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Constraints of a local bosonization in arbitrary dimensions

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I. The problem: describe fermions without Grassmann variables

Pauli principle $\longrightarrow$ antisymmetry $\longrightarrow$ anticommutation rules

Example - simple fermionic Hamiltonian

$$H = i \sum_n \phi(n)^\dagger \phi(n+1) - \phi(n+1)^\dagger \phi(n), \quad \{\phi(m)^\dagger, \phi(n)\} = \delta_{mn}, \tag{1}$$

Equivalent Hamiltonian in terms of spin variables/operators (one space-dimension only (d=1))

$$H = \frac{1}{2} \sum_n \sigma^1(n)\sigma^2(n+1) + \sigma^2(n)\sigma^1(n+1) \tag{2}$$

Proof: Jordan-Wigner transformation.
J-W transforms local fermionic "bilinears" into local spin ones.
In higher dimensions J-W introduces non-local interactions.

Avoiding J-W transformation [Nambu (1950)]

Link operators

$$X(n) = \phi(n)^\dagger + \phi(n), \qquad Y(n) = i(\phi(n)^\dagger - \phi(n))$$
$$S(n) = iX(n)X(n+1), \qquad \tilde{S}(n) = iY(n)Y(n+1)$$

The algebra of link operators

$$[\bar{S}(m), \bar{S}(n)] = 0, \quad m \neq n-1, n+1,$$
$$\{\bar{S}(m), \bar{S}(n)\} = 0, \quad m = n-1, n+1,$$

$$[S(m), \bar{S}(n)] = 0$$

The same algebra is obeyed by the following link operators

$$S(n) = \sigma^1(n)\sigma^2(n+1), \quad \tilde{S}(n) = -\sigma^2(n)\sigma^1(n+1),$$

Which gives (2)

II. Two space dimensions

Two dimensional lattice hamiltonian is given by analogous links in two dimensions.

The algebra: links in fermionic representation anticommute only if they have one site in common.

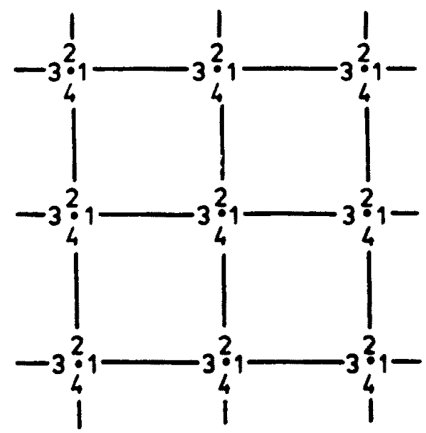
How to find their spin representation ?

HINT: Four links meet at one point $\longrightarrow$ we need four anticommuting matrices [Wosiek (1982)].

$$
\begin{array}{ll}
S(\vec{n},\vec{e}_x) = \Gamma^1(\vec{n})\Gamma^3(\vec{n}+\vec{e}_x), & S^X(\vec{n},\vec{e}_y) = \Gamma^2(\vec{n})\Gamma^4(\vec{n}+\vec{e}_y) \\
\tilde{S}(\vec{n},\vec{e}_x) = \tilde{\Gamma}^1(\vec{n})\tilde{\Gamma}^3(\vec{n}+\vec{e}_x), & \tilde{S}(\vec{n},\vec{e}_y) = \tilde{\Gamma}^2(\vec{n})\tilde{\Gamma}^4(\vec{n}+\vec{e}_y)
\end{array}
\qquad (3)
$$

$$
\tilde{\Gamma}^k = i\Pi_{j\neq k}\Gamma^j
$$

Expect: The two hamiltonians are equivalent, i.e. they have the same spectrum.



3 2 1 4
3 2 1 4
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III. Constraints

For d=1 and with L sites : $\dim(\mathcal{H}_f = 2^L)$

For d=2 and with L^2 sites : $\dim(\mathcal{H}_f = 2^{L^2})$

However : $\dim(\mathcal{H}_s = 4^{L^2})$

Suspect: in higher dimensions spin systems must be constrained.

$\Longrightarrow$ **Plaquette operators** $P(L, I, N, K) = S(L)S(I)S(N)S(K)$.

[Itzykson (1980)]

P=1 in fermionic representation, $P^2 = 1$ **in spin representation.**

$\Longrightarrow L^2$ **constraints** $P = 1 \Longrightarrow 2^{L^2}$ **dimensions remain - OK.**

We therefore impose L^2 constraints in the spin representation

$$P_{\vec{n}} = 1, \tag{4}$$

IV. An explicit implementation

The Hilbert space for $L_x \times L_y$ lattice has $4^{\mathcal{N}}$ dimensions, $\mathcal{N} = L_x L_y$.

States:

$$\{i_1, i_2, \ldots, i_{\mathcal{N}}\}, \quad i_n = 1, \ldots, 4, \quad n = 1, \ldots, \mathcal{N}. \tag{5}$$

Operators: $\mathcal{N}$-fold tensor products of Γ's (and unity)

Sparse matrices - O($4^{\mathcal{N}}$) memory size.

Aviable sizes $\mathcal{N} \sim 16 - 20$.

IV.1 Constraints

Projectors

$$\Sigma_{m,n} = \frac{1}{2}(1 + P_{m,n}), \qquad \Sigma_Z = \frac{1}{2}(1 + \mathcal{L}_Z), \quad Z = x, y, \tag{6}$$

Number of fermions and fermionic density

$$N = \sum_n N(n) = \sum_n \frac{1}{2}(1 - \eta\Gamma^5(n)), \quad \eta = \pm 1. \tag{7}$$

Reduction $\mathcal{H}_{spins} \longrightarrow \mathcal{H}_{fermions}$ done at fixed N or even N(n).

Constraints between constraints

$$\prod_{m,n} P_{m,n} = 1, \qquad \mathcal{L}_x(y+1) = \prod_{adjacent\ row} P_{row}\mathcal{L}_x(y) \tag{8}$$

$$(-1)^N = \eta^{L_x L_y} \left(-\frac{\epsilon'_x}{\epsilon_x}\right)^{L_x} \left(-\frac{\epsilon'_y}{\epsilon_y}\right)^{L_y}, \quad \epsilon,\ \epsilon' = \pm 1. \tag{9}$$

Reduction schemes: (I) at fixed N=p, (II) at fixed {N(n)}

$$2^{\mathcal{N}} \binom{\mathcal{N}}{p} \longrightarrow \binom{\mathcal{N}}{p}, \qquad 2^{\mathcal{N}} \longrightarrow 1. \tag{10}$$

Scheme (I) - at fixed p

p=	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Tr $\boldsymbol{\Sigma_{11}}$	256	2304	9216	21504	32256	32256	21504	9216	2304	256
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}}$	128	1152	4608	10752	16128	16128	10752	4608	1152	128
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}\Sigma_{13}}$	64	576	2304	5376	8064	8064	5376	2304	576	64
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{21}}$	32	288	1152	2688	4032	4032	2688	1152	288	32
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{22}}$	16	144	576	1344	2016	2016	1344	576	144	16
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{23}}$	8	72	288	672	1008	1008	672	288	72	8
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{31}}$	4	36	144	336	504	504	336	144	36	4
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{32}}$	2	18	72	168	252	252	168	72	18	2
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{33}}$	2	18	72	168	252	252	168	72	18	2
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{x}}$	1	9	36	84	126	126	84	36	9	1
Tr $\boldsymbol{\Sigma_{11}\Sigma_{12}...\Sigma_{y}}$	0	9	0	84	0	126	0	36	0	1

Table 1: Reduction of the spin Hilbert space for $\mathbf{3 \times 3}$ lattice in $\boldsymbol{p}$-particle sectors. Periodic boundary conditions are assumed.

Scheme (II) - at fixed positions (of p excitations)

<table>
<tr><td colspan="2">Sector ($\boldsymbol{p}$)</td><td>even, $\mathbf{0} \leq \boldsymbol{p} \leq \mathbf{16}$</td><td>odd, $\mathbf{0} < \boldsymbol{p} < \mathbf{16}$</td></tr>
<tr><td colspan="2">Occupied sites</td><td colspan="2">from # 1 to # p</td></tr>
<tr><td rowspan="18">Hilbert space reduction</td><td>Tr $\mathbf{\Sigma_{11}}$</td><td colspan="2">32768</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}\Sigma_{21}}$</td><td colspan="2">16384</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{31}}$</td><td colspan="2">8192</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{41}}$</td><td colspan="2">4096</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{12}}$</td><td colspan="2">2048</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{22}}$</td><td colspan="2">1024</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{32}}$</td><td colspan="2">512</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{42}}$</td><td colspan="2">256</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{13}}$</td><td colspan="2">128</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{23}}$</td><td colspan="2">64</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{33}}$</td><td colspan="2">32</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{43}}$</td><td colspan="2">16</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{14}}$</td><td colspan="2">8</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{24}}$</td><td colspan="2">4</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma}_{\boldsymbol{x}}$</td><td colspan="2">2</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma}_{\boldsymbol{y}}$</td><td colspan="2">1</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{34}}$</td><td>1</td><td>0</td></tr>
<tr><td>Tr $\mathbf{\Sigma_{11}...\Sigma_{44}}$</td><td>1</td><td>0</td></tr>
</table>

Table 2: Reduction of the spin Hilbert space for subsectors $\mathbf{0} \leq \boldsymbol{p} \leq \mathbf{16}$, and fixed coordinates, on a $\mathbf{4} \times \mathbf{4}$ lattice. Sites of the lattice are ordered lexicographically, thus e.g. sites from #1 to #5 means sites (1,1), (2,1), (3,1), (4,1) and (1,2).

IV.2 The spectrum

Reduction (I) $\longrightarrow \mathcal{N}_p = \begin{pmatrix} \mathcal{N} \\ p \end{pmatrix}$, 1 - eigenvectors, v_i, of all constraints.

They span the Hilbert space of p free, indistinguishable fermions.

The reduced spin Hamiltonian $H_{ij} = \langle v_i | H | v_j \rangle$ is equivalent to the fermionic Hamiltonian (1) in a p-fermion sector.

Closing the circle:

The eigenvalues of H_{ij} agree with the energies of p identical, free fermions.

V. The whole family of constraints - emergent Wegner gauge field

What about all other constraints ($2^{\mathcal{N}} - 1$ of them)?

$$P_n = \pm 1, \qquad 1 \leqslant n \leqslant \mathcal{N}. \tag{11}$$

Couple fermions (1) to an external Z_2 gauge field $U(l)$

$$H_f = i \sum_{\vec{n},\vec{e}} \left(U(\vec{n}, \vec{n}+\vec{e}) \phi(\vec{n})^\dagger \phi(\vec{n}+\vec{e}) - U(\vec{n}, \vec{n}+\vec{e}) \phi(\vec{n}+\vec{e})^\dagger \phi(\vec{n}) \right) \tag{12}$$

$$= \frac{1}{2} \sum_l \left(U(l) S(l) + U(l) \tilde{S}(l) \right), \tag{13}$$

In the spin representation this goes into

$$H_s = \frac{1}{2} \sum_l \left(U(l) S(l) + U(l) \tilde{S}(l) \right). \tag{14}$$

with the same variables $U(l)$, and $S(l)$ given by (3).

H_s describes corresponding spins in an external Z_2 field.

As in the free case H_f and projected H_s should be equivalent.

$\longrightarrow$ **Generalizing the fermion-spin equivalence to systems in external fields [Szczerba (1984)].**

Now watch this: one can absorb the $U(l)$ factors into new link operators

$$S'(l) = U(l)S(l); \quad \tilde{S}'(l) = U(l)\tilde{S}(l), \tag{15}$$

commutation rules of S''s are unchanged [Bochniak, Ruba (2019)].

The new spin hamiltonian does not depend on the external field

$$H'_s = \frac{1}{2}\sum_l \left(S'(l) + \tilde{S}'(l)\right), \tag{16}$$

but the constraints on the new spin variables do.

$$P'_n = \prod_{l\in C_n} U(l). \tag{17}$$

$\Longrightarrow$ Two ways of introducing minimal interaction with an external field:

1) the standard one by putting explicitly link variables into the hamiltonian and imposing "free" form of the constraints (4), and
2) use the free spin hamiltonian (16), but impose the "interacting" constraints (17).

Conclusion: the whole family of possible constraints can be parametrized by an external gauge Z_2 field .

An example

There exists a particular configuration of Wegner variables, namely

$$U_x(x,y) = (-1)^y, \quad U_y(x,y) = 1, \tag{18}$$

for which the fermionic problem can be solved analytically.
The spectrum of the fermionic hamiltonian (13) reads

$$E^{(1)}_{magnetic}(k_x,k_y) = \pm 2\sqrt{\sin\left(\frac{2\pi k_x}{L_x}\right)^2 + \sin\left(\frac{2\pi k_y}{L_y}\right)^2} \quad 1 \leqslant k_x \leqslant L_x, \quad 1 \leqslant k_y \leqslant L_y/2, \tag{19}$$

to be contrasted with the free case

$$E^{(1)}_{free}(k_x,k_y) = 2\sin\left(\frac{2\pi k_x}{L_x}\right) + 2\sin\left(\frac{2\pi k_y}{L_y}\right), \quad 1 \leqslant k_z \leqslant L_z, \quad z = x, y. \tag{20}$$

Configuration (18) can be realized only for an even L_y and results in all plaquettes being equal

$$P_n = -1, \quad 1 \leqslant n \leqslant \mathcal{N}, \tag{21}$$

hence it is a Wegner version of a constant magnetic field.

A punchline: Mathematica exercise for 3×4 lattce ($p = 1$ sector)

1. Upon reduction correct size of $\mathcal{H}$ was obtained
2. Fermionic spectrum (19) was reproduced

VI. Dynamical gauge field - the spectrum of dualities

Each set of constraints { $P_n = \pm 1$ } $\leftrightarrow$ gauge invariant configuration (an orbit) of a Z_2 gauge field.

Complete Hilbert space of $\{Z_2\}$ splits into above classes.

$\Longrightarrow$ **The dynamical Wegner field $\leftrightarrow$ the unconstraint Γ-spins.**

Dualities in 2+1 dimensions (70's – now)

M.Peskin (1978), A.M.Polyakov (1987, 1988), ...,
T. Jaroszewicz (1991),... ,
D. Tong (2016), E. Witten (2016), D. T. Son (2018)...

- **Kramers-Wannier duality: ising spins $\leftrightarrow$ kinks (1+1)**

- **particle-vortex duality (2+1):**

XY spins $\leftrightarrow$ vortices of the phase (+ a gauge field (!))

- **fermion-fermion duality:**

free Dirac field $\leftrightarrow$ fermions coupled to an emergent gauge field

- **fermion-boson dualities $\leftarrow$ flux attachmet: flux+charged boson $\leftrightarrow$ fermion**

An emergent gauge field is always a Chern-Simons field

Our system (Γ-spins ≡ pairs of Ising spins) :

free lattice fermions ↔ Γ-spins with "pure-gauge" constraints

with an external Z_2 field ↔ Γ-spins with Z_2-driven constraints

with a dynamical Z_2 field ↔ unconstraint Γ-spins

- **Our Z_2 field might resemble a CS field (see Błażej's talk).**

•• **Another duality in 2+1 (and higher) dimensons ?**

VII. Summary

• **Fermions can be equivalently represented by ”spins” i.e. discrete bosonic degrees of freedom.**

• **In higher dimensions above spin systems are subject to constraints.**

• **Spin hamiltonian in the constraint space is fully equivalent to the original fermionic hamiltonian (it has the same spectrum).**

• **An interesting, physical interpretation of the complete family of all constraints in terms of the external Z_2 field has been also found.**

• **An explicit realization (and check): constant, magnatic Wegner field.**

• • ... **The intriguing possibility of a new, unknown yet duality.**