Inhomogeneous phases in the $2 + 1$-dimensional Gross-Neveu model in the large-N_f limit

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Gross-Neveu model

- Phase diagram of QCD poses an interesting problem with a lot of open questions
- Particular focus on inhomogeneous order parameters & chiral symmetry breaking $\Rightarrow$ Investigate phase structure of simpler models
- The Gross-Neveu model serves as a toy model with crude similarity to QCD.
  - Interaction realized by a 4-point fermion interaction.
  - Action is invariant under a discrete chiral symmetry.
  - This symmetry can be spontaneously broken.
- An inhomogeneous phase exists in $1+1$-dimensions in the limit of infinite fermion flavors
- Euclidean action of the Gross-Neveu model in $2+1$ dimensions:

$$S_E = \int d^3x \left(\bar{\psi}_f \left(\gamma_\nu \partial_\nu + \gamma_0 \, \mu \right) \psi_f - \frac{\lambda}{2N_f} \left(\bar{\psi}_f \, \psi_f \right)^2 \right).$$

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- Hubbard-Stratonovich transformation

$$Z = \mathcal{N} \int \mathcal{D}\sigma \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[- \int d^3x \left(\frac{N_f}{2\lambda} \sigma^2 + \bar{\psi}_f \underbrace{\left(\gamma_\nu \partial_\nu + \gamma^0 \mu + \sigma \right)}_{\mathrm{Q}} \psi_f \right) \right].$$

- $\langle \bar{\psi}(x) \psi(x) \rangle = -\frac{N_f}{\lambda} \langle \sigma(x) \rangle \Rightarrow$ refer to σ as chiral condensate.

The effective action in the large-N_f limit

- After integrating out fermion fields

$$Z = \mathcal{N} \int \mathcal{D}\sigma \exp\left(-N_f \left[\frac{1}{2\lambda} \int d^3x \sigma^2 - \ln\det\left(\mathrm{Q}\right) \right] \right)$$

- $\det(Q) \in \mathbb{R}$ required $\Rightarrow$ restrict $\sigma(x) = \sigma(x_1, x_2)$[1]
- Limit $N_f \to \infty$ supresses bosonic fluctuations in Z
  $\Rightarrow$ the global minimum of $S_{\text{eff}}[\sigma]$ dominates the path integral!

[1] Alternatively, one could use a reducible fermion representation.

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$\Rightarrow \langle O \rangle = \frac{1}{Z}\int \mathcal{D}\sigma O(\sigma) \mathrm{e}^{-S_{\text{eff}}[\sigma]} = O(\sigma_{\text{min}}(x))$

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Chiral symmetry and order parameter

- Action of Gross-Neveu model invariant under discrete chiral symmetry transformation (only possible if γ_5 exists)

$$\psi \to \gamma_5 \psi,\ \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma.$$

- Characterize phases with σ as order parameter

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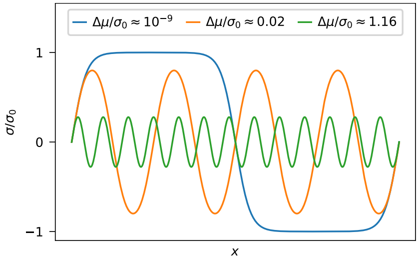
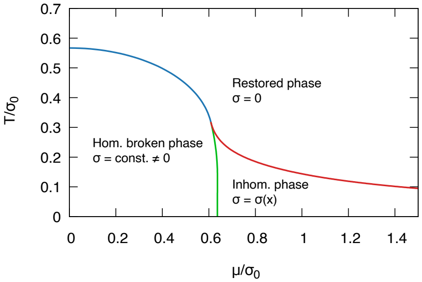
- Characterize phases with σ as order parameter
- Inhomogeneous phase occurs in $1+1$ dimensions in the limit $N_f \to \infty$ [M. Thies and K. Urlichs, Phys. Rev. D **67**, 125015 (2003) [hep-th/0302092]]

The chiral condensate for $T = 0$:



Revised phase diagram:



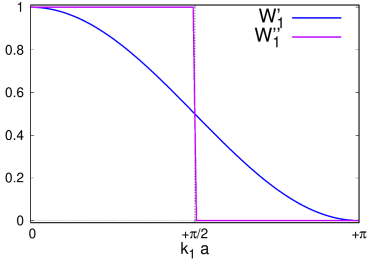
Discretization

$$S_{\text{eff}}[\sigma] = N_f \left[\frac{1}{2\lambda} \int \mathrm{d}^3x \sigma^2 - \ln \det \underbrace{\left(\gamma_\nu \partial_\nu + \gamma^0 \mu + \sigma \right)}_{\mathrm{Q}} \right]$$

- Global minimization of $S_{\text{eff}}(\sigma(x_1, x_2))$ is a challenging task
- Efficient computation of $\det \mathrm{Q}$
- Restriction $\sigma(x_1, x_2) = \sigma(x_1)$
- Finite mode approach in temporal direction & naive discretization in spatial directions, $\sigma = \sigma(x_1)$
- Easy analytic simplifications in x_0-, x_2-direction, μ can be introduced in the continuum
  (lower discretization errors [Lenz et al. (2020) arXiv:2007.08382 [hep-lat]])

Discretization of the effective action

- Discretized interaction term $\sim \bar{\psi}(x_1)\sum_{y_1} W_1(x_1 - y_1)\sigma(y_1)\psi(x_1)$ compared to point interaction in the continuum $\bar{\psi}(x_1)\sigma(x_1)\psi(x_1)$
- $W_1(x_1 - y_1) = \delta_{x_1,y_1}$ as in the continuum leads to incorrect interaction terms, e.g. coupling of different fermion flavors, $\sim \bar{\psi}\gamma_0\sigma\psi$
- Observed by analysis of continuum limit [Lenz et al. (2020) Phys. Rev. D **101**, no.9, 094512]
- $\tilde{W}_1$ needs to suppress momentum of $\tilde{\sigma}$ near "doubler poles"



$\tilde{W}_1' = (1 + \cos(k_1 a))/2$ "soft momentum cutoff", $W_1' \neq 0$ only for neighbored lattice points, and $\tilde{W}_1'' = \Theta(\pi/2a - |k_1|)$ "hard momentum cutoff", W_1'' distributes over the whole lattice

Detection of phase boundaries

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- Instead: Stability analysis of $\sigma(x_1) = 0 \Rightarrow (H)_{xy} = \frac{\partial^2}{\partial \sigma_{x_1} \partial \sigma_{y_1}} \mathsf{S}_{\text{eff}}|_{\sigma=0}$
  - H indefinit $\Rightarrow$ broken chiral symmetry
  - Eigenvectors corresponding to negative eigenvalues indicate directions in σ-space that lower the effective action

(+) Boundary between restored and inhomogeneous phase

(−) Does not detect hom. broken to inhom. phase boundary

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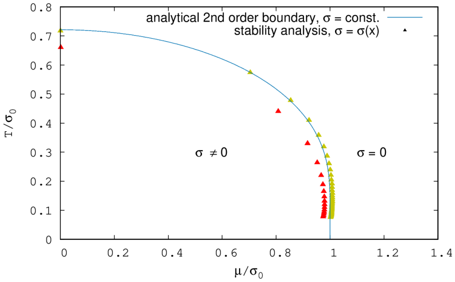
  (−) Does not detect hom. broken to inhom. phase boundary
- Local minimization in σ for given (μ, T) via conjugate gradient algorithm

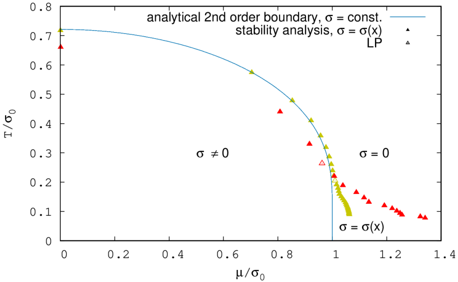
  (+) Could possibly compute missing boundary

  (−) Insecurity about global minimization

$W_1 = W_1'$



$W_1 = W_1''$

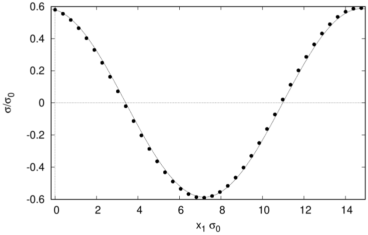


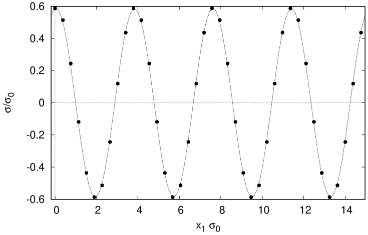
- Two different lattice spacings $a\sigma_0 = 0.379$, $a\sigma_0 = 0.174$
- Analytical 2nd order boundary for $\sigma = \text{const.}$ [K. Klimenko (1988) Z. Phys. C **37**, 457]
- $W_1 = W_1'$: No inhomogeneous phase, stability analysis leads to same boundary as computation with $\sigma = \text{const.}$
- $W_1 = W_1''$: Inhomogeneous phase at finite a^2, starting from Lifshitzpoint (LP), shrinks with decreasing a [R. Narayanan (2020) Phys. Rev. D **101**, no.9, 096001]
- Continuum stability analysis: Inhomogeneous phase at finite cutoff, vanishes when removing the cutoff

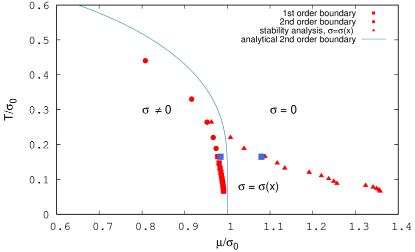
[2] M. Winstel (2019) [arXiv:1909.00064 [hep-lat]]

Shape of eigenvectors

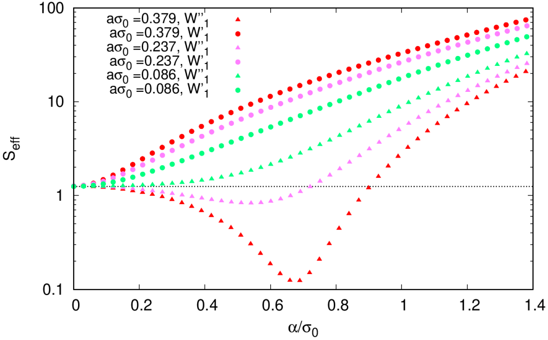
- Shape of eigenvector that leads to a lower action than $\sigma = 0$







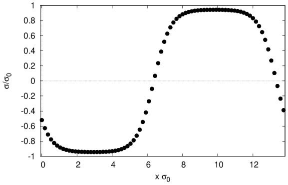
- The effective action for $\sigma(x_1) = \alpha\cos(6\pi x_1/L)$
- Within the inhomogeneous region $(\mu/\sigma_0 = 1.035,\ T/\sigma_0 = 0.11)$



- The actions approach each other in the limit $a \to 0$
- Supports shrinking of the inhomogeneous phase
- Qualitatively similar to [R. Narayanan (2020) Phys. Rev. D **101**, no.9, 096001]

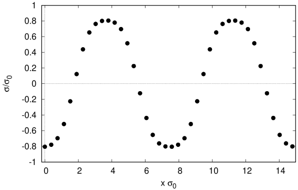
Minimization of the effective action

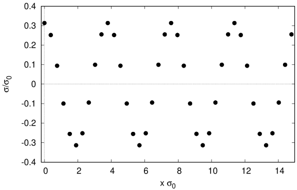
- Does not change the phase diagram and its continuum limit
- Inhomogeneous modulations as local minima within the hom. broken phase
- $T \to 0$ Differences to global minimum decrease (supports [K. Urlichs (2007) Doctoral thesis])



- Global minima found for W_1'' at finite a behave analogous to $1+1$ dimensions

$(T/\sigma_0 = 0.132)$ $\mu/\sigma_0 = 0.97$ $\mu/\sigma_0 = 1.11$





Conclusion & Outlook

- We developed methods to search for inhomogeneous phases without an specific ansatz
- Inhomogeneous phase in the 2+1-dim. Gross-Neveu model in the large-N_f limit found only for finite lattice spacing
- In the continuum limit inhomogeneous region vanishes
  - Cutoff dependence of inhomogeneous phase to be explored
- Maybe a spatial modulation $\sigma(x_1, x_2)$ is favored?
  - Efficient algorithms to evaluate S_{eff}, minimization
- Extend models towards QCD-like scenarios
  - Isospin chemical potential
  - Continuous chiral symmetry
  - Finite N_f simulations

Appendix

Chirality operator

- Calculation performed with irreducible representation of fermions
  Dirac matrices as Pauli matrices

$$\gamma^0 = \sigma_1,\ \gamma^1 = \sigma_2,\ \gamma^2 = \sigma_3$$

$$\gamma^0 = -\sigma_1\ \gamma^1 = -\sigma_2,\ \gamma^2 = -\sigma_3$$

$\Rightarrow$ Non-trivial γ_5 not available

- Which symmetry is spontaneously broken by the condensate ?

Parity in 2+1 dimensions

- Parity as inversion of all spatial coordinates equivalent to rotation

$$(x_0, x_1, x_2)^T \xrightarrow{P} (x_0, x_1, -x_2)^T$$
$$\psi \xrightarrow{P} -\mathrm{i}\gamma_2\psi$$
$$\bar{\psi} \xrightarrow{P} -\mathrm{i}\bar{\psi}\gamma_2$$

- Obtain $\sigma \xrightarrow{P} -\sigma$
- Non-vanishing σ indicates spontaneous breaking of parity

How to obtain a chiral symmetry?

- Use four component spinors via combination of two inequivalent irreducible spinors ($\tau_i \equiv$ Pauli matrices in isospin space)

$$\gamma^\mu = \tau_3 \otimes \sigma_{\mu+1}, \quad \gamma_4 = \tau_1 \otimes \mathbb{1}$$
$$\gamma_5 = -\tau_2 \otimes \mathbb{1}, \quad \gamma_{45} = i\gamma_4\gamma_5$$

- Parity to be defined in isospin space via tensor product with τ_1
- Mass term $\bar{\psi}\psi$ now invariant under parity
- Obtain the same lagrangian as before, but now with fermion doubling

Chiral transformations

- Symmetries of free massless fermions in 2+1 dimensions $\big(U(2N_f)\big)$

$$\psi \to e^{i\theta\Gamma}\psi \qquad \Gamma \in \{\mathbb{1}, \gamma_{45}, \gamma_4, \gamma_5\}$$

- For the Gross-Neveu model only a subgroup is realized

$$\psi \to \gamma_5\psi,\ \bar{\psi} \to -\bar{\psi}\gamma_5$$

- Together with this "GN-typical" term we have a continuous symmetry

$$\psi \to e^{i\phi\gamma_{45}}\psi,\ \bar{\psi} \to \bar{\psi}e^{-i\phi\gamma_{45}}$$

- Found a GN-type model that is invariant under a chiral symmetry

Fermion representation and symmetries

- Use four component spinors via combination of two inequivalent irreducible spinors
- Chiral symmetry transformation recovered
- Interpretation of σ as chiral order parameter
- Obtain the same action as before, but now with fermion doubling
- Additionally, one can show the Gross-Neveu model with the assumption $\sigma(x) = \sigma(x_1, x_2)$ using properties of the Dirac operator Q

$$\det(Q_{\text{4-comp.}}) = \det(Q_{\text{2-comp.}})^2 \Rightarrow S_{\text{eff,4-comp.}}[\sigma] = 2S_{\text{eff,2-comp.}}[\sigma].$$

- For the GN model we can use the irreducible representation and obtain the same physics
- Be careful about other $2+1$ dimensional field theories

Discretization of the effective action

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$$S_{\text{eff}}[\sigma] = N_f \left[\frac{1}{2\lambda} \int \mathrm{d}^3 x \sigma^2 - \ln \det \big(\underbrace{\gamma_\nu \partial_\nu + \gamma^0 \mu + \sigma}_{\mathrm{Q}} \big) \right]$$

- Easy analytic simplifications in x_0-, x_2-direction, μ can be introduced in the continuum

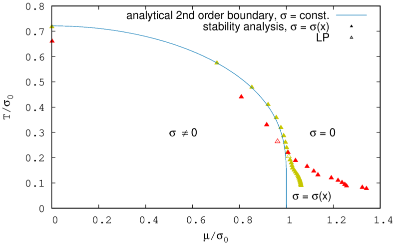
$$\mathrm{Q}_{n_0,n_2;n_0',n_2'}(x_1;x_1') = \delta_{n_0,n_0'}\delta_{n_2,n_2'} \Big(\gamma_0 (\mathrm{i}\omega_{n_0} + \mu)\delta_{x_1,x_1'} + \gamma_1 \tilde{\partial}_{x_1;x_1'}$$

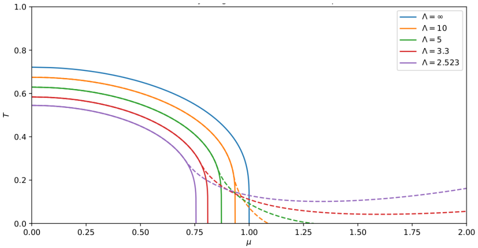
$$+ \delta_{x_1,x_1'} \gamma_2 \mathrm{i} \sin(k_{n_2}) + \delta_{x_1,x_1'} \sum_{y_1} W_1(x_1 - y_1)\sigma(y_1) \Big)$$

$$S_{\text{eff}}[\sigma] = N_f \left[\frac{\beta L}{2\lambda} \sum_{x_1} \sigma^2(x_1) - \frac{1}{4} \sum_{n_0} \sum_{n_2} \ln \det \Big(\mathrm{Q}_{n_0,n_2;n_0,n_2}(x_1;x_1') \Big) \right].$$

- Naive, symmetric lattice derivative $\tilde{\partial}_{x_1;x_1'} = \frac{\delta_{x_1+1,x_1'} - \delta_{x_1-1,x_1'}}{2}$

Comparison with analytical stability analysis





Comparison with continuum results of M. Buballa, L. Kurth

2d-structures?

- Phase boundary only for $\sigma(x_1)$
- Inhomogeneous structures with $\sigma = \sigma(x_1, x_2)$ for small volume and larger spacing