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Dirac eigenvalue spectrum and its relation to $U(1)_A$ symmetry breaking in high temperature N_f=2+1 QCD

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Outline

- Motivation
- Chiral & $U(1)_A$ symmetry and Dirac eigenvalue spectrum
- Lattice Setup
- Results
- Summary and Outlook

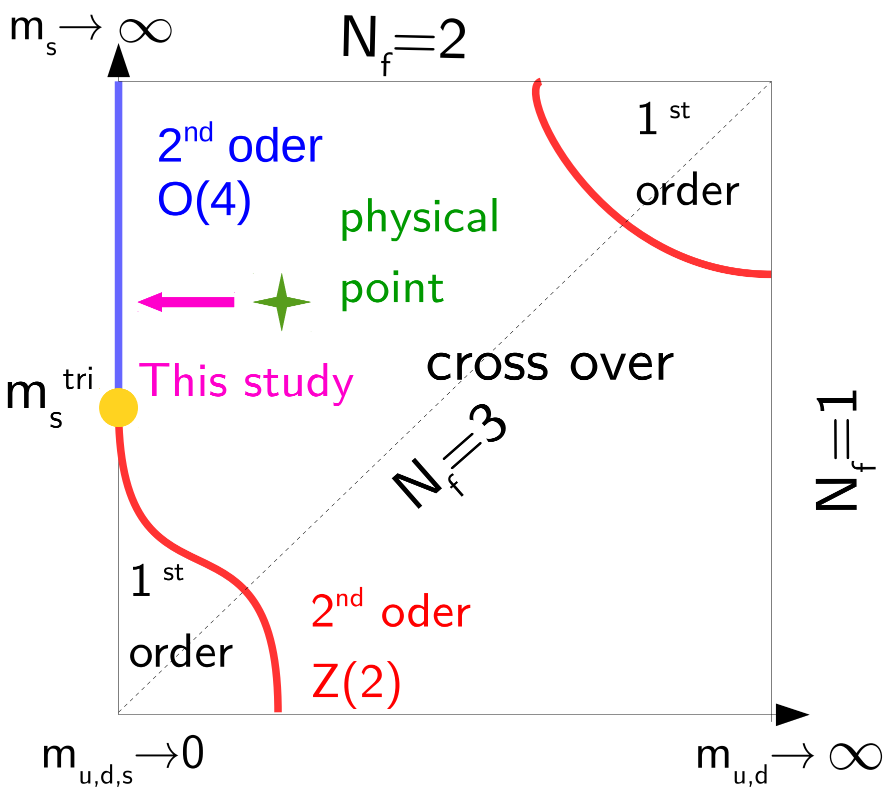
Motivation

At $T > T_c$, chiral symmetry is restored.

How about the fate of $U(1)_A$ symmetry?

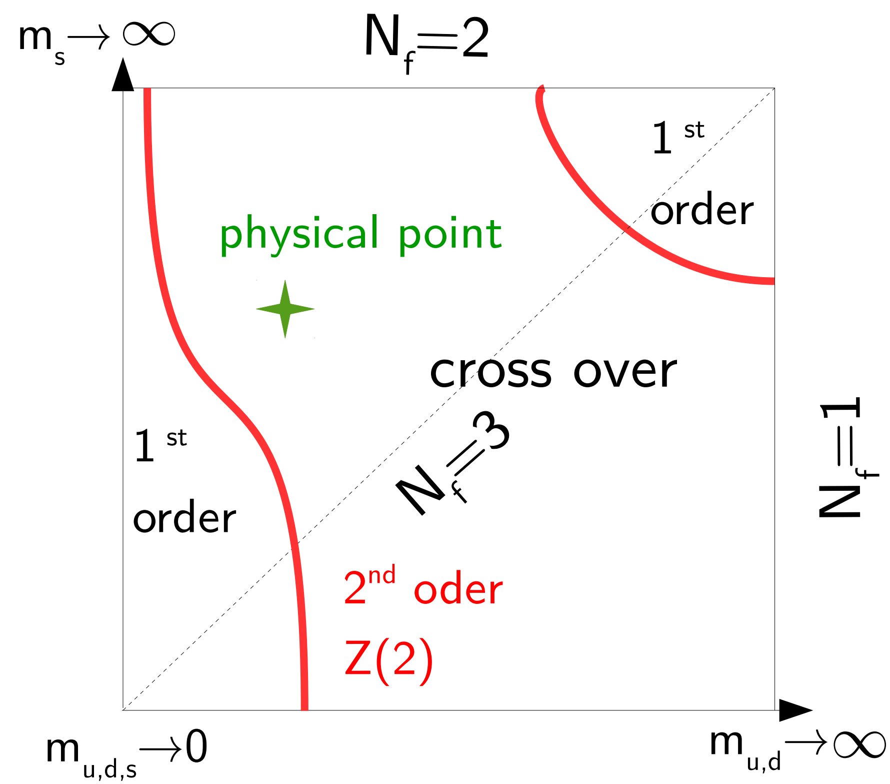
Two possible scenarios:

Substantial $U(1)_A$ breaking



Pisarski, Wilczek (1984)

Negligible $U(1)_A$ breaking



Philipsen, Pinke, PRD 93 (2016) 114507

Chiral & U(1)$_A$ symmetry and susceptibilities

$$
\begin{tikzcd}
\chi_{5,\mathrm{con}} & \pi : \bar{q}\gamma_5 \frac{\tau}{2} q \arrow[r, leftrightarrow, "\mathrm{SU(2)_L \times SU(2)_R}"] \arrow[d, leftrightarrow, "\mathrm{U(1)_A}"'] & \sigma : \bar{q}q \arrow[d, leftrightarrow, "\mathrm{U(1)_A}"] & \chi_{\mathrm{con}} + \chi_{\mathrm{disc}} \\
\chi_{\mathrm{con}} & \delta : \bar{q}\frac{\tau}{2} q \arrow[r, leftrightarrow, "\mathrm{SU(2)_L \times SU(2)_R}"'] & \eta : \bar{q}\gamma_5 q & \chi_{5,\mathrm{con}} - \chi_{5,\mathrm{disc}}
\end{tikzcd}
$$

$$\chi_\pi = \frac{N_f}{4}\frac{1}{V}\left\langle \mathrm{Tr}\left(M^{-1}\gamma_5 M^{-1}\gamma_5\right)\right\rangle \qquad \chi_\delta = \frac{N_f}{4}\frac{1}{V}\left\langle \mathrm{Tr}\left(M^{-1}M^{-1}\right)\right\rangle$$

$$\chi_{\mathrm{disc}} = \left(\frac{N_f}{4}\right)^2 \frac{1}{V}\left[\left\langle(\mathrm{Tr}M^{-1})^2\right\rangle - \left\langle \mathrm{Tr}M^{-1}\right\rangle^2\right]$$

in the chirally symmetric phase, χ_{disc} can be used to probe the restoration of the U(1)$_A$ symmetry

[See talk by Fukaya, Tues. 17:20]

signatures from Dirac eigenvalue spectrum

$$\boxed{\langle\bar{\psi}\psi\rangle = \frac{N_f}{4}\int_0^\infty d\lambda \frac{2m\rho(\lambda,m)}{\lambda^2+m^2} \xrightarrow{m\to 0} \pi\rho(0)}$$

T.Banks and A.Casher (1980)

$$\boxed{\chi_\pi - \chi_\delta = \frac{N_f}{4}\int_0^\infty \mathrm{d}\lambda \frac{4m^2\rho(\lambda,m)}{(\lambda^2+m^2)^2}}$$

- the restoration of $SU(2)_L \times SU(2)_R$ symmetry
  - $\rho(0) = 0$
  - Dilute Instanton gas approximation

Gross, Yaffe & Pisarski, Rev. Mod. Phys. '81

- the restoration of $U(1)_A$ symmetry
  - A gap in the near-zero modes

Cohen, nucl-th/980106

  - At high T, $\rho(\lambda)$ obey the Poisson distribution

Kanazawa and Yamamoto, arXiv: 1508.02416

we aim on determining the mass dependence of $\rho(\lambda,m)$, so we propose to look at the derivatives of $\rho(\lambda,m)$ with respect to the mass

Observables representation in terms of ρ(λ)

$$\boxed{\langle\bar{\psi}\psi\rangle = \frac{N_f}{4}\int_0^\infty d\lambda \frac{2m\rho(\lambda,m)}{\lambda^2+m^2}}$$

$$\frac{\mathrm{d}\langle\bar{\psi}\psi\rangle}{\mathrm{d}m} = \frac{N_f}{4}\int_0^\infty \mathrm{d}\lambda\rho(\lambda,m)\frac{\partial}{\partial m}\left(\frac{2m}{\lambda^2+m^2}\right) + \frac{N_f}{4}\int_0^\infty \mathrm{d}\lambda\frac{2m\frac{\partial\rho(\lambda,m)}{\partial m}}{m^2+\lambda^2}$$
$$= \chi_{\mathrm{con}} + \chi_{\mathrm{disc}}$$

$$\chi_{\mathrm{disc}} = \frac{N_f}{4}\int_0^\infty \mathrm{d}\lambda\frac{2m\frac{\partial\rho(\lambda,m)}{\partial m}}{\lambda^2+m^2}$$

$$\text{part of } \frac{\mathrm{d}\chi_{\mathrm{disc}}}{\mathrm{d}m} = \frac{N_f}{4}\int_0^\infty \mathrm{d}\lambda\frac{2m\frac{\partial^2\rho(\lambda,m)}{\partial m^2}}{\lambda^2+m^2}$$

$\partial\rho(\lambda,m)/\partial m$ and $\partial^2\rho(\lambda,m)/\partial m^2$ are related to χ_{disc} and $\mathrm{d}\chi_{\mathrm{disc}}/\mathrm{d}m$

The derivatives of ρ(λ) with respect to the quark mass

$$\boxed{\frac{\partial \rho(\lambda, m)}{\partial m} = \frac{N_f V}{4} \int_0^\infty \mathrm{d}\lambda' \frac{2m C_2(\lambda, \lambda')}{\lambda'^2 + m^2}}$$

$$\boxed{\begin{aligned} \frac{\partial^2 \rho(\lambda, m)}{\partial m^2} &= \left(\frac{N_f V}{4}\right)^2 \int_0^\infty \mathrm{d}\lambda'' \int_0^\infty \mathrm{d}\lambda' \frac{4m^2 C_3(\lambda, \lambda', \lambda'')}{(\lambda''^2 + m^2)(\lambda'^2 + m^2)} \\ &+ \frac{N_f V}{4} \int_0^\infty \mathrm{d}\lambda' \frac{2(\lambda'^2 - m^2) C_2(\lambda, \lambda')}{(\lambda'^2 + m^2)^2} \end{aligned}}$$

$$C_2(\lambda, \lambda') = \langle \rho_u(\lambda) \rho_u(\lambda') \rangle - \langle \rho_u(\lambda') \rangle \langle \rho_u(\lambda) \rangle$$

$C_3(\lambda, \lambda', \lambda'')$: three-point correlations of ρ(λ) in λ

We establish an analytic relation between the mass derivatives of ρ(λ) to the multi-point correlation functions of ρ(λ) between different λ

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- **Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number**

mode number:

$$\bar{n}[s,t] \approx \frac{1}{N_r}\sum_{r=1}^{N_r}\sum_{j=0}^{p} g_j^p \boxed{\gamma_j} \langle \xi_r^\dagger \boxed{T_j}(A)\xi_r \rangle$$

spectrum:

$$\rho(\lambda,\delta) = \frac{1}{V}\frac{\bar{n}[\lambda-\delta/2, \lambda+\delta/2]}{\delta} \qquad (\lambda \geq \delta/2)$$

T_j : Chebyshev polynomial

γ_j : coefficient

p : polynomial order

Yu Zhang, Lattice 19', arXiv: 2001.05217
Giusti,Luscher, arXiv: 0812.3638
A.Patella, arXiv:1204.4432
Di Napoli et al., arXiv: 1308.4275
Itou et al, arXiv:1411.1155
Fodor et al., arXiv: 1605.08091
Cossu et al., arXiv: 1601.00744

Lattice Setup

- **Actions:**

  Tree level improved gauge action

  Highly improved staggered quark action

- **Lattice size:**

  $N_\tau = 8, \; N_\sigma = 32, 40, 56$

  $N_\tau = 12, N_\sigma = 48, 60, 72$

  $N_\tau = 16, N_\sigma = 64$

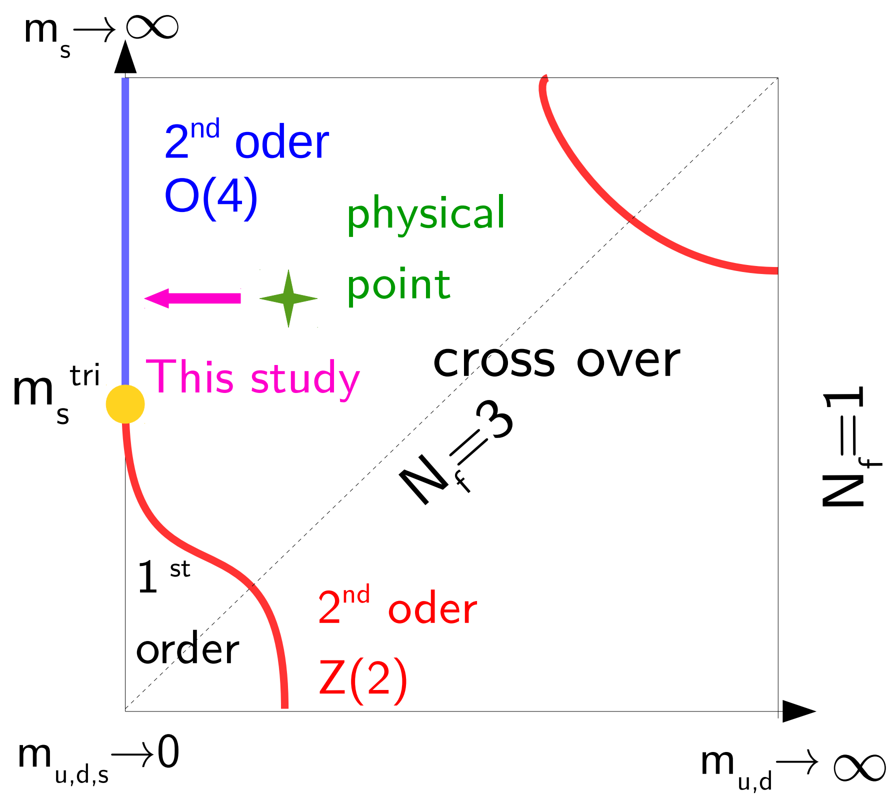
- **Quark mass:**

  m_s : set to its physical value

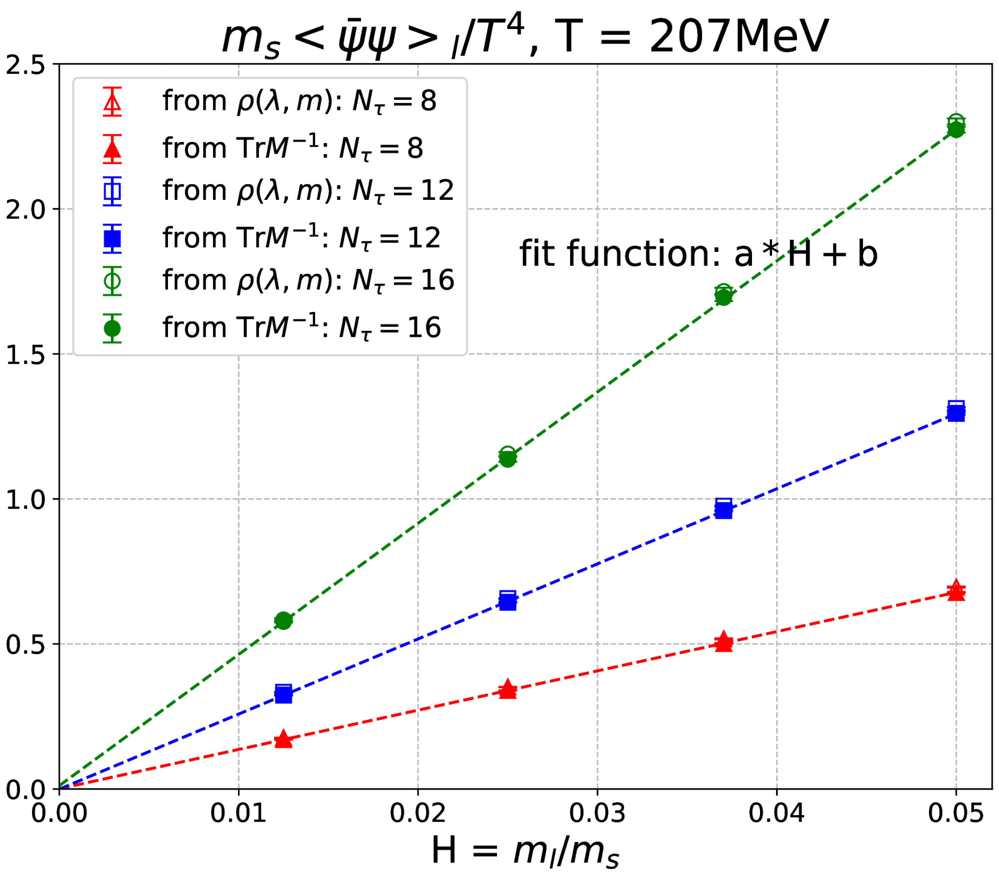
  $m_l/m_s = 1/20, 1/27, 1/40, 1/80$ ($m_\pi = 160, 140, 110, 80$MeV)

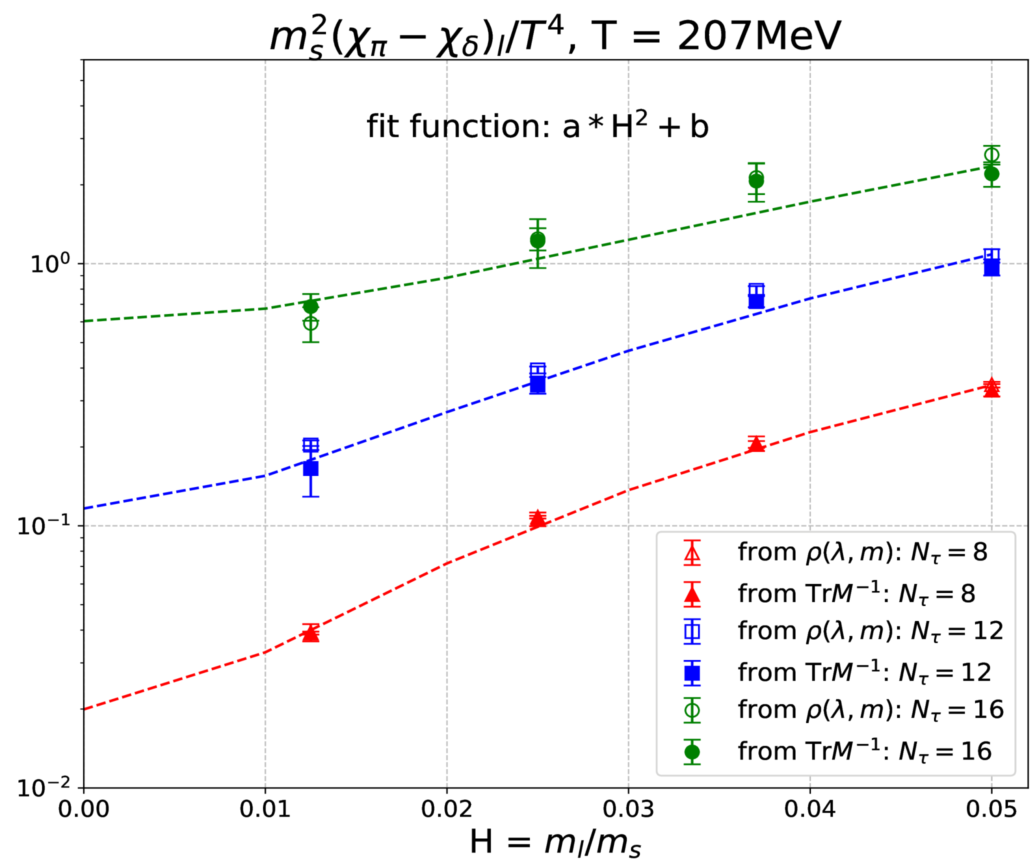
- **Temperature:**

  $T \approx 207$MeV



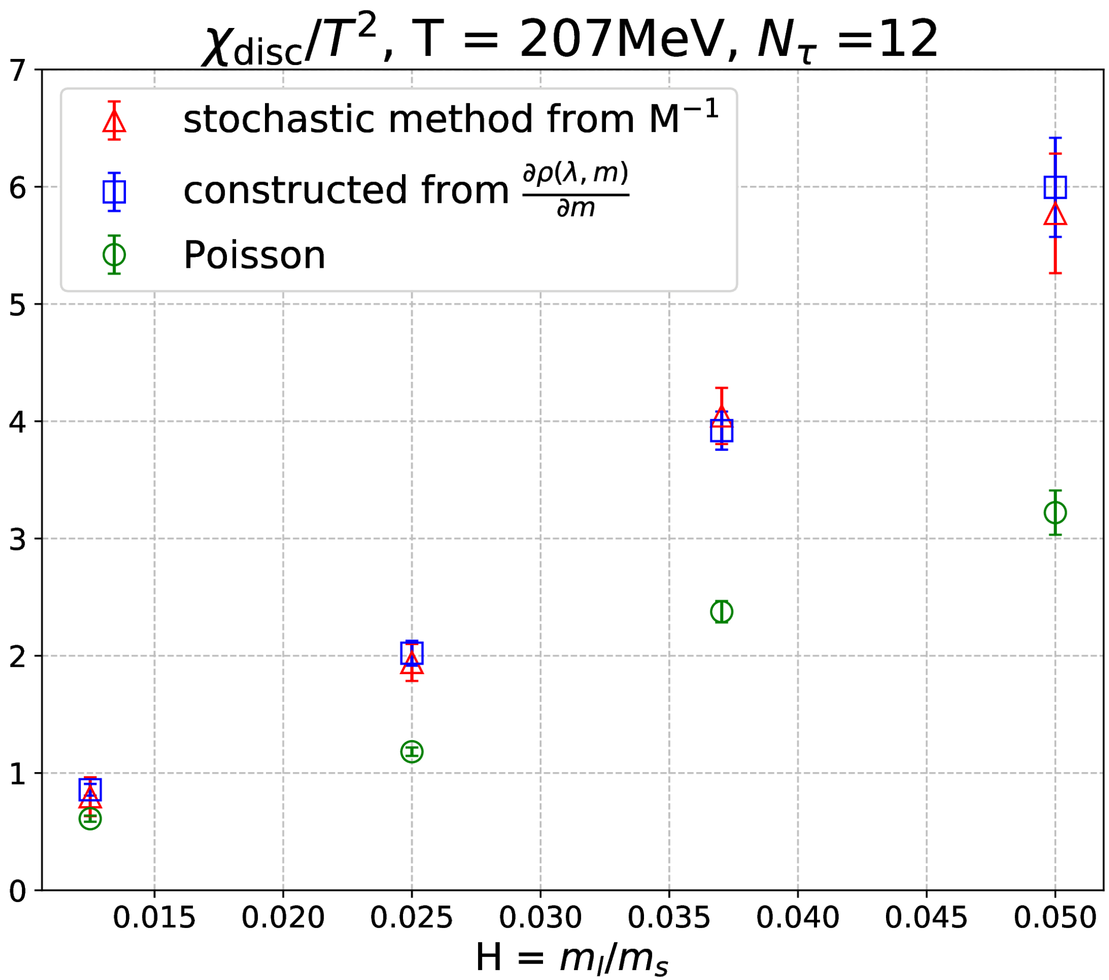
Chiral observables





- $\bar{\psi}\psi$ and $\chi_\pi - \chi_\delta$ can be reproduced well from the eigenvalue spectrum
- In the chiral limit, chiral symmetry is restored but $U(1)_A$ symmetry is not at T≈207MeV
- $\chi_\pi - \chi_\delta$ becomes larger towards continuum limit

disconnected chiral susceptibilities

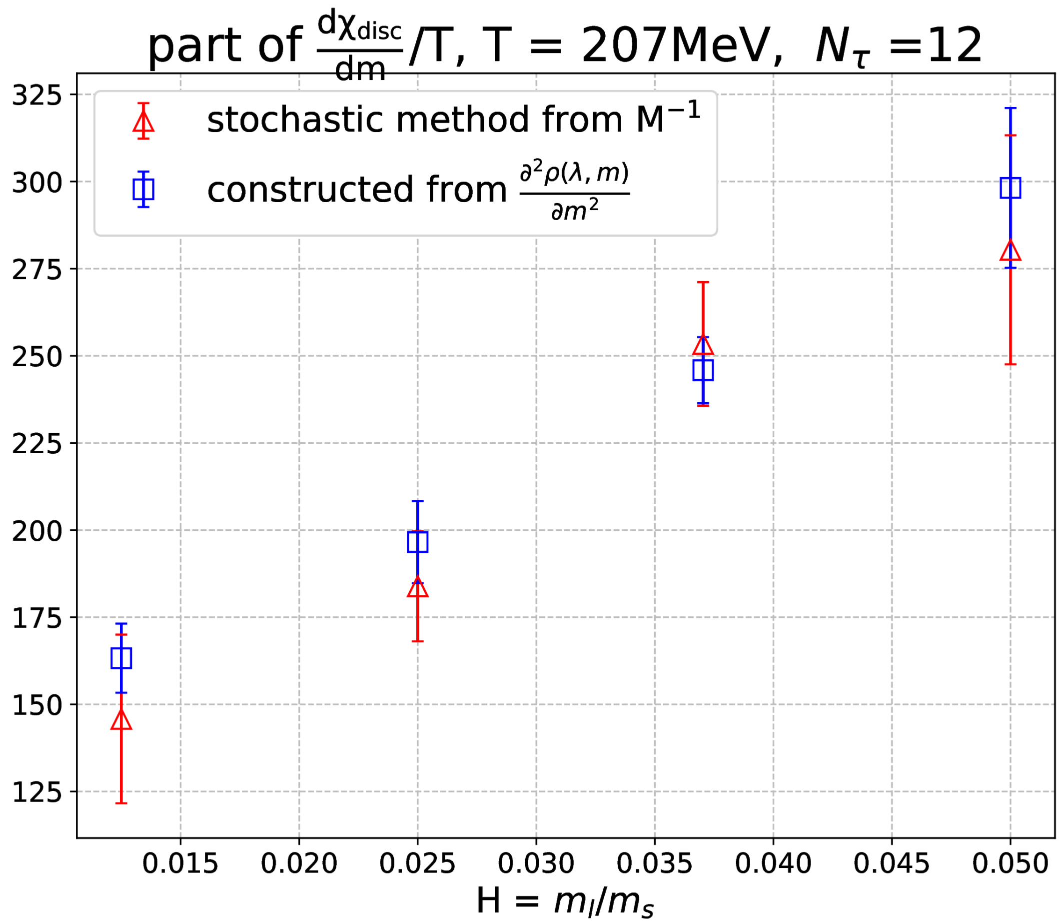


in the chiral limit, $\chi_{\rm disc} \neq 0$

$U(1)_A$ symmetry is not restored

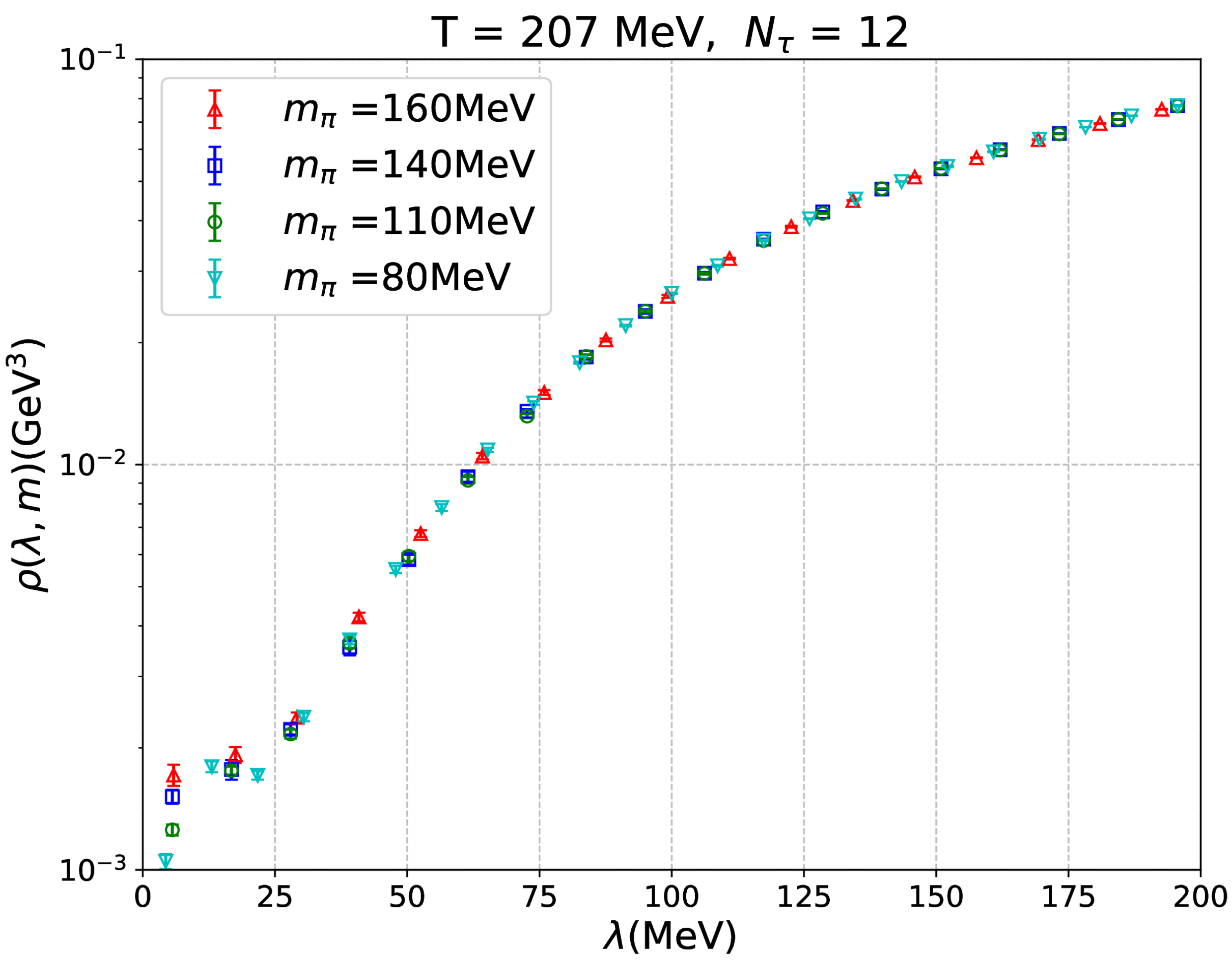
$\chi_{\rm disc}$ can only be reproduced from the correlation function of eigenvalues but not from Poisson distribution

Part of quark mass derivative of the disconnected chiral susceptibilities



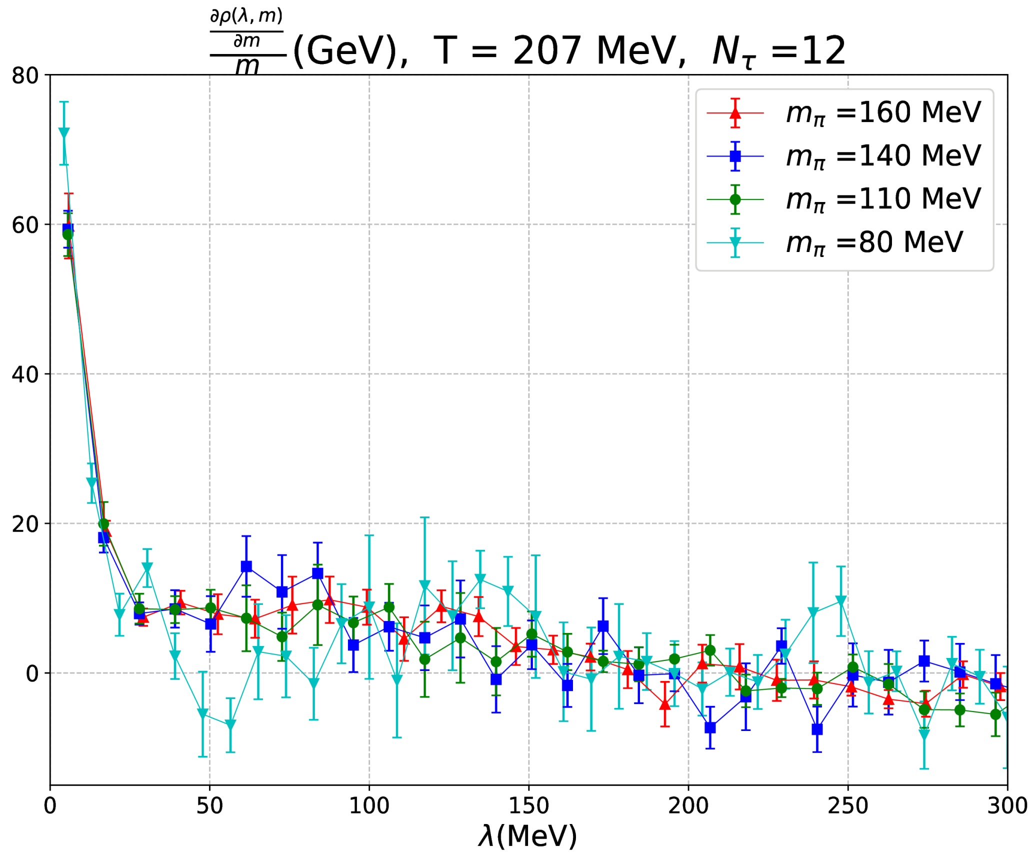
The consistent result from two methods make us confident to extract information on $\frac{\partial^2\rho(\lambda,m)}{\partial m^2}$

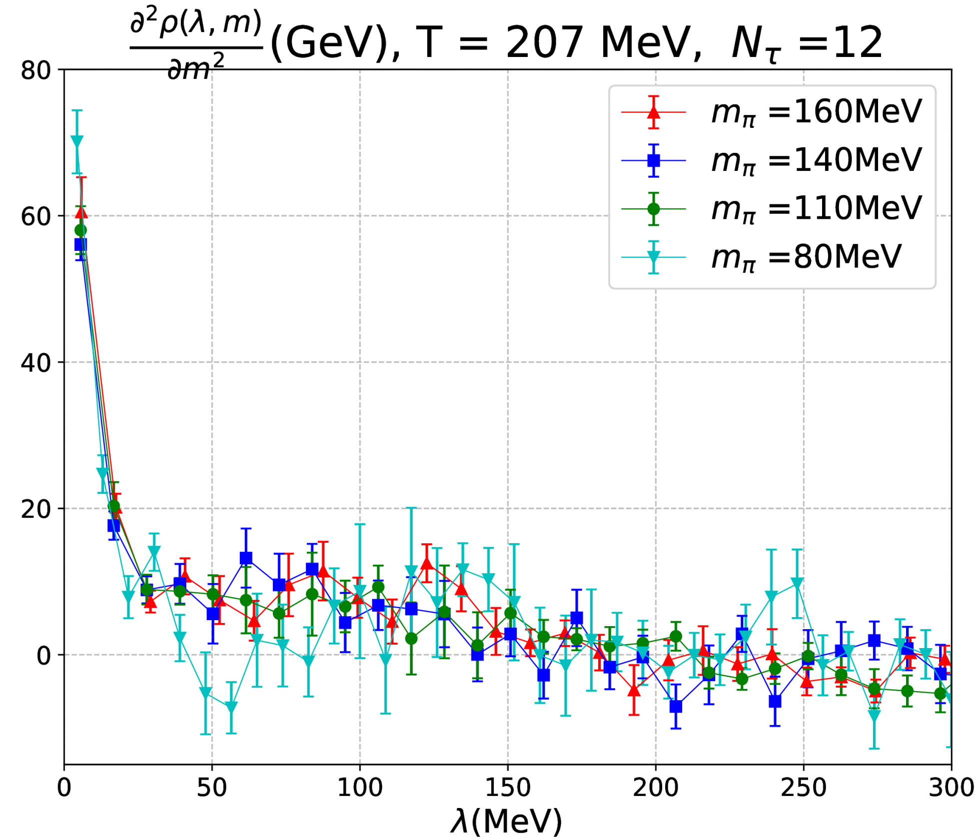
quark mass dependence of ρ(λ)



In the near-zero modes, quark mass dependence is significant

quark mass derivatives of ρ(λ)





$$\frac{\frac{\partial\rho(\lambda,m)}{\partial m}}{m} = \frac{\partial^2\rho(\lambda,m)}{\partial m^2} \Longrightarrow \rho(\lambda,m) \propto m^2$$

a small peak in the near-zero modes, like delta function $\delta(\lambda)$

In the near-zero modes, $\rho(\lambda,m) \approx m^2\delta(\lambda)$ have the same behavior as dilute instanton gas approximation (DIGA)

Summary

- We established the relation between the spectrum derivatives with respect to quark mass and the correlation function of the eigenvalues
- $U(1)_A$ symmetry is still broken at $T \approx 207$MeV as the non-vanishing disconnected susceptibility
- The spectrum does not obey the Poisson distribution
- The spectrum behaves as $m^2\delta(\lambda)$ in the near-zero modes, as indicated from the dilute instanton gas approximation

Outlook

- Check the volume dependence
- Goes to the continuum limit

backup

possible behaviors for ρ(λ,m)

At high T, if ρ(λ,m) obey the Poisson distribution

$$\boxed{C_2(\lambda,\lambda') = \frac{1}{V}\langle\rho_u(\lambda)\rangle\,\delta(\lambda-\lambda') - \frac{1}{N}\langle\rho_u(\lambda)\rangle\,\langle\rho_u(\lambda')\rangle}$$

then

$$\boxed{\chi_{\rm disc} = \frac{N_f}{4}(\chi_\pi - \chi_\delta) - \frac{V}{N}\langle\bar{\psi}\psi\rangle^2}$$

(N is the number of nonzero Dirac eigenvalue pairs)

Poisson distribution

$$
\begin{aligned}
C_2(\lambda,\lambda') &= \langle \rho_u(\lambda)\rho_u(\lambda')\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \sum_{l=1}^{N} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k)\delta(\lambda'-\lambda_k) \right\rangle + (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle \quad (1) \\
&= \boxed{\frac{1}{V}\langle \rho_u(\lambda)\rangle \delta(\lambda-\lambda') + (\frac{1}{V})^2 \left\langle \sum_{k=1}^{N} \delta(\lambda-\lambda_k) \right\rangle \left\langle \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle - \langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle}
\end{aligned}
$$

$$
\frac{1}{V} \left\langle \sum_{l\neq k} \delta(\lambda'-\lambda_l) \right\rangle = \frac{N-1}{N}\langle \rho_u(\lambda')\rangle \qquad (N = V/2)
$$

$$
\boxed{C_2(\lambda,\lambda') = \frac{1}{V}\langle \rho_u(\lambda)\rangle \delta(\lambda-\lambda') - \frac{1}{N}\langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle}
$$

Three point correlations of ρ(λ)

$$
\begin{aligned}
C_3(\lambda, \lambda', \lambda'') = & \Big\langle \rho_u(\lambda)\rho_u(\lambda')\rho_u(\lambda'')\Big\rangle - \langle \rho_u(\lambda)\rangle\langle\rho_u(\lambda')\rho_u(\lambda'')\rangle \\
& - \langle \rho_u(\lambda')\rangle\langle\rho_u(\lambda)\rho_u(\lambda'')\rangle - \langle \rho_u(\lambda'')\rangle\langle\rho_u(\lambda)\rho_u(\lambda')\rangle (1) \\
& + 2\langle \rho_u(\lambda)\rangle\langle \rho_u(\lambda')\rangle\langle \rho_u(\lambda'')\rangle
\end{aligned}
$$

Time history of the topological charge