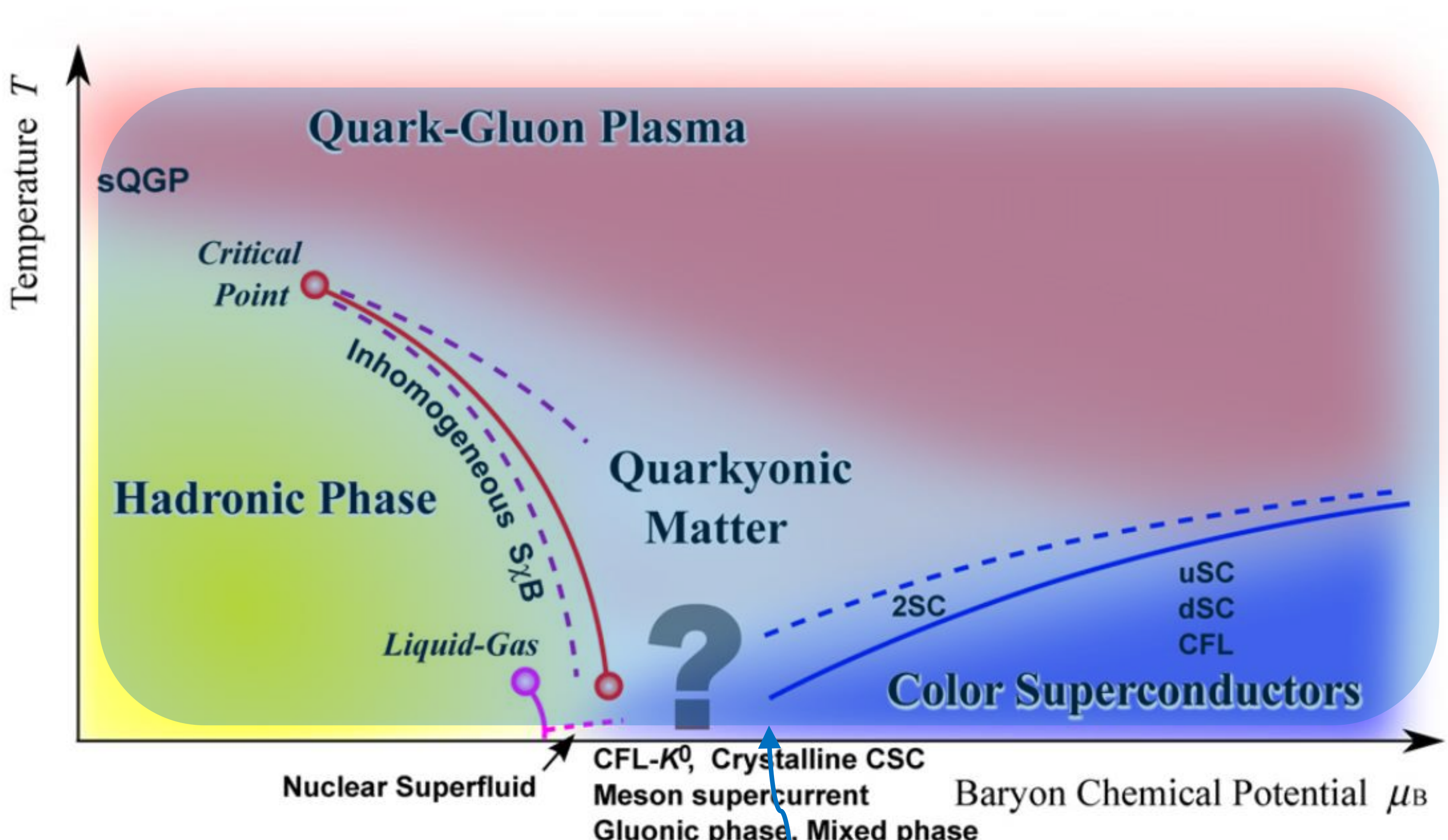
Color superconductivity in lattice QCD

Talk at KEK Theory workshop 2020
Tsukuba, Japan, December 15-18, 2020

Jun Nishimura (KEK & SOKENDAI)

Ref.) Ito, Matsufuru, Namekawa, J.N., Shimasaki, Tsuchiya, Tsutsui, JHEP 10 (2020) 144, 2007.08778 [hep-lat],
Yokota, Asano, Ito, Kaneko, Matsufuru, Namekawa, J.N., Tsuchiya, Tsutsui, work in progress

QCD phase diagram at finite T and μ



Fukushima-Hatsuda RPP(2011)

First principle calculations are difficult due to the sign problem

The sign problem in Monte Carlo methods

- At finite baryon number density,

$$Z = \int dU\, d\Psi\, \mathrm{e}^{-S[U,\Psi]}$$
$$= \int dU\, \mathrm{e}^{-S_\mathrm{g}[U]} \det\mathcal{M}[U]$$

The fermion determinant becomes complex in general.

$$\det\mathcal{M}[U] = |\det\mathcal{M}[U]|\, \mathrm{e}^{i\Gamma[U]}$$

Generate configurations *U* with the probability $\mathrm{e}^{-S_\mathrm{g}[U]}\, |\det\mathcal{M}[U]|$
and calculate

$$\langle \mathcal{O}[U] \rangle = \frac{\langle \mathcal{O}[U]\, \mathrm{e}^{i\Gamma[U]} \rangle_0}{\langle \mathrm{e}^{i\Gamma[U]} \rangle_0}$$ (reweighting)

become exponentially small as the volume increases
due to violent fluctuations of the phase Γ

Number of configurations needed to evaluate <O> increases exponentially.

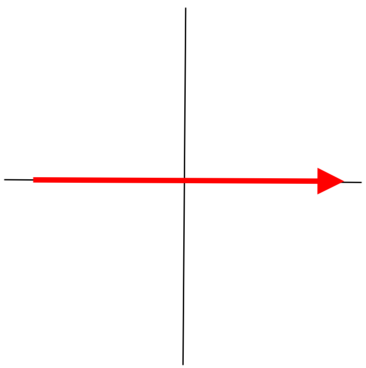
"sign problem"

A new development toward solution to the sign problem   2011～

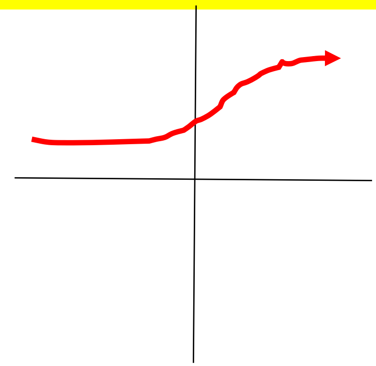
Key : complexification of dynamical variables

The original path integral

$$Z = \int dx \, w(x)$$



The phase of $w(x)$ oscillates violently (sign problem)



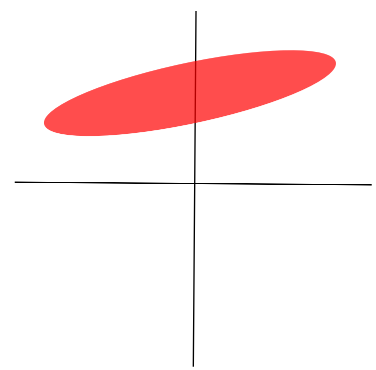
$$Z = \int dz \, w(z)$$

Minimize the sign problem by deforming the integration contour

Lefschetz thimble approach

path optimization method

This talk



An equivalent stochastic process of the complexified variables (no sign problem !)

complex Langevin method

The equivalence to the original path integral holds only under certain conditions.

Plan of the talk

1. Complex Langevin method (CLM)
2. Application to lattice QCD at finite density
3. Color superconductivity on the lattice
4. Summary and future prospects

1. Complex Langevin method

The real Langevin method

$$Z = \int dx\, w(x) \qquad w(x) \geq 0$$

Parisi-Wu ('81)
Damgaard-Huffel ('87)

View this as the stationary distribution of a stochastic process.

Langevin eq. $$\frac{d}{dt}x^{(\eta)}(t) = v(x^{(\eta)}(t)) + \eta(t)$$

"drift term" — Gaussian noise

$$v(x) \equiv \frac{1}{w(x)}\frac{\partial w(x)}{\partial x}$$

$$\lim_{t\to\infty} \langle \mathcal{O}(x^{(\eta)}(t))\rangle_\eta = \frac{1}{Z}\int dx\, \mathcal{O}(x) w(x) \qquad \langle \cdots \rangle_\eta = \frac{\int \mathcal{D}\eta \cdots \mathrm{e}^{-\frac{1}{4}\int dt\, \eta^2(t)}}{\int \mathcal{D}\eta\, \mathrm{e}^{-\frac{1}{4}\int dt\, \eta^2(t)}}$$

Proof

$$= \int dx\, \mathcal{O}(x) P(x,t)$$

$$P(x,t) = \langle \delta(x - x^{(\eta)}(t))\rangle_\eta$$

Probability distribution of $x^{(\eta)}(t)$

Fokker-Planck eq. $$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} - \frac{1}{w(x)}\frac{\partial w(x)}{\partial x}\right) P \qquad \lim_{t\to\infty} P(x,t) = \frac{1}{Z} w(x)$$

The complex Langevin method

Parisi ('83), Klauder ('83)

$$Z = \int dx\, w(x)$$

complex

$v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$ also becomes complex.

Complexify the dynamical variables, and consider their (fictitious) time evolution :

$$z^{(\eta)}(t) = x^{(\eta)}(t) + i\, y^{(\eta)}(t)$$

defined by the complex Langevin equation

$$\frac{d}{dt} z^{(\eta)}(t) = v(z^{(\eta)}(t)) + \eta(t)$$

Gaussian noise (real)

probability $\propto e^{-\frac{1}{4}\int dt\, \eta(t)^2}$

$$\lim_{t\to\infty} \langle \mathcal{O}(z^{(\eta)}(t)) \rangle_\eta \stackrel{?}{=} \frac{1}{Z} \int dx\, \mathcal{O}(x) w(x)$$

Rem 1 : When $w(x)$ is real positive, it reduces to the real Langevin method.

Rem 2 : The drift term $v(x) \equiv \frac{1}{w(x)} \frac{\partial w(x)}{\partial x}$ and the observables $\mathcal{O}(x)$ should be evaluated for complexified variables by analytic continuation.

2. Application to lattice QCD at low temperature and high density

Ref.) Ito, Matsufuru, Namekawa, J.N., Shimasaki, Tsuchiya, Tsutsui,
JHEP 10 (2020) 144, 2007.08778 [hep-lat]

complex Langevin method for finite density QCD

$$w(U) = \mathrm{e}^{-S_{\mathrm{plaq}}[U]} \det M[U]$$ ← complex !

$$S_{\mathrm{plaq}}(U) = -\beta \sum_n \sum_{\mu\neq\nu} \mathrm{tr}\,(U_{n\mu}U_{n+\hat{\mu},\nu}U^{-1}_{n+\hat{\nu},\mu}U^{-1}_{n\nu})$$

$$v_{an\mu}(U) = \frac{1}{w(U)} D_{an\mu} w(U)$$

$$D_{an\mu} f(U) = \frac{\partial}{\partial x} f(\mathrm{e}^{ixt_a} U_{n\mu})\Big|_{x=0}$$

t_a: generators of SU(3)

Complexification of dynamical variables : $U_{n\mu} \mapsto \mathcal{U}_{n\mu} \in \mathrm{SL}(3,\mathbb{C})$

Discretized version of complex Langevin eq.

$$\mathcal{U}^{(\eta)}_{n\mu}(t+\epsilon) = \exp\left\{ i \sum_a \left(\epsilon\, v_{an\mu}(\mathcal{U}) + \sqrt{\epsilon}\, \eta_{an\mu}(t) \right) t_a \right\} \mathcal{U}^{(\eta)}_{n\mu}(t)$$

The drift term can become large when :

1) link variables $\mathcal{U}_{n\mu}$ become far from unitary (excursion problem)

"gauge cooling"

Seiler-Sexty-Stamatescu, PLB 723 (2013) 213

Nagata-J.N.-Shimasaki, Phys.Rev. D94 (2016) no.11, 114515

2) $M[\mathcal{U}]$ has eigenvalues close to zero (singular drift problem)

Rem.) The fermion determinant gives rise to a drift $\mathrm{tr}\,(M[\mathcal{U}]^{-1}\mathcal{D}_{an\mu}M[\mathcal{U}])$

Mollgaard-Splittorff, Phys.Rev. D88 (2013) no.11, 116007

J.N.-Shimasaki, Phys.Rev. D92 (2015) no.1, 011501

Simulation setup

Ito, Matsufuru, Namekawa, J.N., Shimasaki, Tsuchiya, Tsutsui, JHEP 10 (2020) 144, 2007.08778 [hep-lat]

- lattice size : $8^3 \times 16,\ 16^3 \times 32$
- plaquette action with $\beta = 5.7$
- staggered fermion (4 quark flavors)
- CLM is valid for:
  $5.2 \le \mu_q/T \le 7.2 \quad (8^3 \times 16)$
  $1.6 \le \mu_q/T \le 9.6 \quad (16^3 \times 32)$
- quark mass : $ma = 0.01$
- total Langevin time $= 70 \sim 140 \quad (8^3 \times 16)$
  $10 \sim 20 \quad (16^3 \times 32)$
  with stepsize $\epsilon = 10^{-4}$

$$a = 0.042\text{fm}$$

Determined from Sommer scale by HMC at $\mu_q = 0$ on $24^3 \times 48$ lattice

Note :	spatial extent	temperature (T)
$8^3 \times 16$	0.042fm × 8 = 0.34 fm	290 MeV
$16^3 \times 32$	0.042fm × 16 = 0.68 fm	145 MeV
	cf) $\Lambda_{\rm LQCD}^{-1} \sim 1\text{fm}$	$T_c \sim 170\text{MeV}$

Previous study on a $4^3 \times 8$ lattice
Nagata, J.N., Shimasaki, Phys.Rev. D98 (2018) no.11, 114513, 1805.03964 [hep-lat]

Histogram of the drift term

$8^3 \times 16, \quad N_f = 4$

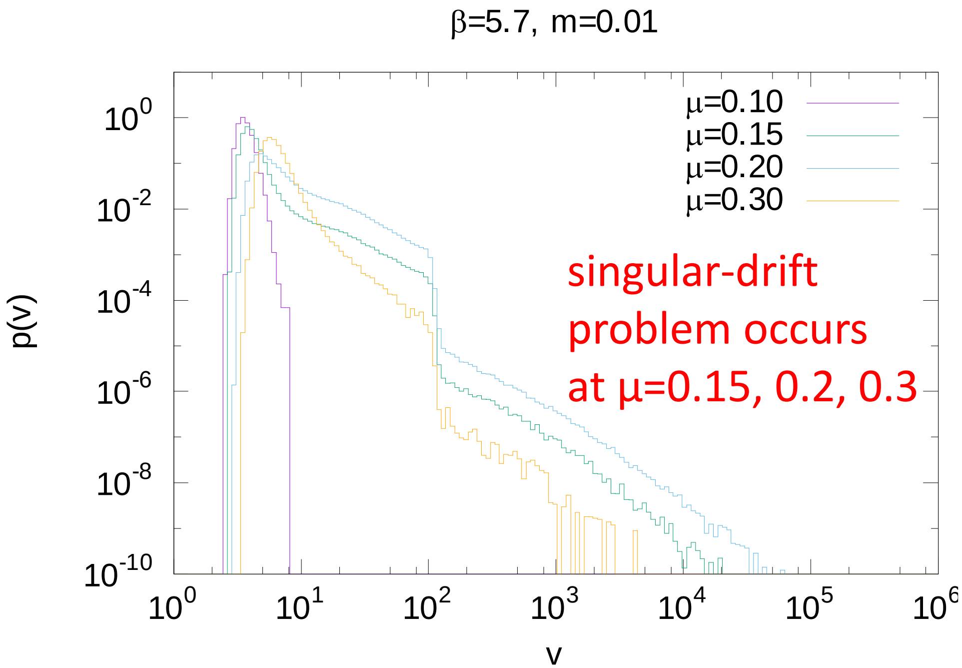
$\beta = 5.7$

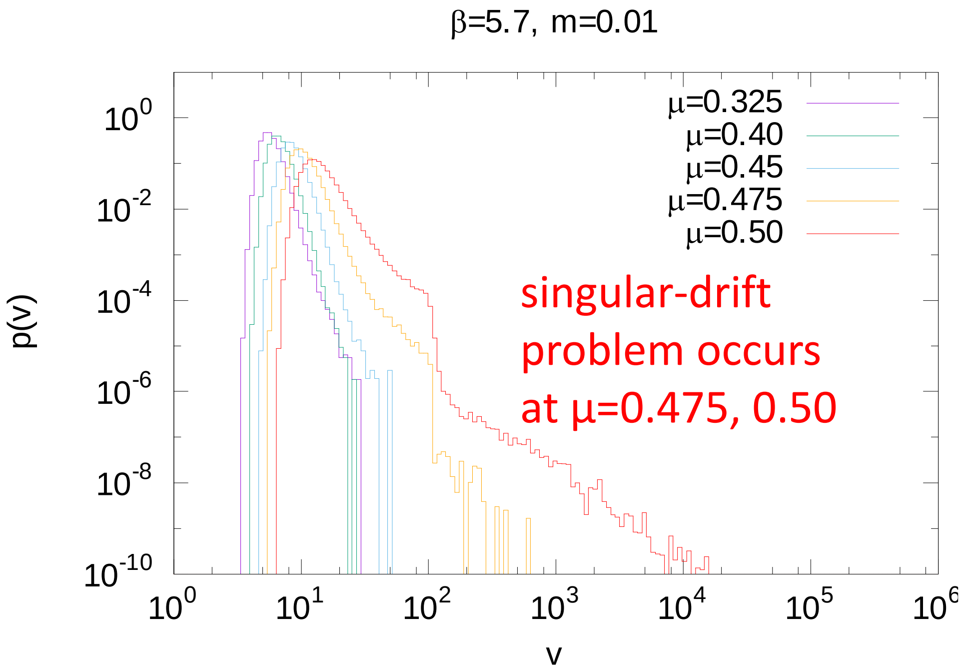
$m = 0.01$

Reliable regions :

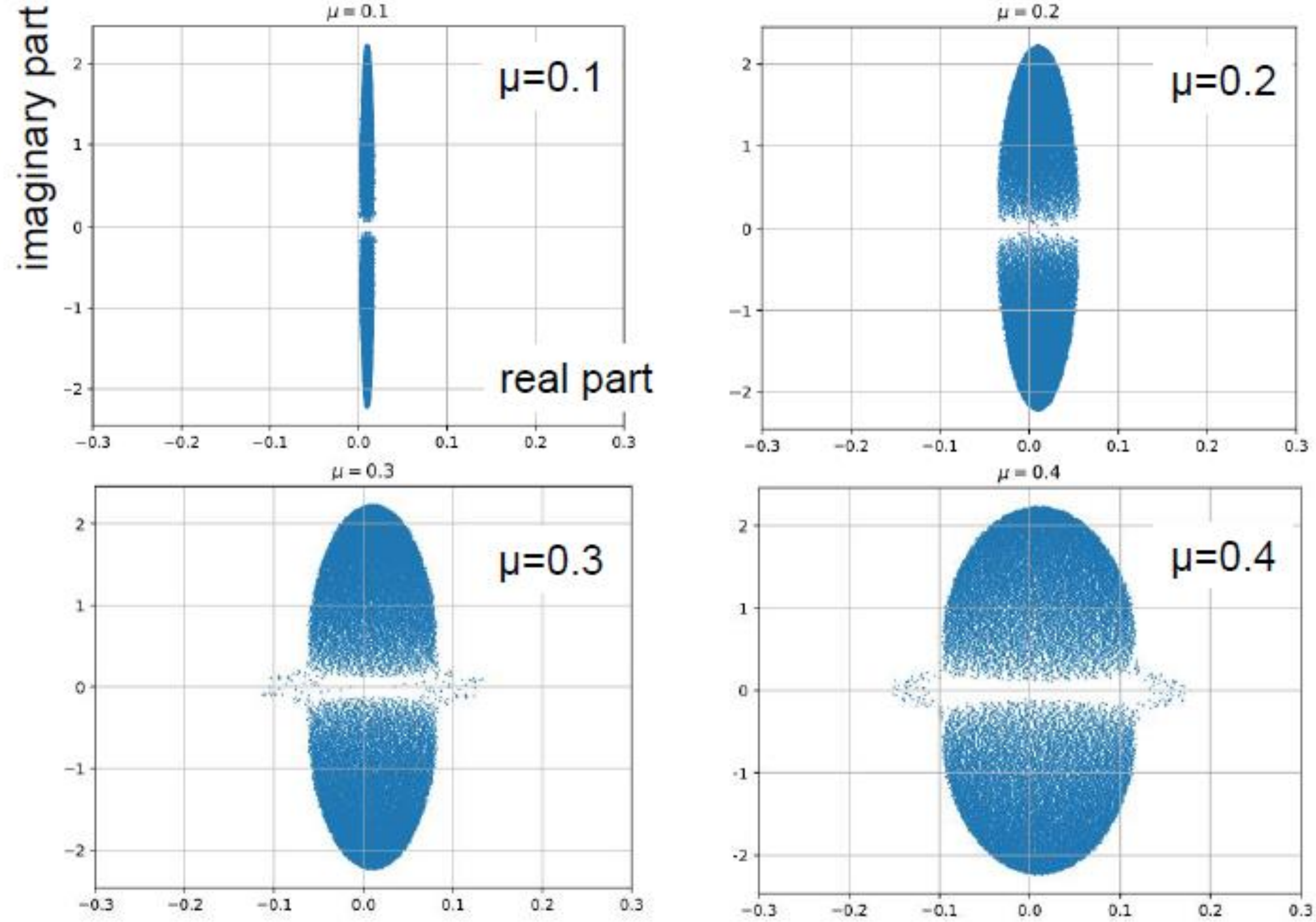
$\mu_q = 0.1$

$0.325 \leq \mu_q \leq 0.45$





Eigenvalue distribution of (D+m)

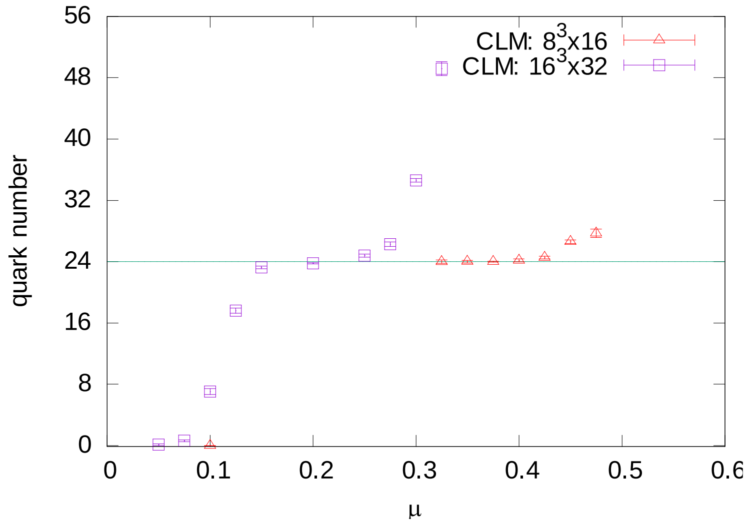


We always have a gap due to finite spatial volume effects, but still the singular-drift problem occurs at $\mu_q = 0.2$ and $\mu_q = 0.3$ possibly due to large fluctuation associated with the phase transition.

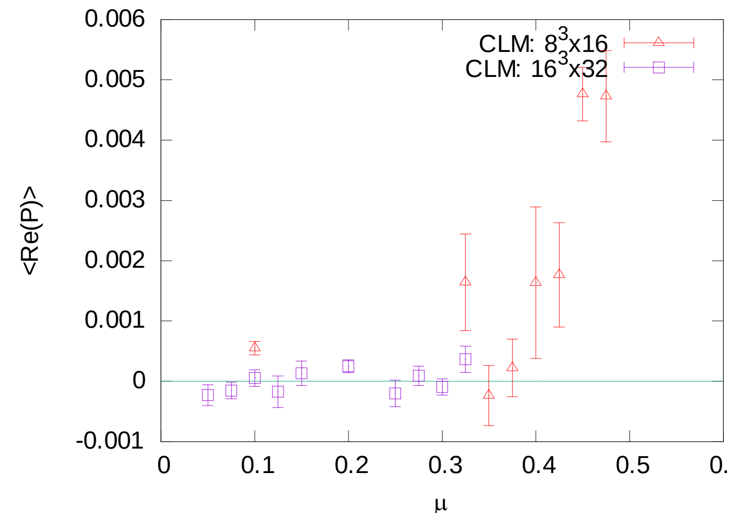
Results for various observables

Ito, Matsufuru, Namekawa, J.N., Shimasaki, Tsuchiya, Tsutsui,
JHEP 10 (2020) 144, 2007.08778[hep-lat]

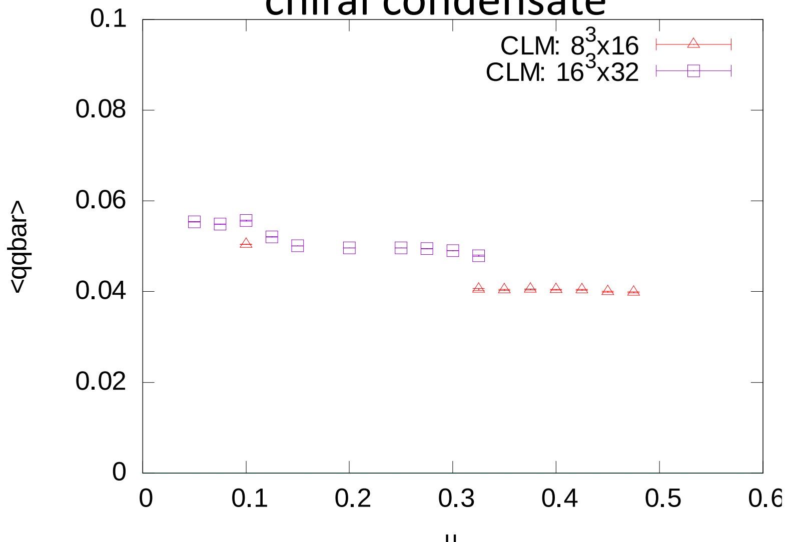
quark number



Polyakov line



chiral condensate



- The plateau appears with the height 24 for the quark number.
- The plateau shifts to the left on the larger lattice.
- Polyakov line remains small for all the values of μ . ("low T behaviors")

Interpretation of the plateau

Note : the spatial extent of our lattice : 0.042fm × 8 = 0.34 fm
0.042fm × 16 = 0.68 fm

Still too small to form a baryon.
(The nuclear matter phase does not exist in this setup.)

Formation of the Fermi sphere ?

$$E = \sqrt{\mathbf{p}^2 + m^2} \sim \frac{2\pi}{L}|\mathbf{n}| = \begin{cases} 0 & \mathbf{n} = (0,0,0) \\ \frac{2\pi}{L} & \mathbf{n} = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \\ \vdots & \end{cases}$$

$$\frac{2\pi}{L} = \begin{cases} 0.79 & (L = 8) \\ 0.39 & (L = 16) \end{cases}$$

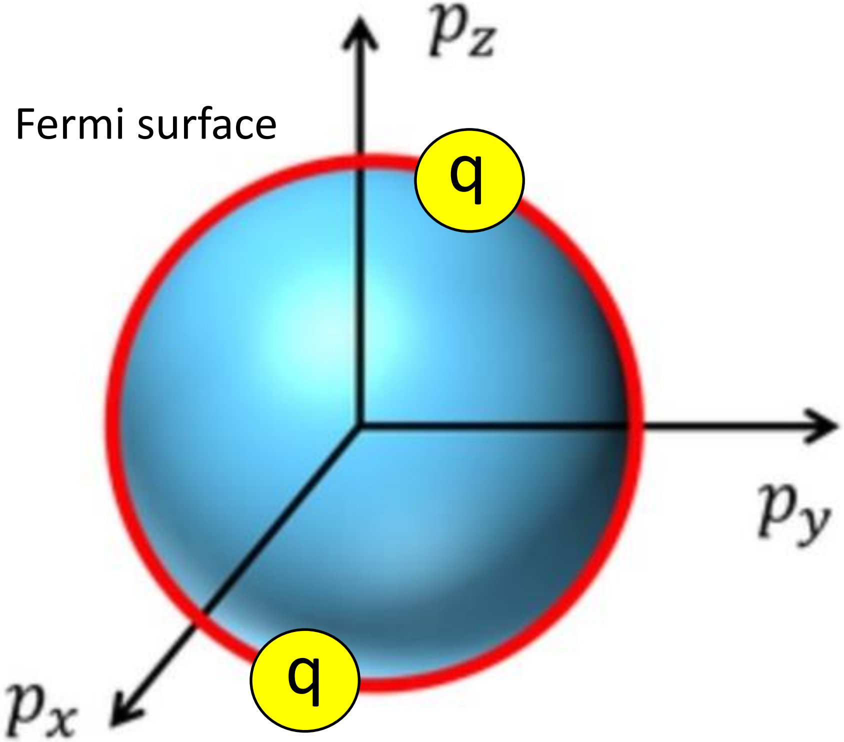
For $\mu_q \lesssim \frac{2\pi}{L}$, only zero modes condense.

height of the plateau $= 3 \times 4 \times 2 = 24$
(color × flavor × spin)

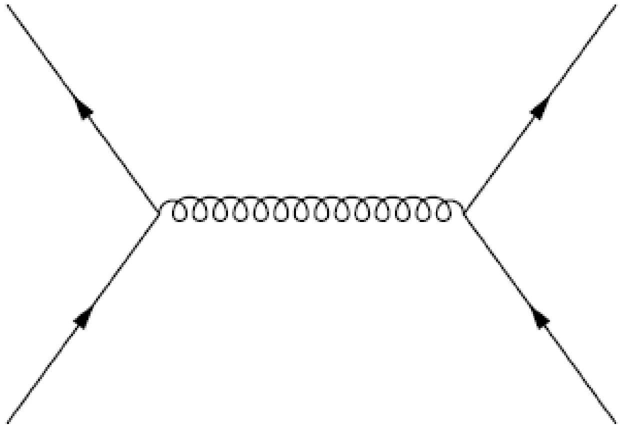
At $\mu_q \sim \frac{2\pi}{L}$, the 2nd plateau with the height $24 \times (1 + 6) = 168$ should appear.

Can we see color superconductivity ?

Barrois NPB (1977), Frautschi (1978),
Bailin, Love, PR (1984),
Alford, Rajagopal, Wilczek, PLB (1998),
Rapp, Schäfer, Shurayk, Velkovsky, PRL(1998)



Cooper pairing ?



attractive in color anti-triplet channel

3 Color superconductivity on the lattice

Ref.) Yokota, Asano, Ito, Kaneko, Matsufuru, Namekawa, J.N., Tsuchiya, Tsutsui,
work in progress

Let's start with a perturbative regime!

Convensionally, color superconductivity has been studied by considering QCD at very high density. $(\mu_q \gtrsim 100\,\mathrm{TeV})$

Quarks come closer, and the effective coupling becomes weak due to asymptotic freedom.

Instead, we consider QCD in a very small box.

Perturbative calculations become valid even at small μ_q.
Nonperturbative effects can be incorporated just by increasing the volume. (suitable for lattice calculations)

c.f.) dense quark matter in a very small box

- NJL model:
  Amore, Birse, McGovern, Walet, PRD (2002), Hands, Walters, PLB (2002)
- Two-color QCD:
  Hands, Hollowood, Myers, JHEP (2010)
- QCD on $S^1 \times S^3$:
  Hands, Hollowood, Myers, JHEP (2010)

To our knowledge, this is the first study of color superconductivity in lattice QCD using perturbation theory and the complex Langevin simulation.

Gap equation for fermion bilinear condensation

$$S_{\text{kin}} = \sum_x \bar{\psi}(x) D(m,\mu) \psi(x) = \sum_x \bar{\Psi}(x) \mathbf{D} \Psi(x)$$

In order to discuss the situation in which fermion number is not conserved,

$$\langle \psi(x)\psi(y) \rangle \neq 0$$

we use Nambu-Gor'kov representation

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \bar{\psi}(x) \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} D(m,\mu) & 0 \\ 0 & D(-m,-\mu) \end{pmatrix}$$

$$\bar{\Psi}(x) = (\bar{\psi}(x), \psi(x))$$

full propagator

$$\begin{aligned} S(x,y) &= \langle \Psi(x) \bar{\Psi}(y) \rangle \\ &= \frac{1}{\mathbf{D} + \Sigma} \end{aligned} \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

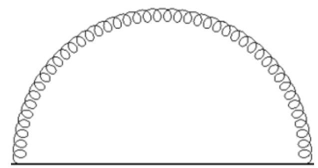
Fermion bilinear condensation implies : $\Sigma_{12}, \ \Sigma_{21} \neq 0$

Bailin, Love PR(1984), …

Gap equation for fermion bilinear condensation (cont'd)

Gap eq. (at the leading order of perturbation theory)

$$\Sigma =$$



$$S = (\mathbf{D} + \Sigma)^{-1}$$

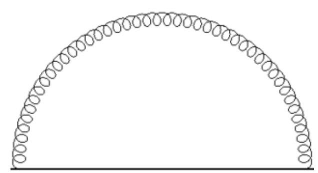
$$\Sigma = \begin{pmatrix} 0 & \Sigma_{12} \\ \Sigma_{21} & 0 \end{pmatrix}$$

At the transition point,

$$\Sigma_{12},\ \Sigma_{21} \ll \Lambda_{\mathrm{QCD}}$$

(continuous transition assumed)

$$\Sigma_{12} =$$



$$S_{12} \sim D_{11}^{-1} \Sigma_{12} D_{22}^{-1}$$

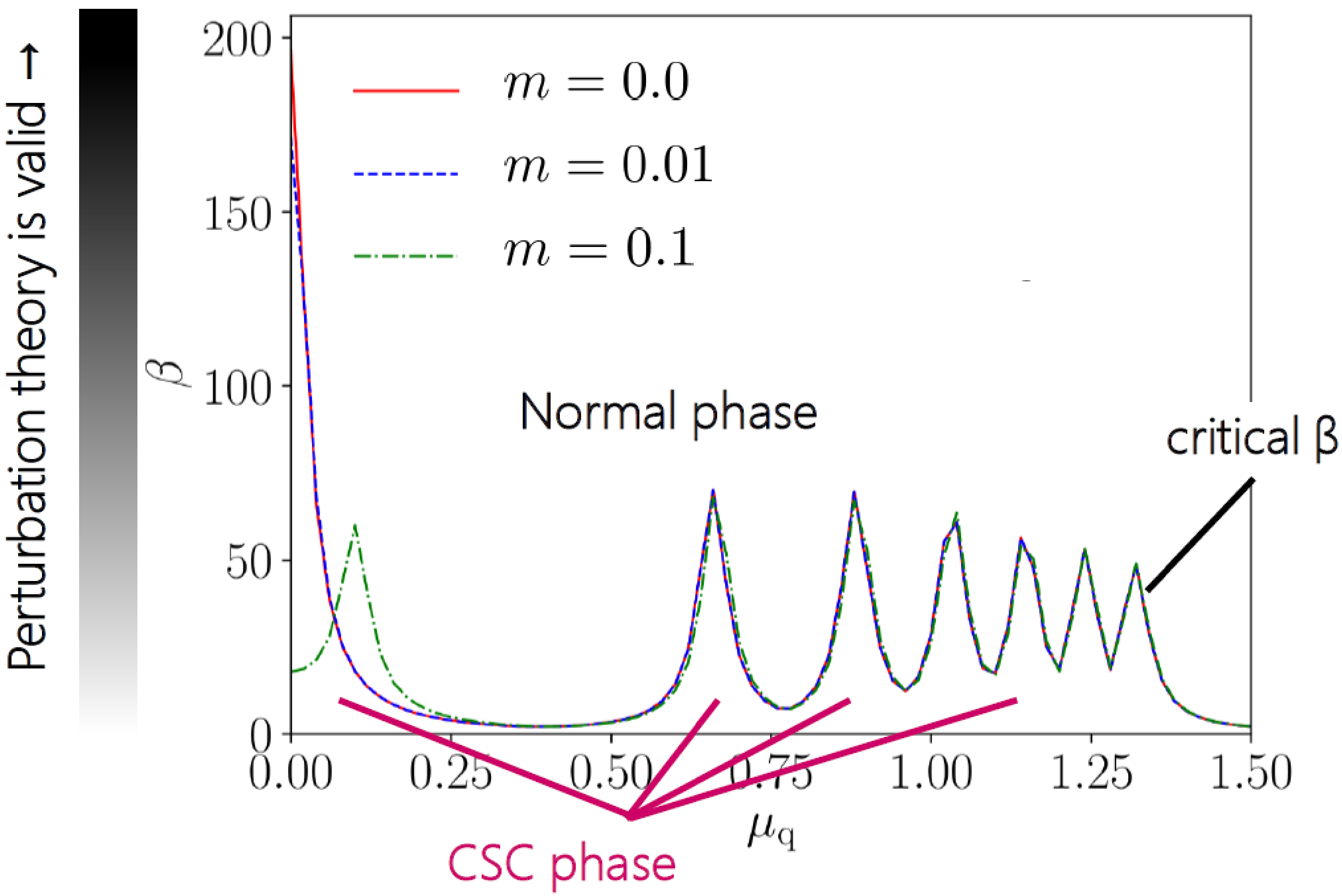
Gap eq. is reduced to a linear eq.: $\mathcal{M}\Sigma_{12} = 0$

$\Sigma_{12} \neq 0$ requires $\det\mathcal{M} = 0$

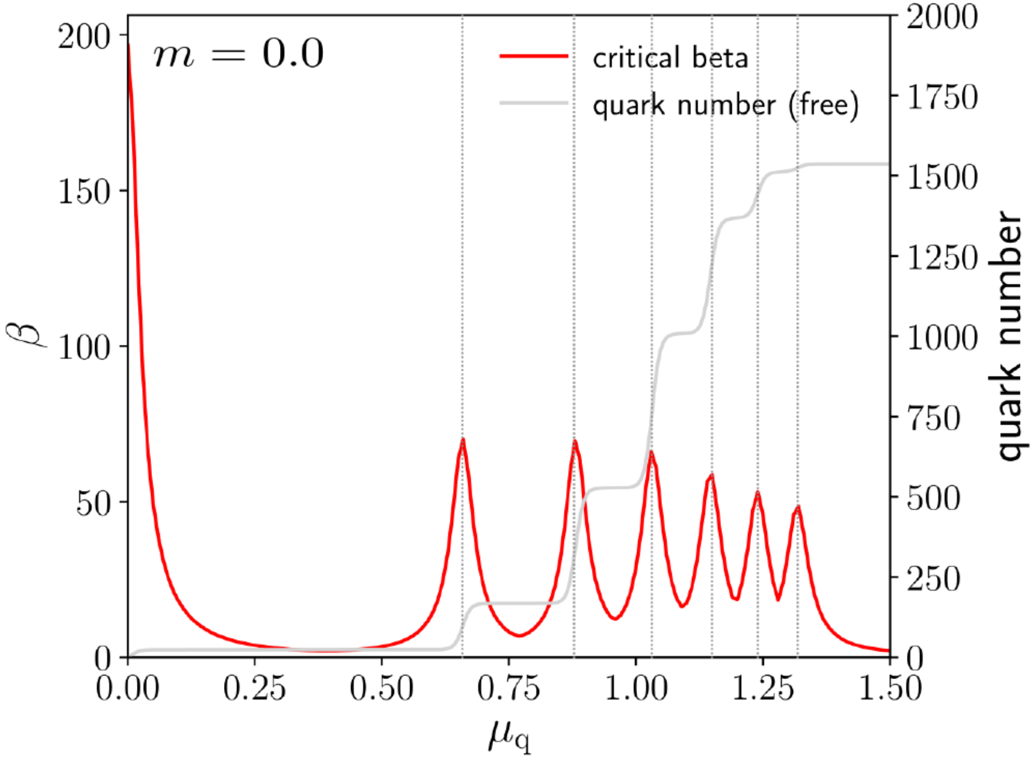
("Thouless criterion" known in a more general context)

Thouless, AP (1960)

Results for lattice QCD with staggered fermions on a $8^3 \times 128$ lattice



The appearance of the fermion bilinear condensates at discrete values of μ_q



When the Fermi surface crosses discrete points in the momentum space, those modes can form a Cooper pair without costing extra energy.

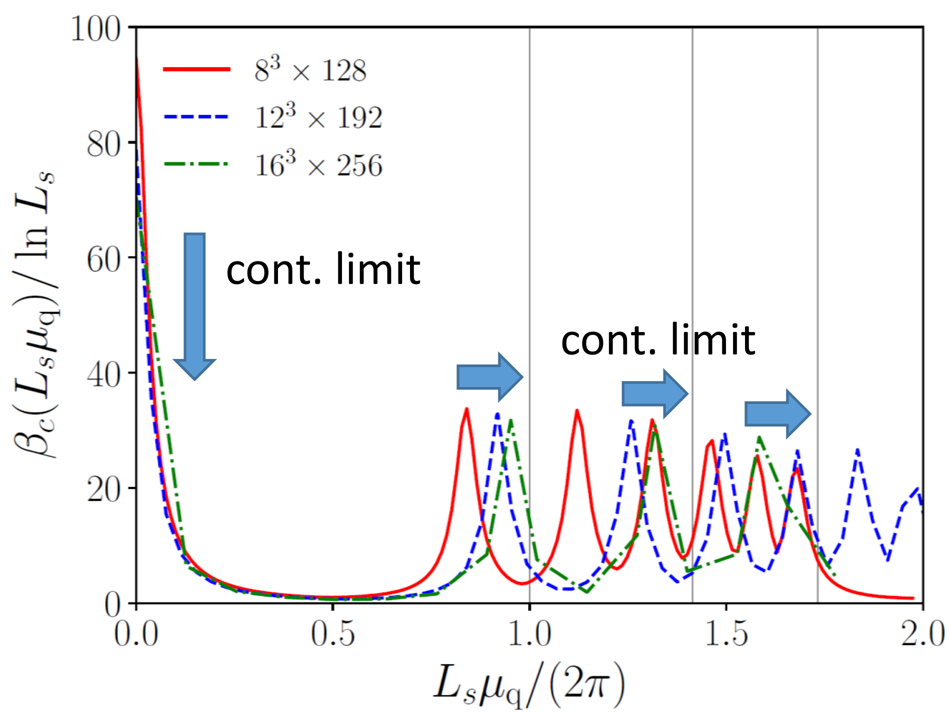
Cf) similar observation in NJL model:
Amore, Birse, McGovern, Walet, PRD (2002)

Critical line in the continuum limit with a fixed physical volume

cont. limit : $a \to 0$

$$aL_s = 1 \quad \text{(fixed)}$$
$$aL_t = T^{-1} \quad \text{(fixed)}$$
$$m = 0$$

$$\begin{cases} \tilde{\mu} = a^{-1}\mu_q = L_s\mu_q \\ \beta \propto \log \frac{1}{a} = \log L_s \end{cases}$$



The 1st peak at $\mu_q = 0$ disappears in the cont. limit.

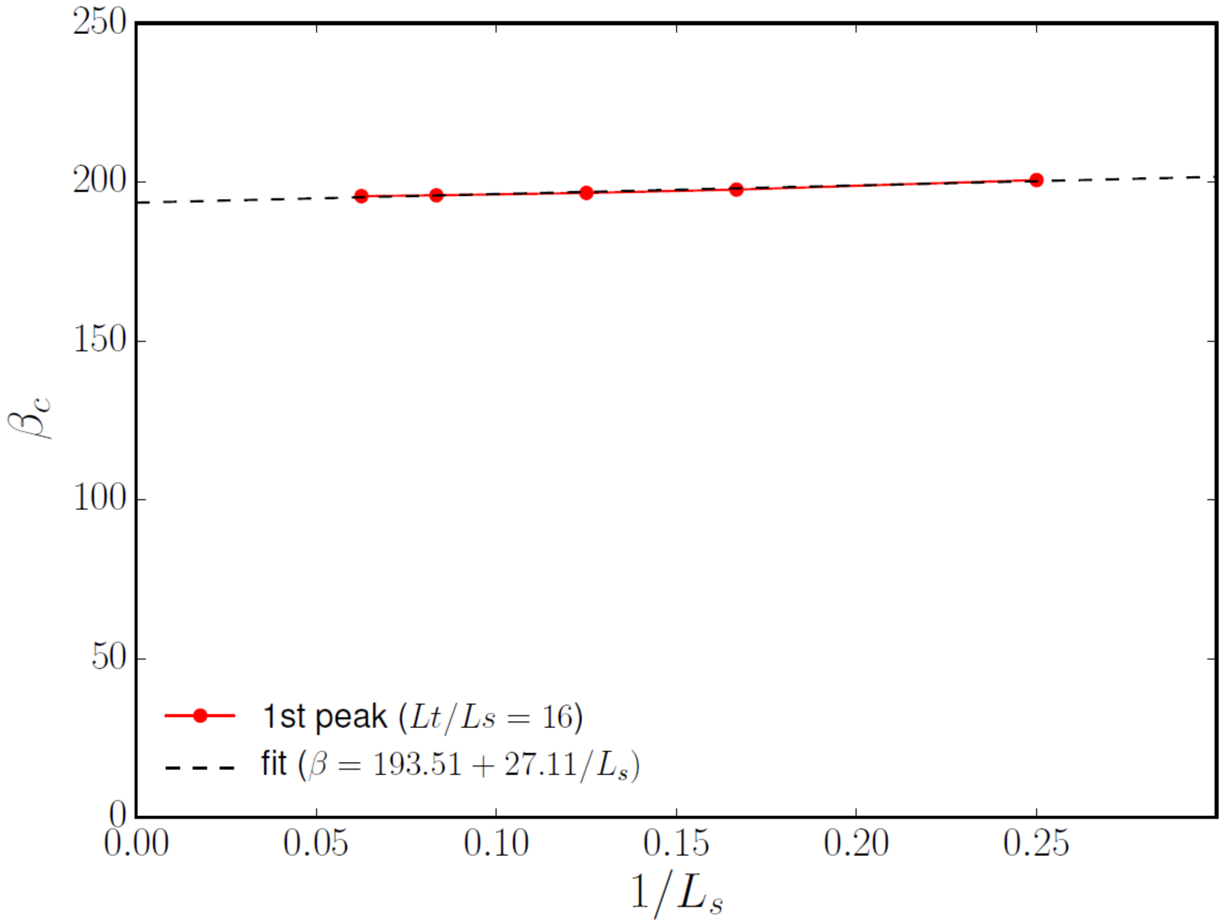
(See next page.)

Zero momentum modes cannot form a Cooper pair.

The other peaks survive and approach $1, \ \sqrt{2}, \ \sqrt{3}, \ 2, \cdots$ in the cont. limit.

$$\mathbf{p} = (1,0,0), \ (1,1,0), \ (1,1,1), \ (2,0,0), \cdots$$

The 1st peak corresponding to zero momentum pairing

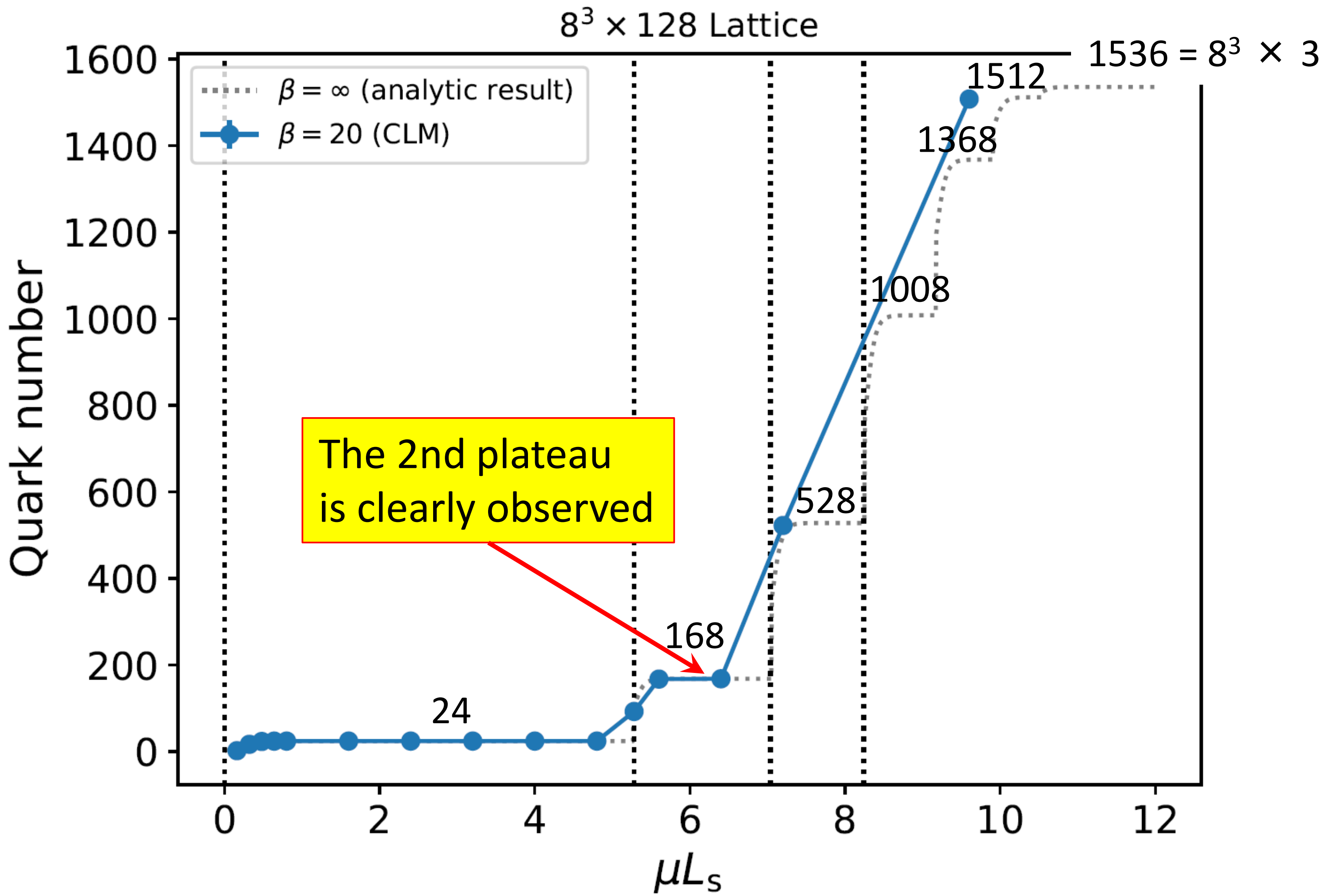


Disappears in the cont. limit $(\beta \sim \log L_s)$

Note: The other peaks grow as $\propto \log L_s$

CL simulations at large β (preliminary)

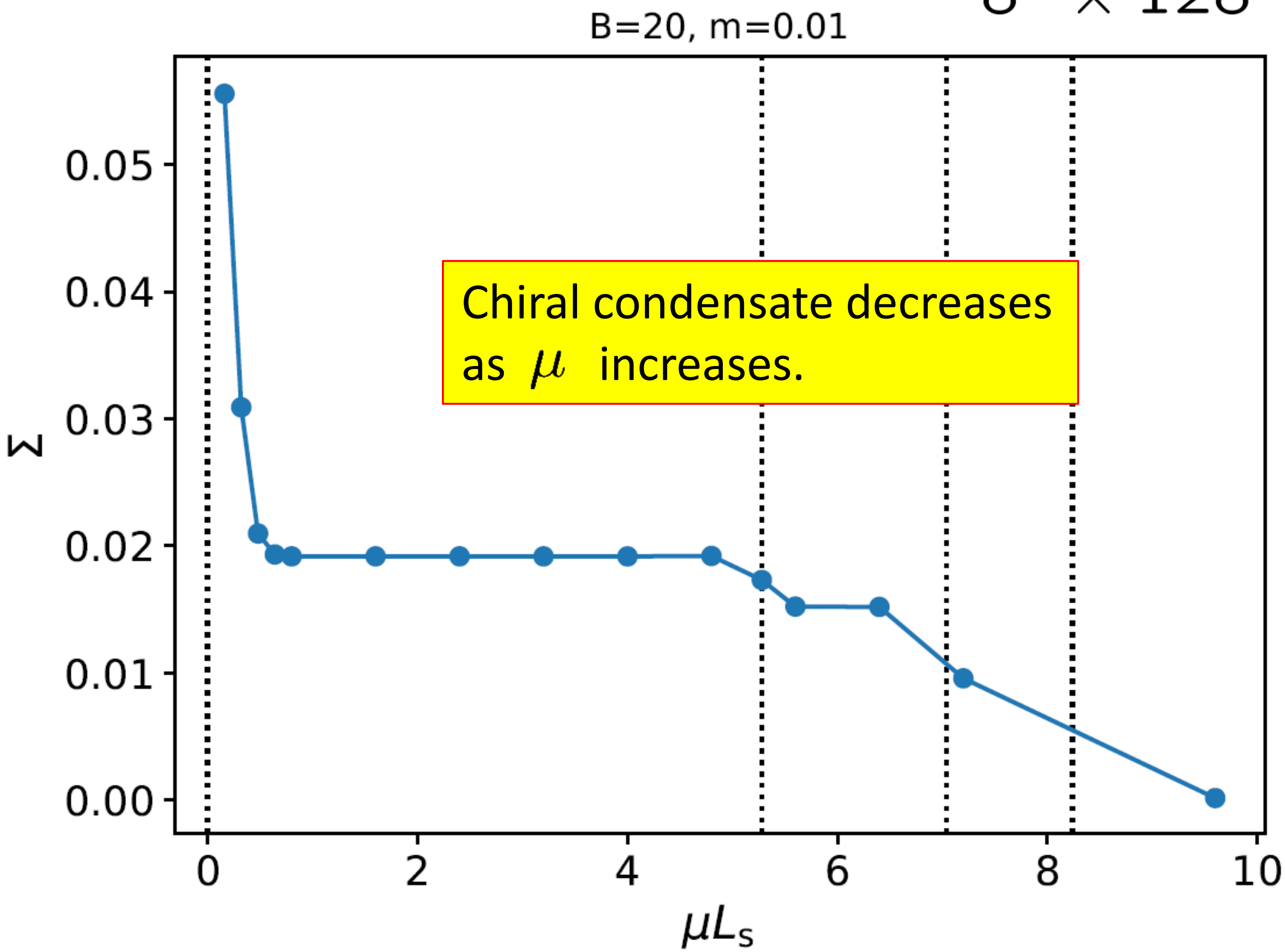
quark number $\beta = 20,\ m = 0.01$



CL simulations at large β (preliminary)

chiral condensate

$\beta = 20,\ m = 0.01$
$8^3 \times 128$



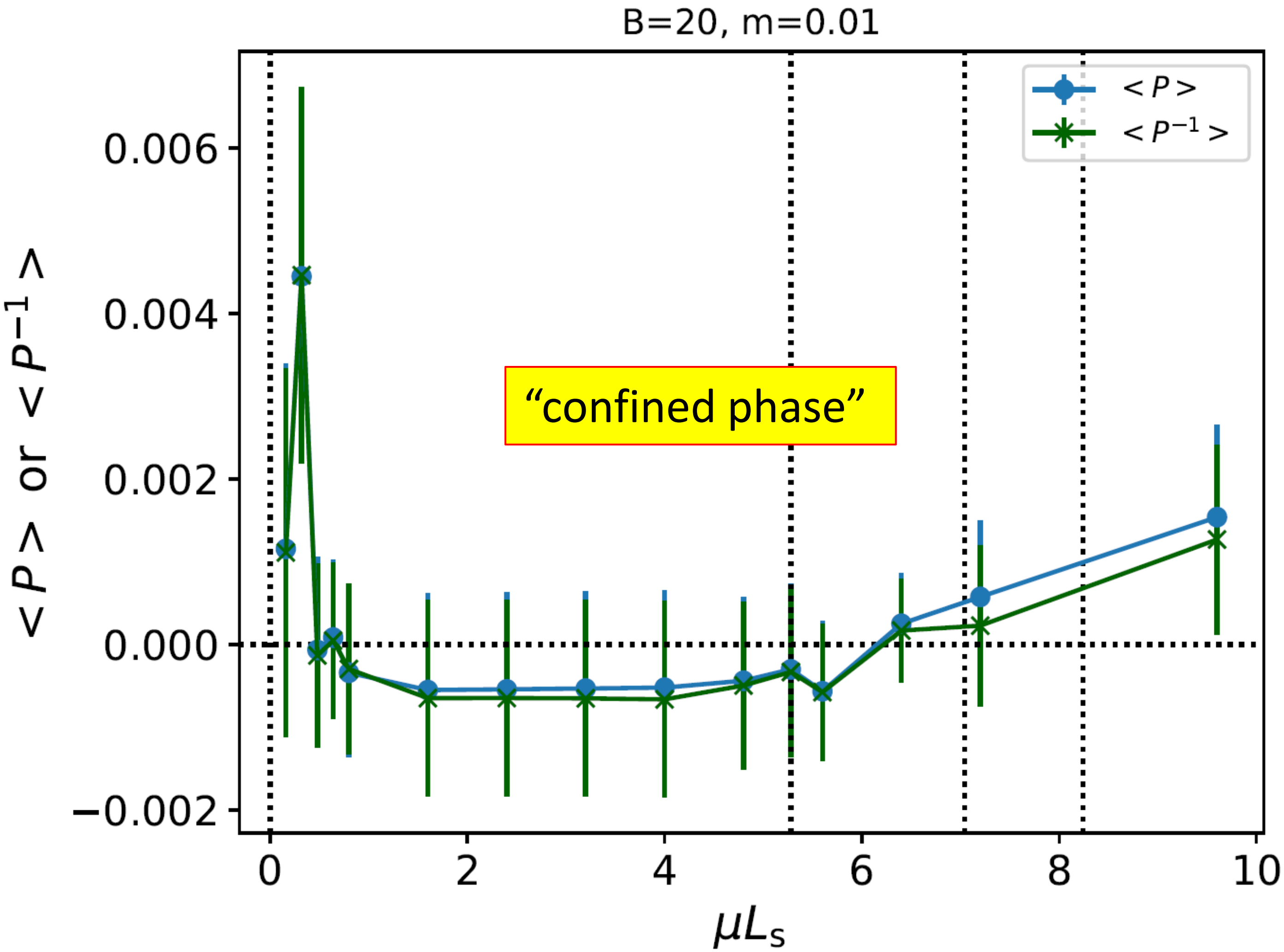
Chiral condensate decreases as μ increases.

CL simulations at large β (preliminary)

Polyakov line

$\beta = 20, \; m = 0.01$

$8^3 \times 128$



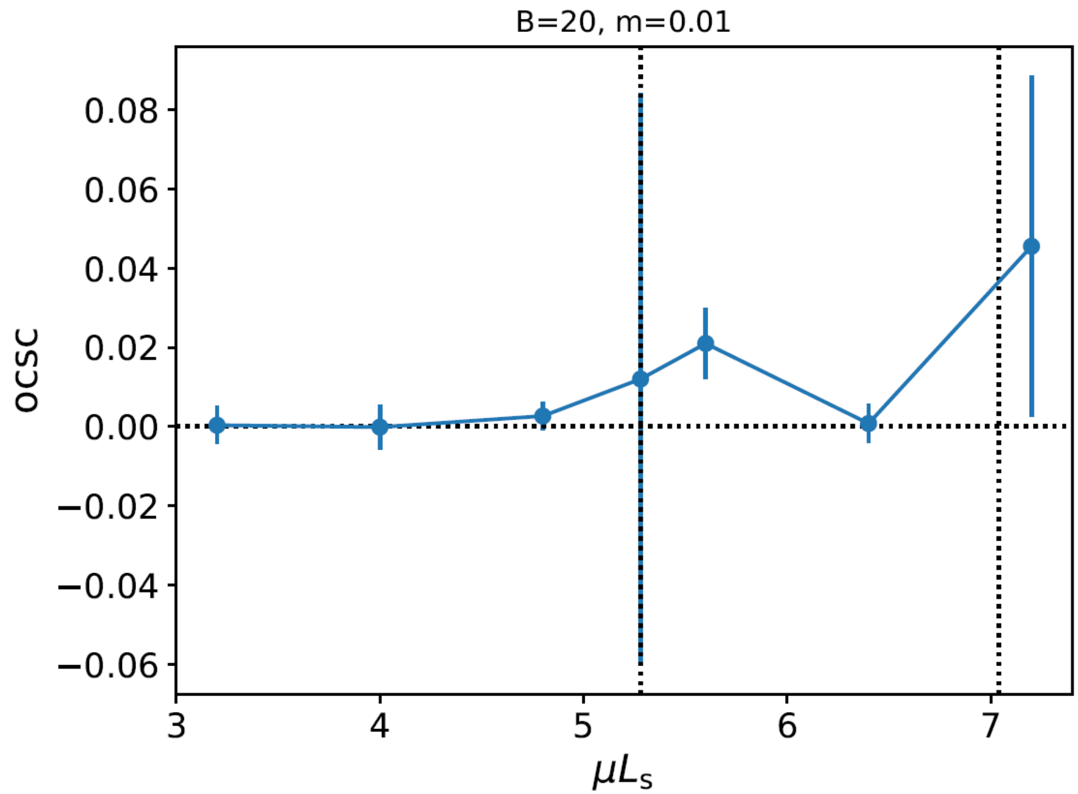
CL simulations at large β (preliminary)

order parameter for
color superconductivity

$\beta = 20, \ m = 0.01$
$8^3 \times 128$

$$\mathcal{O}_{\rm CSC} = \sum_x \varphi_a^\dagger(x)\,\varphi_a(x) \sim \sum_x \bar{\chi}_a(x)\chi_a(x)\bar{\chi}_b(x)\chi_b(x)$$

$$\varphi_a = \epsilon_{abc}{\rm tr}(C^{-1}\Psi_b^{\top} C\Psi_c)$$



Large fluctuation observed
near the peak of β_C

Simulations with larger volume
are expected to confirm
the color superconductivity.

4. Summary and future prospects

Summary and future prospects

- The complex Langevin method is a powerful tool to investigate finite density QCD !
  - The condition for correct convergence is satisfied in the low temperature high density region.
- Color superconductivity can be studied in lattice QCD !
  - The plateau corresponding to the formation of the Fermi sphere has been observed.
  - Perturbative calculations suggest that fermion bilinear condensates appear when the quark number jumps to the next plateau.
  - CLM will be able to reproduce such predictions for QCD in a small box.
- Future directions
  - Larger lattice with smaller β in order to incorporate nonperturbative effects.
  - The hadronic phase should appear at small μ when the box size exceeds ～1 fm. Does the CLM still work in that case ?
  - Extending these studies to 2 quark flavors using Wilson fermions, which are more realistic and have clearer flavor structure. (ongoing)