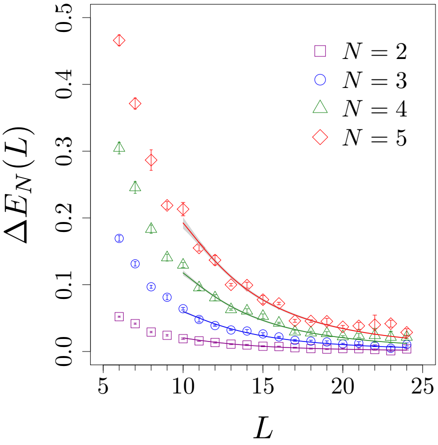
- Find set of parameters $\{a_0, r_0, \bar{\mathcal{T}}\}$ that minimizes $\chi^2_{\text{global}}(\vec{p})$

# Numerical Results – Global Energy Shift Fit

**Global $\Delta E_N(L)$-fit**

with $N = 2, 3, 4, 5$ to $\mathcal{O}(L^{-6})$

- Fit interval $[L_{\text{min}}, L_{\text{max}}] = [10, 24]$



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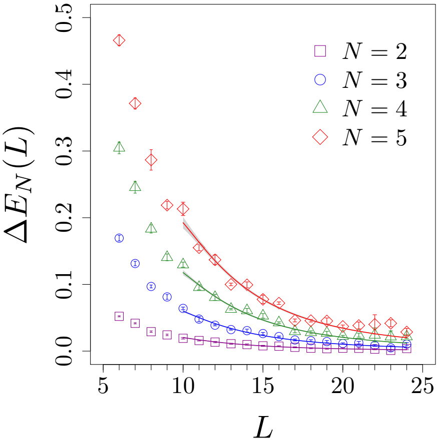
- Find best fit: Fix $L_{\text{max}}$, vary $L_{\text{min}}$



# Numerical Results – Global Energy Shift Fit

## Resulting parameters

from Global $\Delta E_N(L)$-fit to $\mathcal{O}(L^{-6})$

| exp. c. | rel. c. | $a_0$ | $r_0$ | $\bar{\mathcal{T}}$ | $p$-value |
|---|---|---|---|---|---|
| $\times$ | $\times$ | 0.438(15) | $-320(21)$ | $-362327(52137)$ | 0.63 |
| $\times$ | $\checkmark$ | 0.438(15) | $-292(21)$ | $-265422(46637)$ | 0.63 |
| $\checkmark$ | $\times$ | 0.439(15) | $-255(22)$ | $-286929(47814)$ | 0.65 |
| $\checkmark$ | $\checkmark$ | 0.439(15) | $-227(22)$ | $-189799(42507)$ | 0.65 |

$\hookrightarrow$ Reliable results? $\Rightarrow$ compare with perturbation theory

# Numerical Results – Comparison with Perturbation Theory

**Perturbation theory:**

One-loop values of the scattering length, threshold amplitude

$$r_0 = -\frac{1}{M_\varphi^2 a_0} + \frac{20}{3\pi M_\varphi} \qquad \Rightarrow r_0 \approx -50$$

$$\bar{\mathcal{T}} = \frac{288\pi^2 a_0^2}{M_\varphi^3} \qquad \Rightarrow \bar{\mathcal{T}} \approx +57000$$

$\hookrightarrow$ This does not match the previous results!

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- Fit to order $L^{-6}$ within $[L_{\mathrm{min}}, L_{\mathrm{max}}] = [12, 24]$

| priors | $a_0$ | $\bar{\mathcal{T}}$ | $p$-value | $r_0$ |
|---|---|---|---|---|
| – | 0.412(11) | 42559(12695) | 0.46 | −48(2) |
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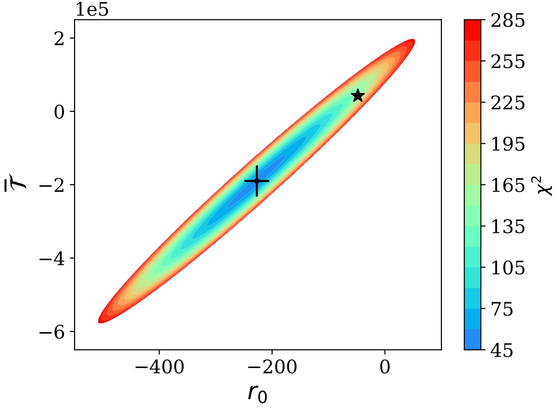
$\hookrightarrow$ **Best agreement with perturbation theory predictions**

$$r_0 \approx -50, \ \bar{\mathcal{T}} \approx 57000$$

# $\chi^2$ heat map for the global fit to order $L^{-6}$ without priors




$\hookrightarrow$ Discrepancy between global fit results and predictions from perturbation theory

# Conclusion

- **Results**
  - ★ Important to take **relativistic** and **exponentially suppressed corrections** into account
  - ★ $a_0$ **reproducible** in continuum perturbation theory
  - ★ Significant **discrepancy** between global fit results and predictions from perturbation theory for $r_0$ and $\bar{\mathcal{T}}$
  - ★ $r_0$ and $\bar{\mathcal{T}}$ **correlated** $\rightarrow$ if $r_0$ is fixed properly then also $\bar{\mathcal{T}}$ is correct

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- **Interpretation**
  - ★ Continuum perturbation theory does not describe lattice $\varphi^4$-theory well
  - ★ Large discretization effects must be expected

# What else?

- Investigate excited states
- Investigate moving frames
- Lattice QCD

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*Thanks for your attention.*

# Appendix – Number of Field Configurations with $T = 48$

## Finite hyper-cubic lattice

$$V = 48 \times L^3$$

| $L$ | $n_{\text{conf}}$ | $L$ | $n_{\text{conf}}$ |
|---|---|---|---|
| 6 | 100,000 | 16 | 100,000 |
| 7 | 100,000 | 17 | 100,000 |
| 8 | 100,000 | 18 | 100,000 |
| 9 | 100,000 | 19 | 100,000 |
| 10 | 100,000 | 20 | 80,000 |
| 11 | 100,000 | 21 | 70,000 |
| 12 | 100,000 | 22 | 50,000 |
| 13 | 100,000 | 23 | 50,000 |
| 14 | 100,000 | 24 | 30,000 |
| 15 | 100,000 | | |

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