Introduction to higher form symmetries

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References

Generalized Global Symmetries

Gaiotto, Kapustin, Seiberg, Willett, JHEP 02 (2015) 172

Lecture on anomalies and topological phases (2019)
by Yuji Tachikawa

https://member.ipmu.jp/yuji.tachikawa/lectures/2019-top-anom/

Outline

Generalized global symmetries

- Ordinary symmetries
- Higher form symmetries

Application

- Spontaneous symmetry breaking
- ’t Hooft anomaly
- Symmetry protected topological phases

Summary

Notation

Spacetime dimensions

$$D = d + 1$$

spatial dimensions

$$A, B, C, \cdots$$ **Background gauge fields**

$$a, b, c, \cdots$$ **Dynamical gauge fields**

Differential form

Wedge product

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$$

p-form

$$A^{(p)} = \frac{1}{p\,!} A_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Exterior derivative d

$$dA^{(p)} = \frac{1}{p\,!} \partial_\mu A_{\mu_1 \cdots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

which satisfies $d(dA^{(p)}) = 0$

Hodge dual

$$(\star A)^{(D-p)} = \frac{\sqrt{|g|}}{p\,!(D-p)\,!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{D-p}} A_{\mu_1 \cdots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{D-p}}$$

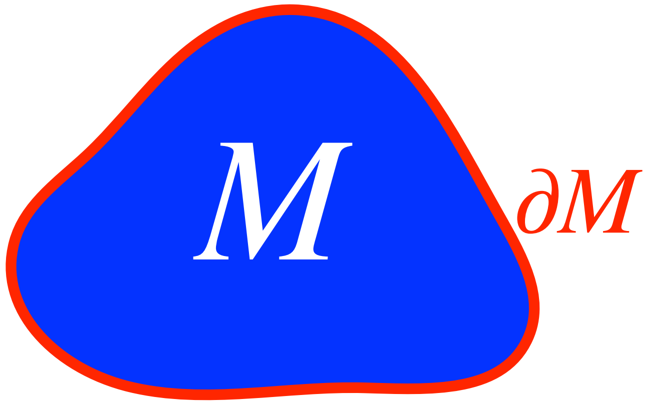
Differential form

f **is closed if** $df = 0$

f **is exact if** f **is express as** $f = dg$

(dg is always closed because $d^2 = 0$)

Stokes theorem $\int_M df = \int_{\partial M} f$



Conservation law

$$\partial_\mu j^\mu = 0 \iff dj = 0$$

$$j := \frac{1}{d!}\epsilon_{\mu_1\cdots\mu_d} j^{\mu_1} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_d}$$

$$\int_{\partial C} j = \int_C dj = 0$$

Ordinary symmetry

Ex) $U(1)$ symmetry

$U(1)$ **charge** $Q = \int d^d x j^0 = \int_{M^d} j$

Time independence

$$\frac{d}{dt}Q = \int d^d x \partial_0 j^0 = - \int d^d x \nabla_i j^i = 0$$

Unitary operator

$$U_g(M^d) = e^{i\alpha Q} \quad (g = e^{i\alpha})$$

$\phi(x)$: charged field

$$U_g(M^d)\phi(x)U_g^{-1}(M^d) = e^{iq\alpha}\phi(x) = V_g\phi(x)$$

Ordinary symmetry

$U_g(M^d) = e^{i\alpha Q}$ $(g = e^{i\alpha})$ **satisfies**

Product $U_g U_{g'} = U_{gg'}$

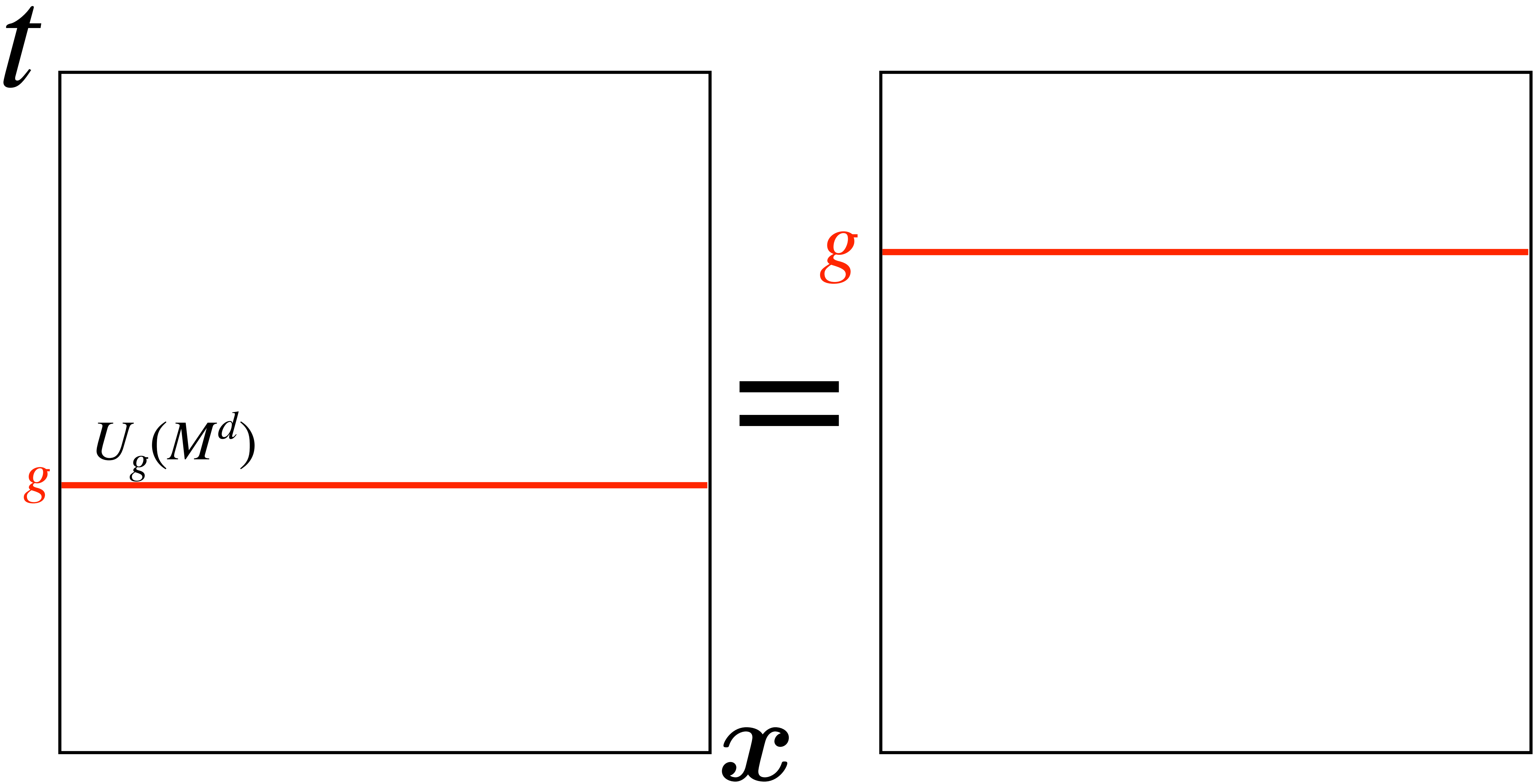
Unit object $1 := U_{e=1}$ $U_g \times 1 = 1 \times U_g = U_g$

Inverse $U_g U_{g^{-1}} = U_{g^{-1}} U_g = 1$

Associativity $U_g(U_{g'} U_{g''}) = (U_g U_{g'}) U_{gg''}$

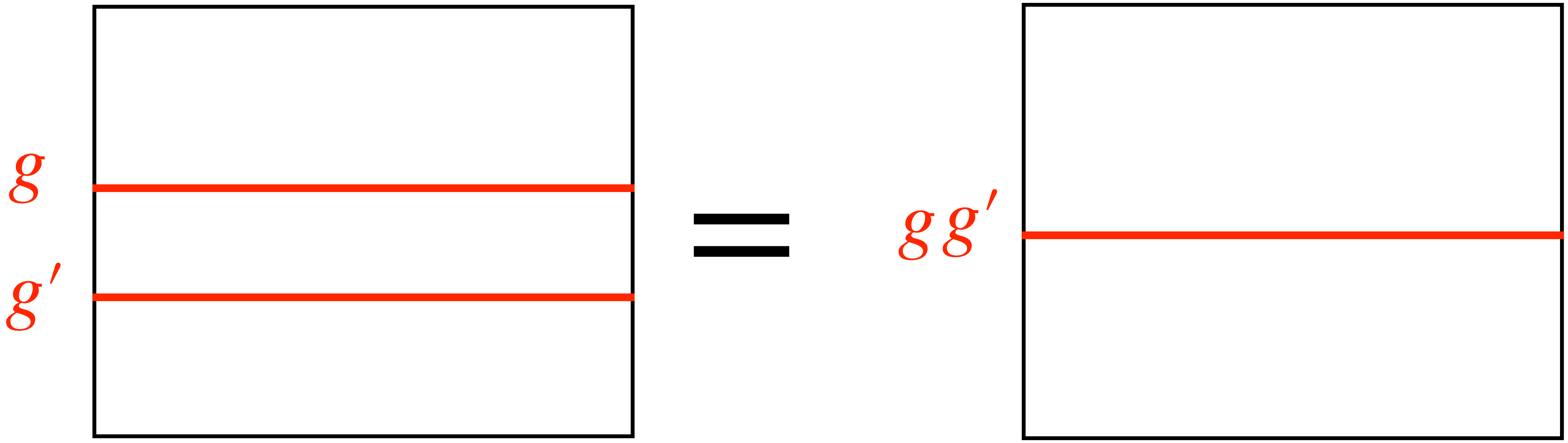
This works for general group G

Graphical representation

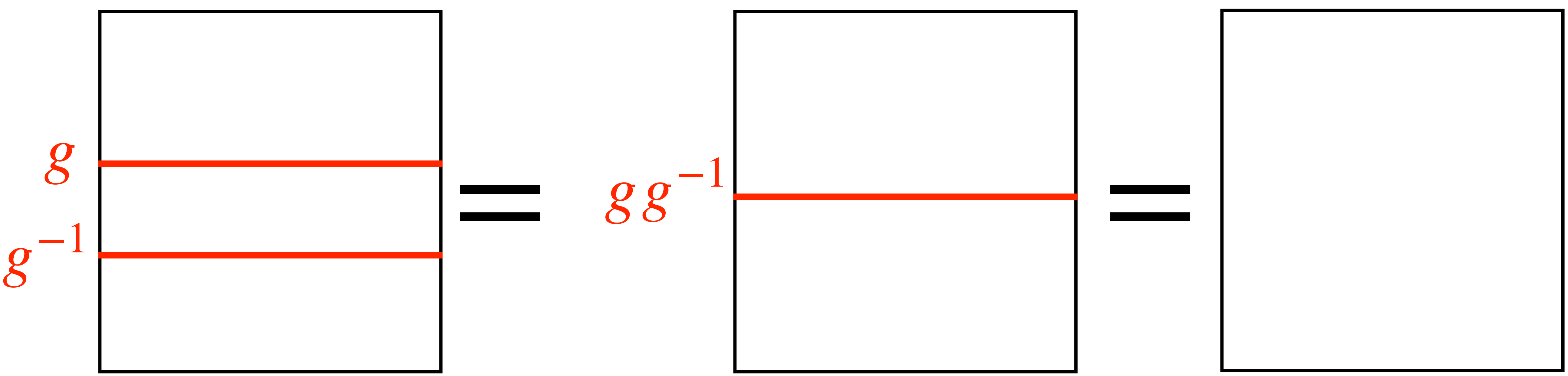


Time independence

Graphical representation

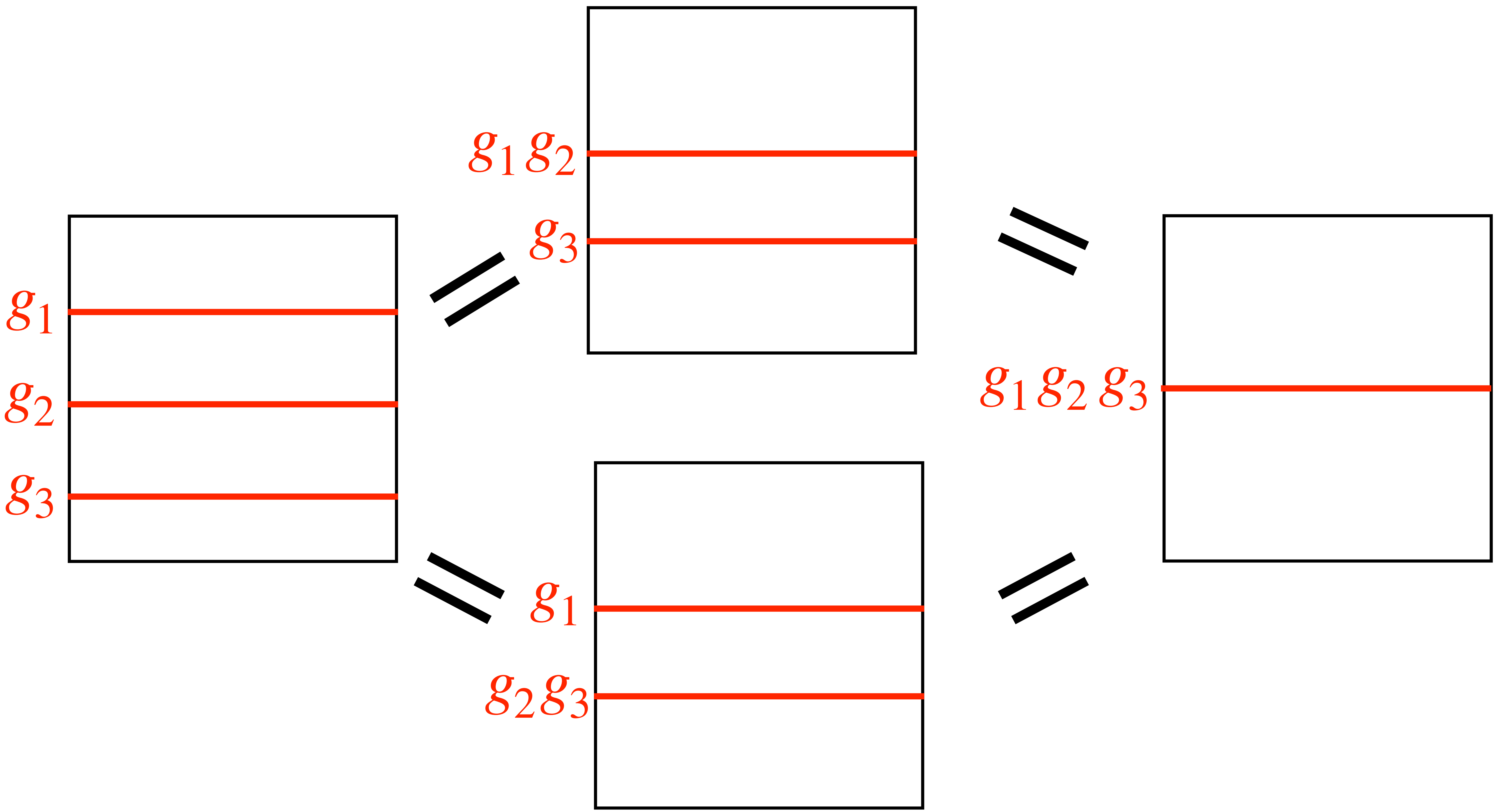


Product $U_g(M^d)U_{g'}(M^d) = U_{gg'}(M^d)$



Inverse $U_g(M^d)U_{g^{-1}}(M^d) = U_{gg^{-1}}(M^d) = 1$

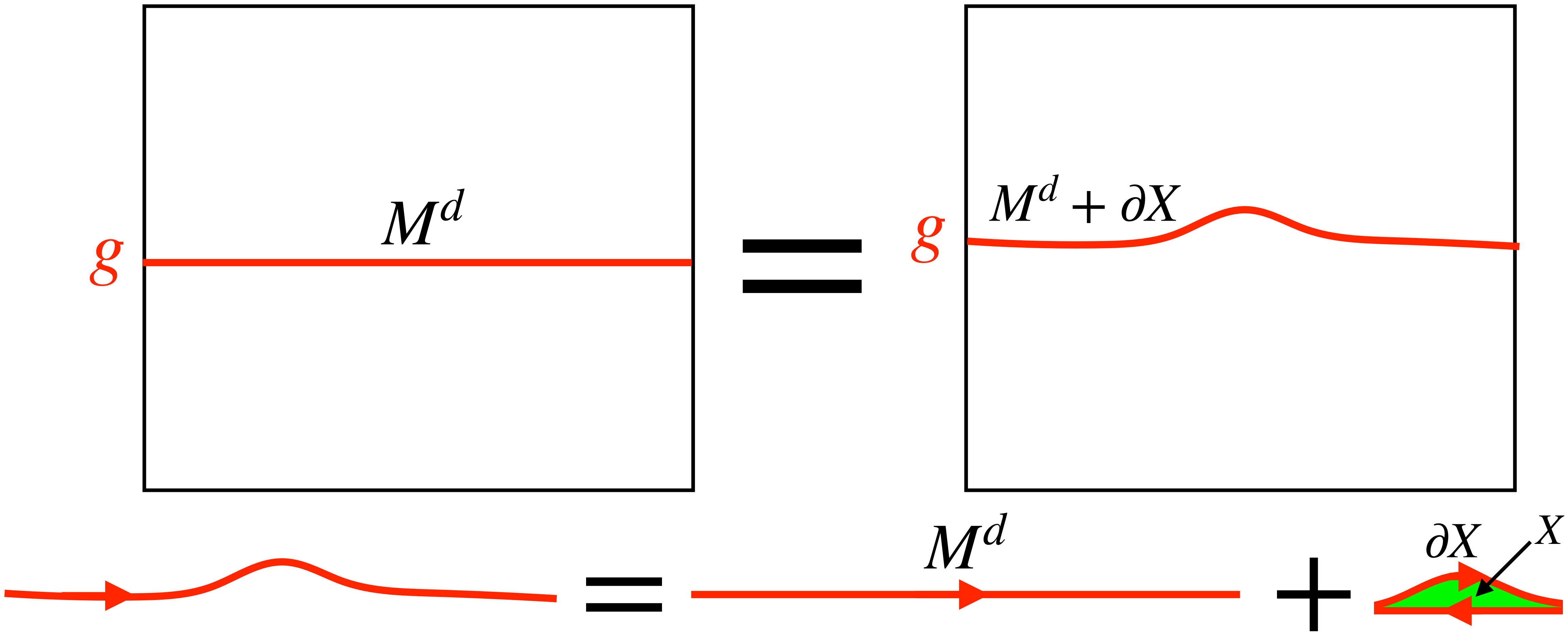
Graphical representation



Associativity: $(U_{g_1}U_{g_2})U_{g_3} = U_{g_1}(U_{g_2}U_{g_3})$

Graphical representation

Symmetry generator is topological



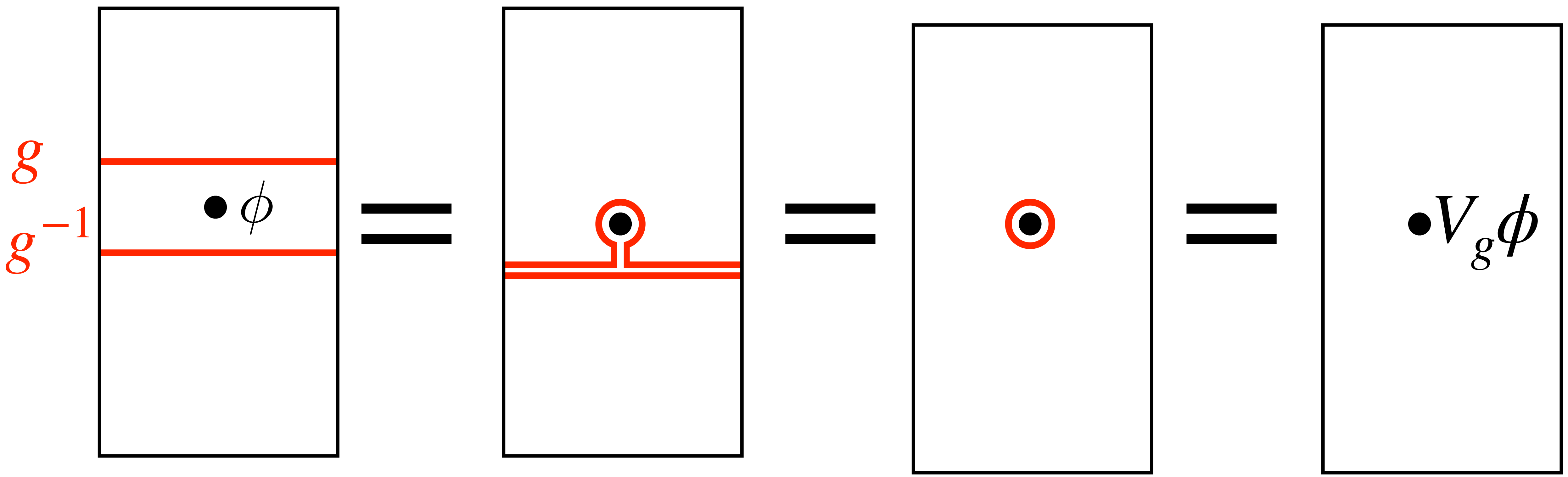
$$\int_{M^d+\partial X} j = \int_{M^d} j + \int_{\partial X} j = \int_{M^d} j + \int_{X} dj = \int_{M^d} j$$

Graphical representation

Charged object

$$U_g \phi(x) U_{g^{-1}} = V_g \phi(x)$$

representation matrix



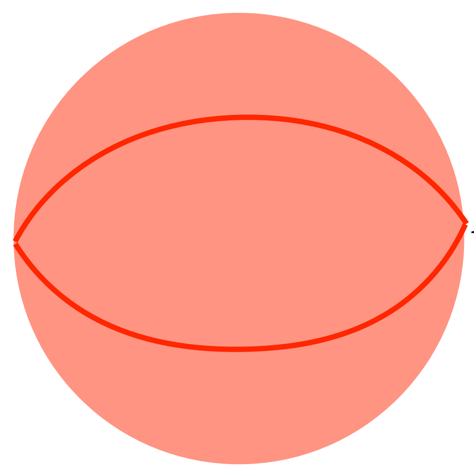
Brief summary

Symmetry generators

=d dimensional topological objects labeled by group elements

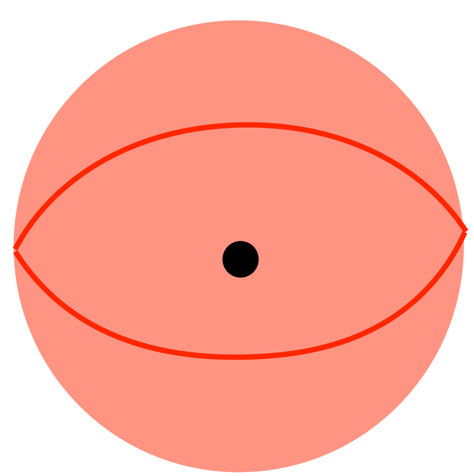
Charged objects

= 0-dimensional objects transforms under G

$U_g(M^d)$ 

$\bullet\, \phi(x)$

Charged object transforms under G

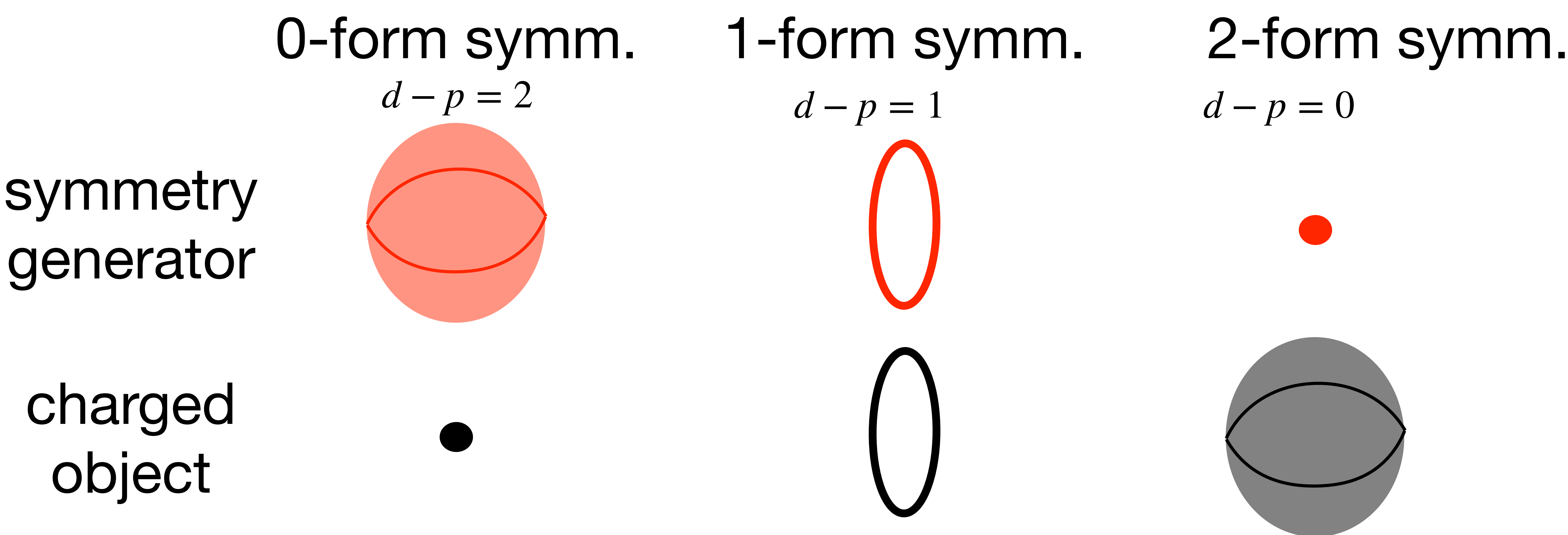
 $= V_g \bullet$

p-form symmetry

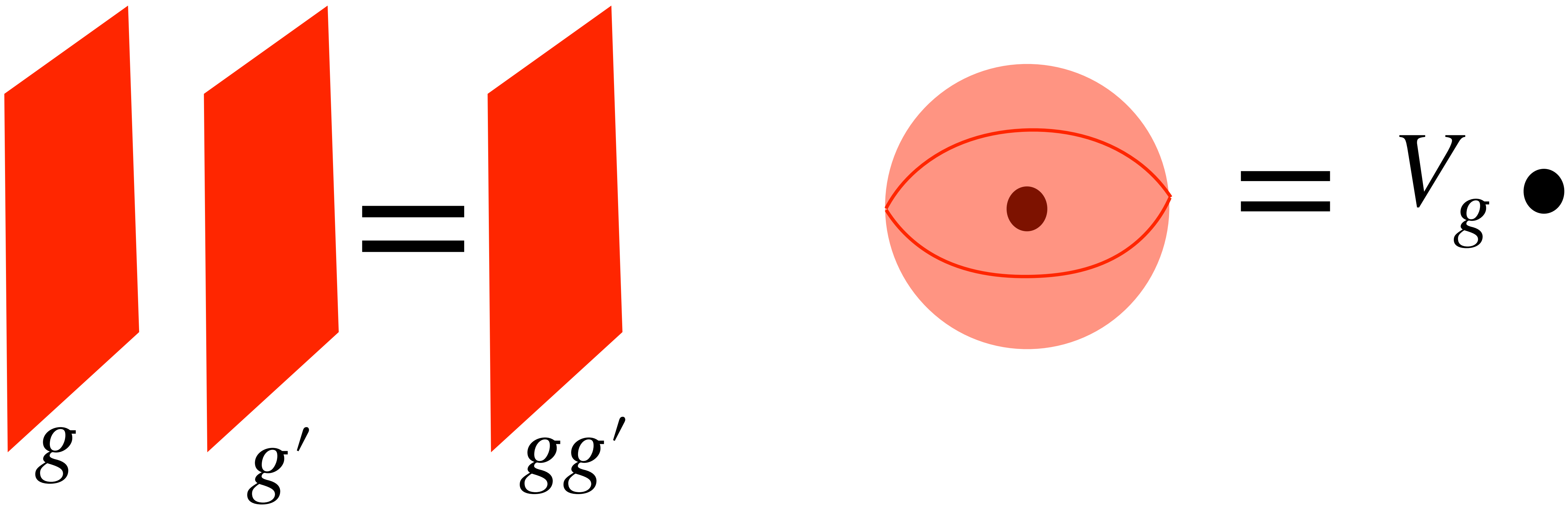
Charged object: p dimensional object

Symmetry generators:
$(d-p)$ dimensional topological objects labeled by group elements.

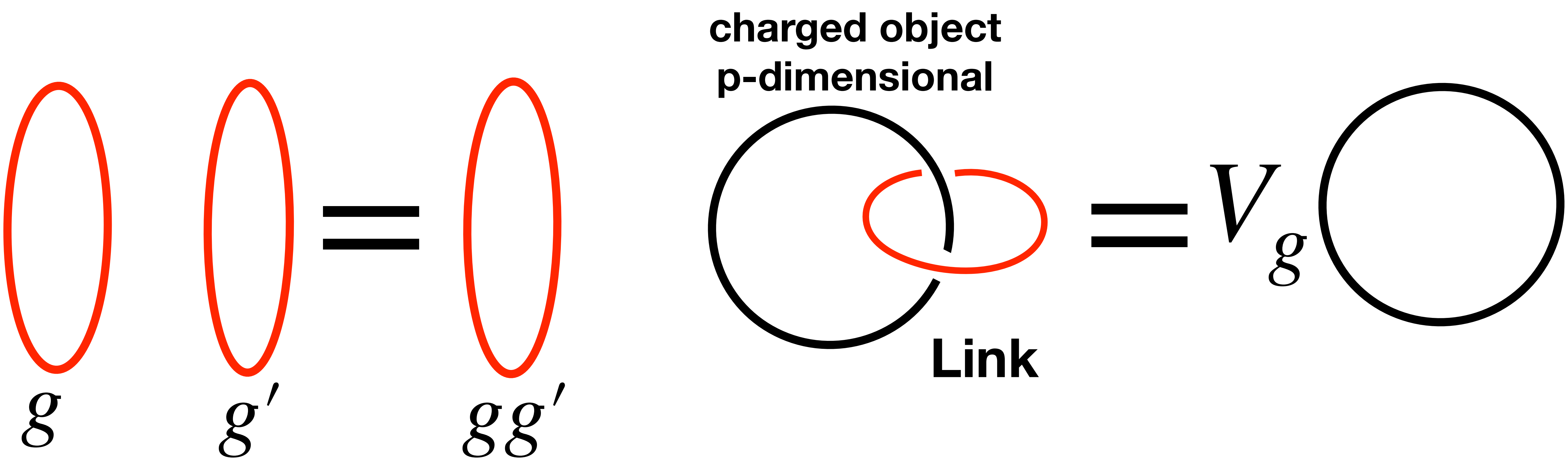
Ex) In 2+1 dimensions



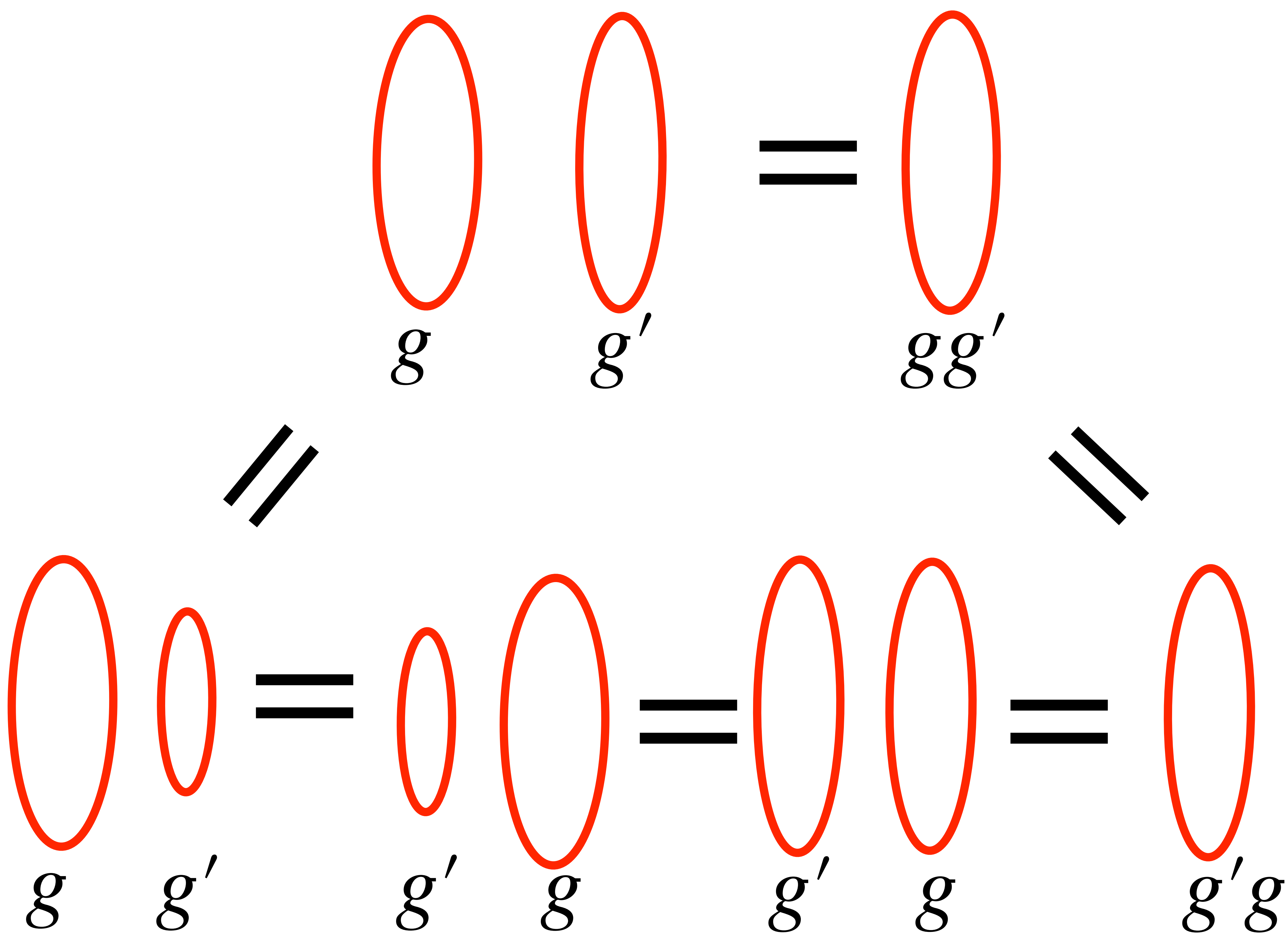
0-form symmetry



p-form symmetry



p-form symmetry($p \geq 1$) is abelian



Ex) U(1) gauge theory

$$S = -\int d^4x \frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} = -\int \frac{1}{2e^2} f \wedge \star f$$

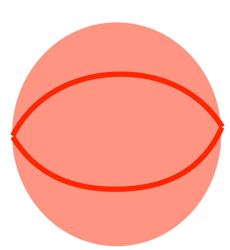
Maxwell equations

$$\partial_\mu f^{\mu\nu} = 0 \quad \Rightarrow \quad d \star f = 0$$

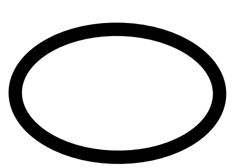
$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu f_{\nu\rho} = 0 \quad \Rightarrow \quad df = 0$$

Conservation of electric and magnetic fluxes

$U(1)_E^{[1]} \times U(1)_M^{[1]}$ symmetries



$$U_E = e^{i \frac{\theta_E}{e^2} \int_S \star f} \qquad U_M = e^{i \frac{\theta_M}{2\pi} \int_S f}$$



$$W = e^{i \int_C a} \qquad T = e^{i \int_C \tilde{a}}$$

Application

Spontaneous symmetry breaking (SSB)

- Topological order (abelian type) as higher form SSB
- Photons as Nambu-Goldstone bosons

Anomaly and symmetry protected topological phases

Nambu-Goldstone bosons

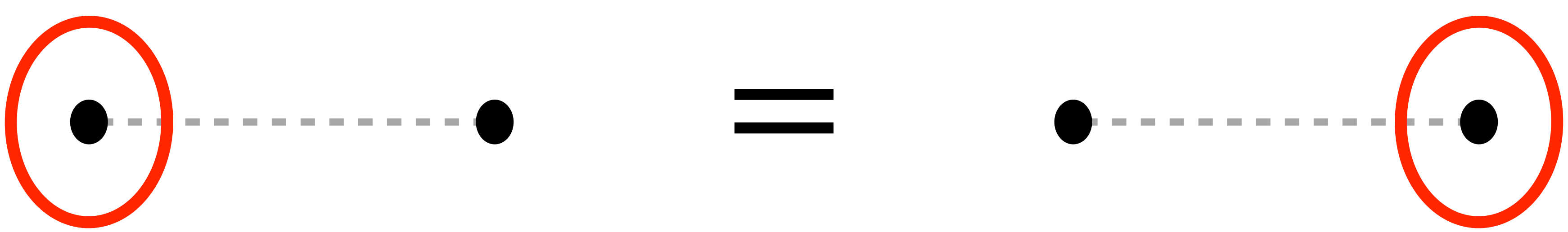
Spontaneous symmetry breaking

0-form symmetry breaking

$$\lim_{x\to\infty}\langle\phi^\dagger(x)\phi(0)\rangle \simeq \langle\phi^\dagger(x)\rangle\langle\phi(0)\rangle \neq 0$$

Off-Diagonal Long-Range Order

two points are a boundary of line



$$\langle e^{i\theta}\phi^\dagger(x)\phi(y)\rangle = \langle\phi(x)e^{i\theta}\phi(y)\rangle$$

Phase of different point is correlated.

Nambu-Goldstone bosons

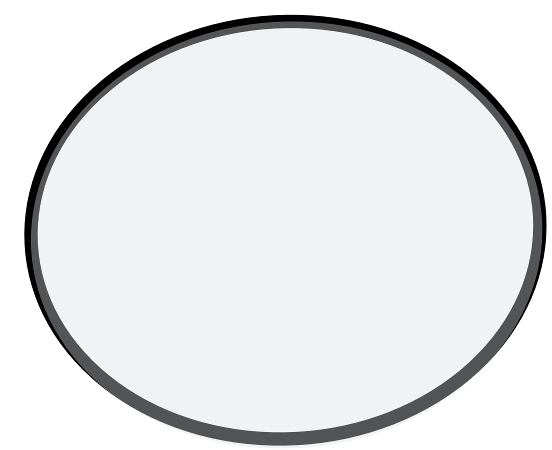
Spontaneous symmetry breaking

0-form symmetry breaking

$$\lim_{x\to\infty}\langle\phi^\dagger(x)\phi(0)\rangle \neq 0$$

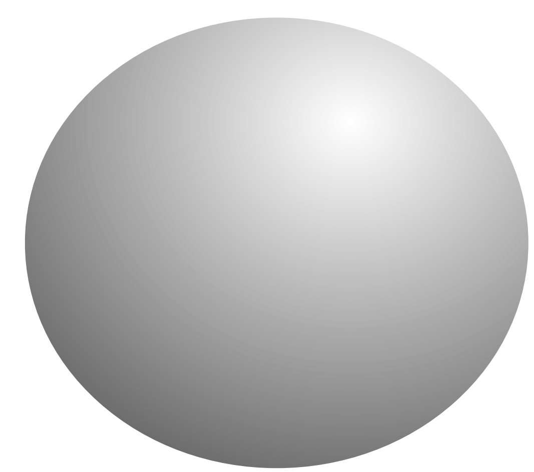
1-form symmetry breaking

$$\lim_{C\to\infty}\langle W(C)\rangle \neq 0$$



p-form symmetry breaking

$$\lim_{M^p\to\infty}\langle W(M^p)\rangle \neq 0$$



Nambu-Goldstone bosons

Spontaneous symmetry breaking $G \to H$

0-form symmetry breaking

Coset variables G/H

Mauler-Cartan form

0-form breaking ($U(1)$)

$$e^{i\varphi(x)}$$

$$e^{i\varphi(x+\delta x)} e^{-i\varphi(x)} = e^{i\int_{\partial C}\varphi} = e^{i\int_C j}$$

$$j = d\varphi$$

Redundancy: $\varphi(x) \to \varphi(x) + 2\pi$

Effective theory $S = -\int \left(f_\pi^2 j \wedge \star j + \cdots \right)$

Current conservation law

$d \star j = d \star d\pi = 0 \Longrightarrow$ **gapless**

Nambu-Goldstone bosons

Spontaneous symmetry breaking $G \to H$

p-form symmetry breaking

Coset variables G/H

$$e^{i\int_M a^{(p)}}$$

Mauler-Cartan form

$$e^{i\int_{\partial X} a^{(p)}} = e^{i\int_X j}$$

$$j = da^{(p)}$$

Redundancy: $a^{(p)} \to a^{(p)} + d\lambda$

Effective theory $S = -\int \left(f_\pi^2 j \wedge \star j + \cdots \right)$

Current conservation law

$d \star j = d \star da = 0 \implies$ **gapless**

Ex) U(1) gauge theory

$$\lim_{C\to\infty} \langle e^{i\int_C a} \rangle \neq 0 \qquad U(1)_E^{[1]}$$ **is broken**

Photons are NG bosons

Low-energy effective theory

$$S = -\int f_\pi^2 j \wedge \star j = -\int \frac{1}{2e^2} f \wedge \star f$$

where $f_\pi^2 = \dfrac{1}{2e^2}$

Ex) $SU(N)$ gauge theory

$$S = -\int d^4x \frac{1}{2g^2} \mathrm{tr} f \wedge \star f \qquad f = da - ia \wedge a$$

has $\mathbb{Z}_N$ 1-form symmetry.

Order parameter

$$\langle W \rangle = \langle \mathrm{tr} e^{i\int_C a} \rangle = \begin{cases} \text{Area law (unbroken)} \\ \text{Perimeter law (broken)} \end{cases}$$

$\mathbb{Z}_N$ is a discrete symmetry,
so that no NG modes exit,
when $\mathbb{Z}_N$ is spontaneously broken.

Topological order

Topological order

Characterization of topological order

- Degeneracy of ground state depending on topology
- Anyon statistics
- Long range entanglement
- Stability of local perturbation

Low energy effective theory
= topological gauge theory, like BF theory

$$S = \frac{k}{2\pi} \int b \wedge da$$

A typical topological order has a higher-form symmetry and it is broken.

Example: Fractional quantum Hall system

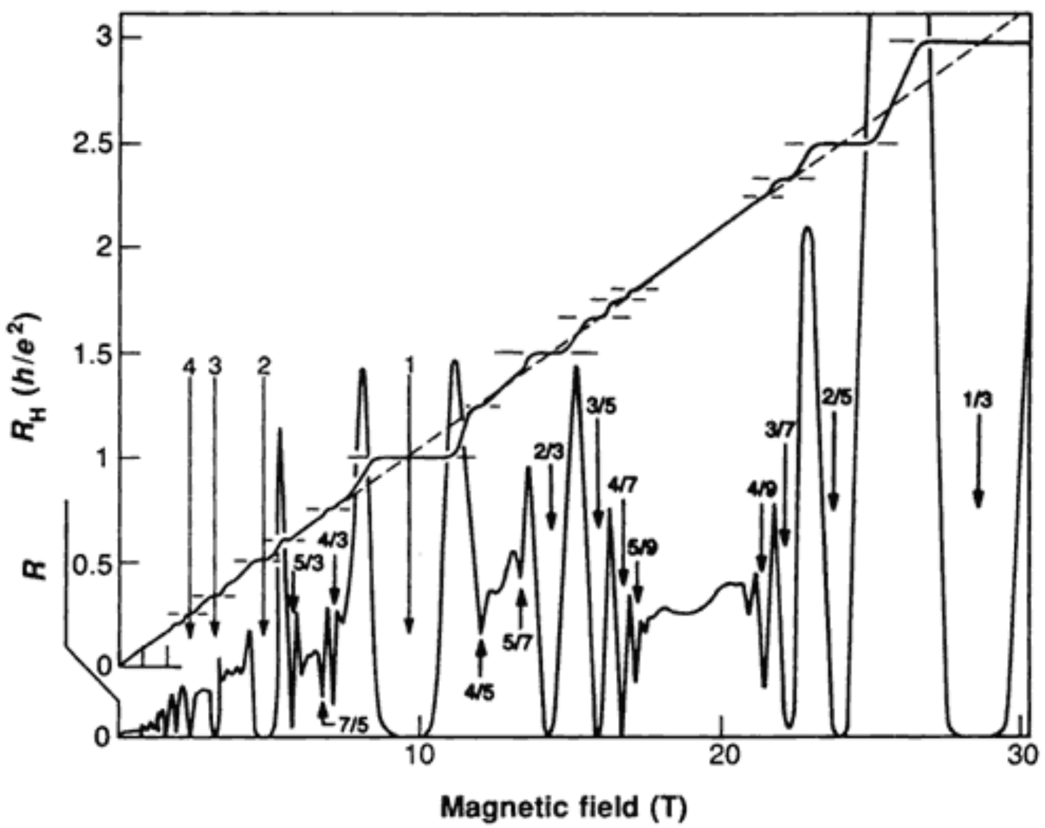


Figure from Nobelprize.org

$$S_{\rm eff} = -\frac{k}{4\pi}\int a \wedge da + \frac{1}{2\pi}\int A \wedge da$$

a: **dynamical one form gauge field**

A: **external U(1) gauge field**

k: **integer**

Equation of motion: $$-\frac{k}{2\pi}da + \frac{dA}{2\pi} = 0$$

Current: $$J = \frac{\delta S_{\rm eff}}{\delta A} = \frac{1}{2\pi}da = \frac{1}{k}\frac{dA}{2\pi}$$ **fractional Hall effect**

Example: Fractional quantum Hall system

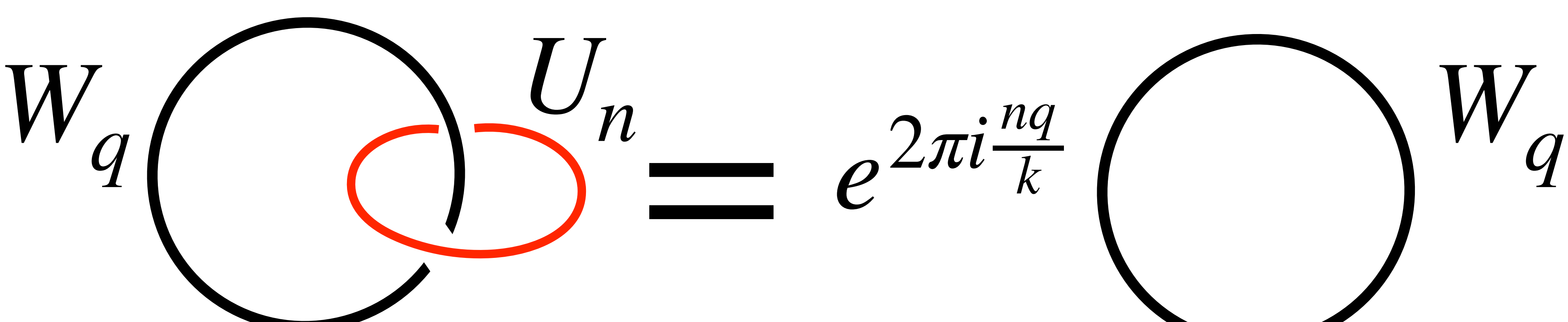
Effective theory: Cherns-Simons

$$S = -\frac{k}{4\pi}\int a \wedge da$$

one-form $\mathbb{Z}_k$ symmetry $a \to a + \frac{\lambda}{k} \qquad d\lambda = 0 \qquad \int \lambda \in 2\pi\mathbb{Z}$

Charged object $W_q = e^{iq\int a}$

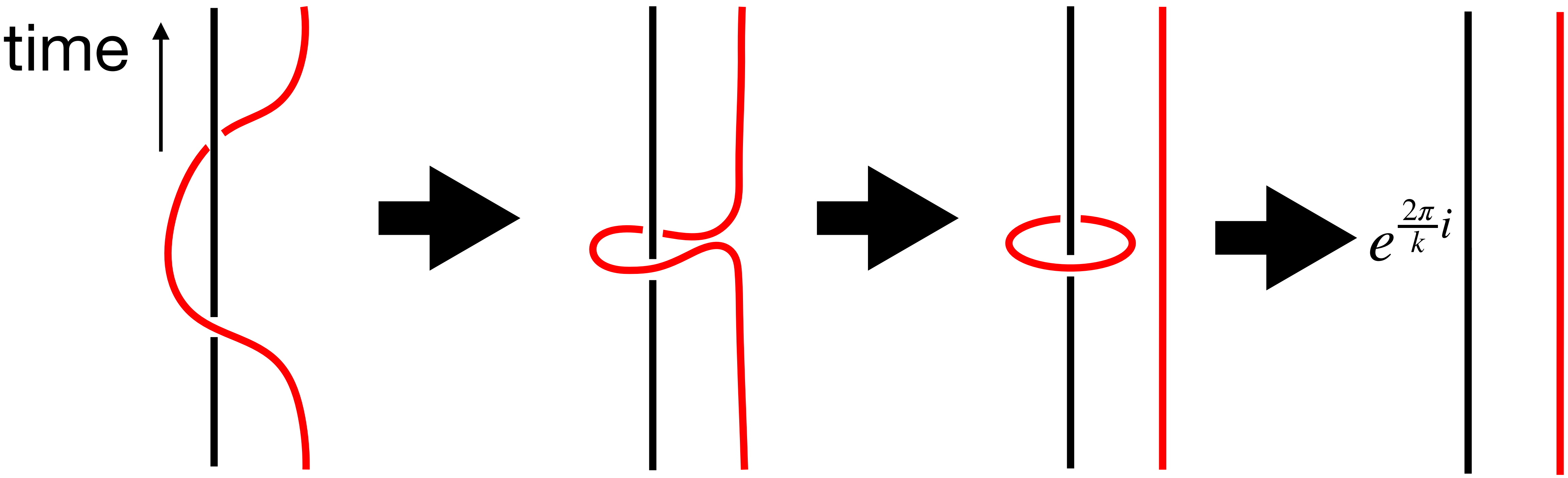
Symmetry generator $U_n = e^{in\int a}$



$$W_q \, U_n = e^{2\pi i \frac{nq}{k}} \, W_q$$

Example: Fractional quantum Hall system

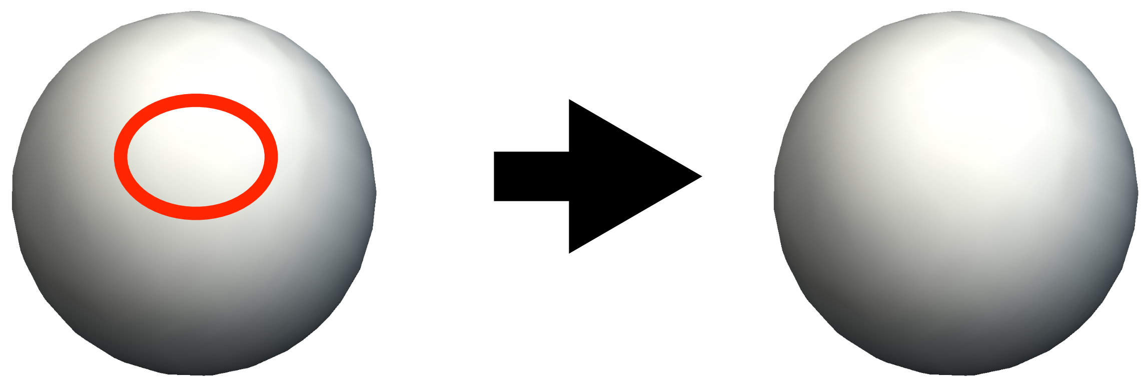
Anyon statistics



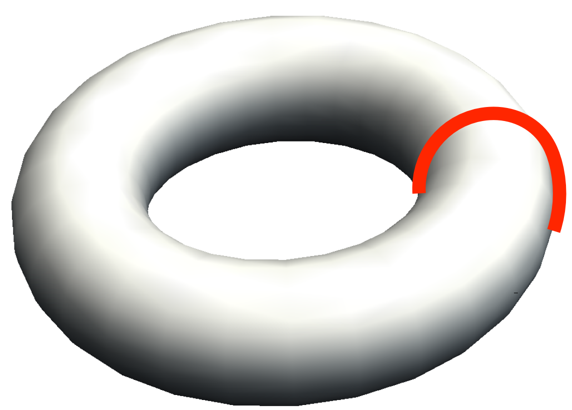
$e^{\frac{\pi}{k}i}$ **is exchange phase**

Example: Fractional quantum Hall system

Ground state degeneracy

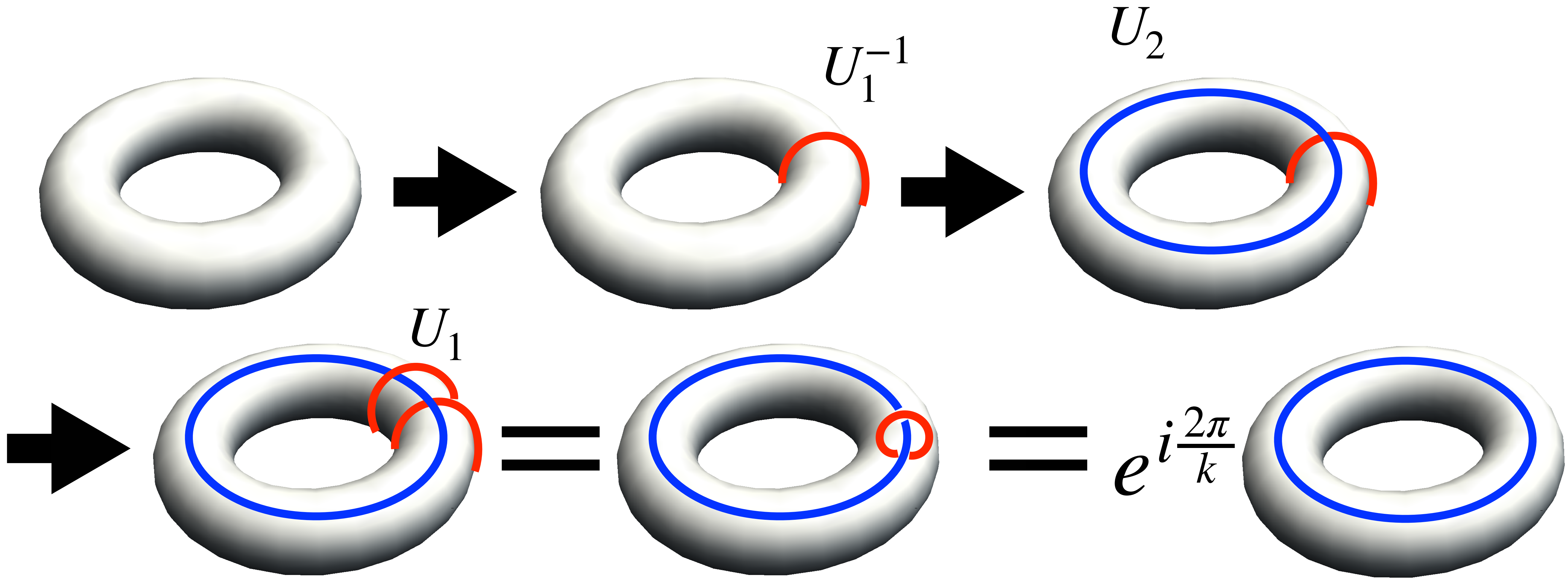


trivial



can be nontrivial

Example: Fractional quantum Hall system



$$\Rightarrow U_1 U_2 U_1^{-1} |\Omega\rangle = e^{i\frac{2\pi}{k}} U_2 |\Omega\rangle$$

$$U_1 U_2 U_1^{-1} |\Omega\rangle = e^{i\frac{2\pi}{k}} U_2 |\Omega\rangle$$

implies the ground state degeneracy

Suppose $U_1^{-1} |\Omega\rangle = e^{i\theta} |\Omega\rangle$

$$\langle\Omega | U_2 | \Omega\rangle = \langle\Omega | U_1 U_2 U_1^{-1} | \Omega\rangle = e^{i\frac{2\pi}{k}} \langle\Omega | U_2 | \Omega\rangle$$

➡ $\langle\Omega | U_2 | \Omega\rangle = 0$

$|\Omega\rangle$ **and** $U_2 |\Omega\rangle$ **are different state**

(ground state degeneracy k^g **fold)**

Example: Superconductor

$$S_{\text{eff}} = v^2 \int (d\varphi - ka) \wedge \star (d\varphi - ka)$$

$k = 2$**: charge of Cooper pair**

Low energy effective theory is $\mathbb{Z}_k$ gauge theory

$$v^2 \to \infty \quad \Rightarrow \quad d\varphi - ka = 0$$

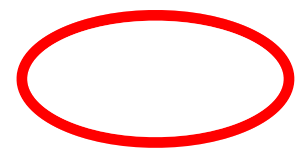
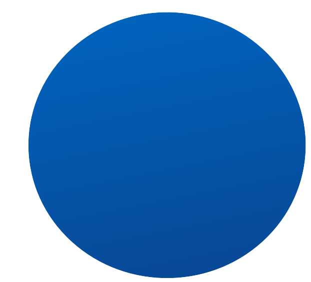
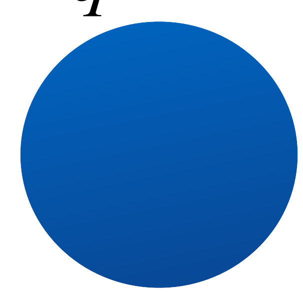
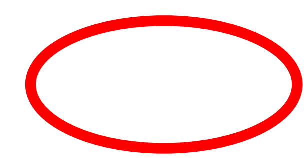
$$S_{\text{eff}} = \frac{1}{2\pi} \int c \wedge (d\varphi - ka)$$

EOM of φ $\quad dc = 0 \quad \Rightarrow \quad c = db$

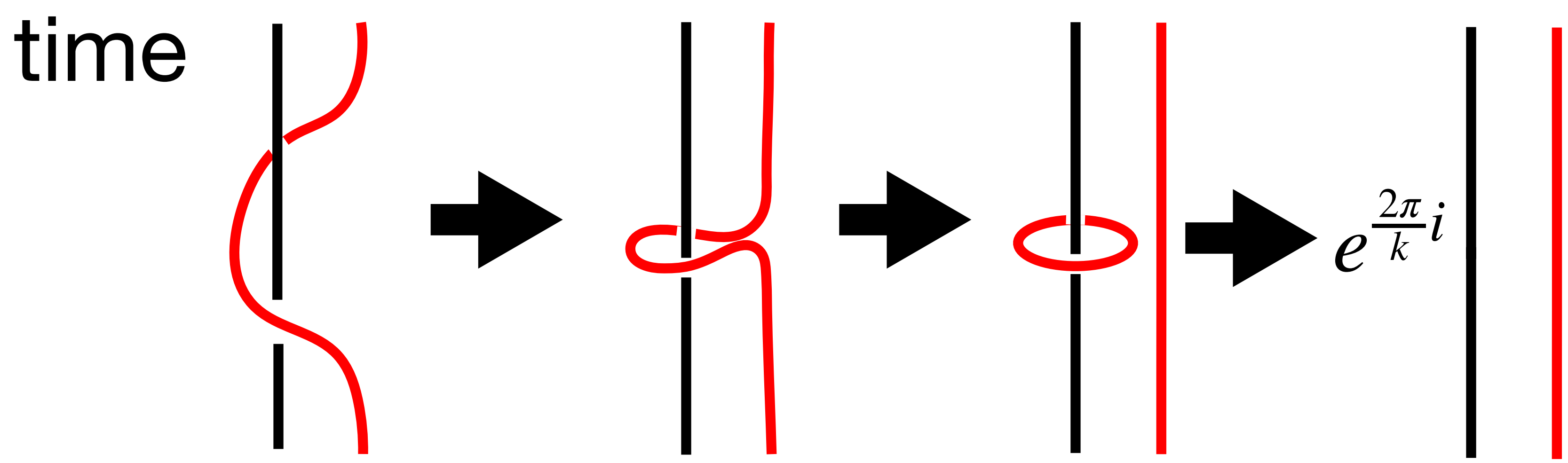
$$S_{\text{eff}} = \frac{-k}{2\pi} \int db \wedge a = \frac{k}{2\pi} \int b \wedge da$$

Example: Superconductor

$$S_{\text{eff}} = \frac{k}{2\pi}\int b \wedge da$$

	charged object	**symmetry generator**
One form symmetry $a \to a + \frac{\lambda^{(1)}}{k}$	 $W_q = e^{iq\int a}$	 $U_n = e^{in\int b}$
Two form symmetry $b \to b + \frac{\lambda^{(2)}}{k}$	 U_n	 W_q

Brading statistics(Aharonov–Bohm phase)



Anomaly

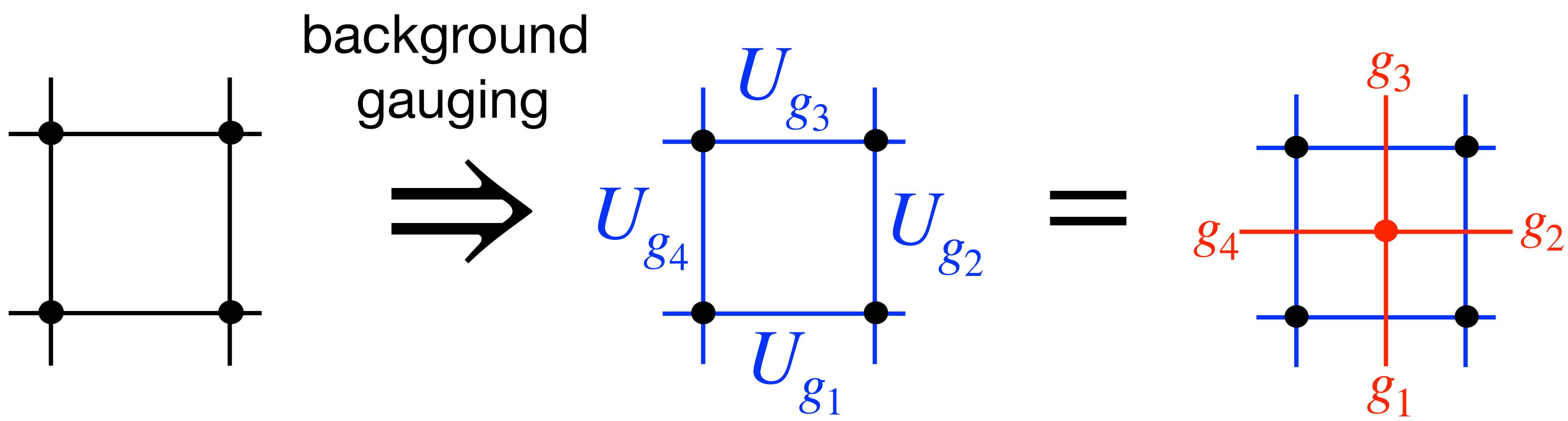
Under a background gauge field,

a current is not conserved

Background gauging =Network of symmetry defects

Couples to background gauge field

In a lattice theory, gauge field corresponds to G valued link variable $U = e^{i\int a}$



equivalent to network of symmetry defects (Poincare dual)

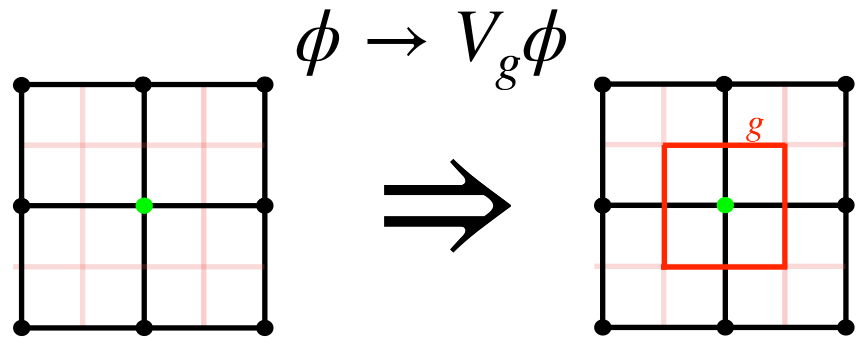
For a discrete symmetry,

$U_{g_1}U_{g_2}U_{g_3}^{-1}U_{g_4}^{-1} = 1$ **is necessary (flat connection),**

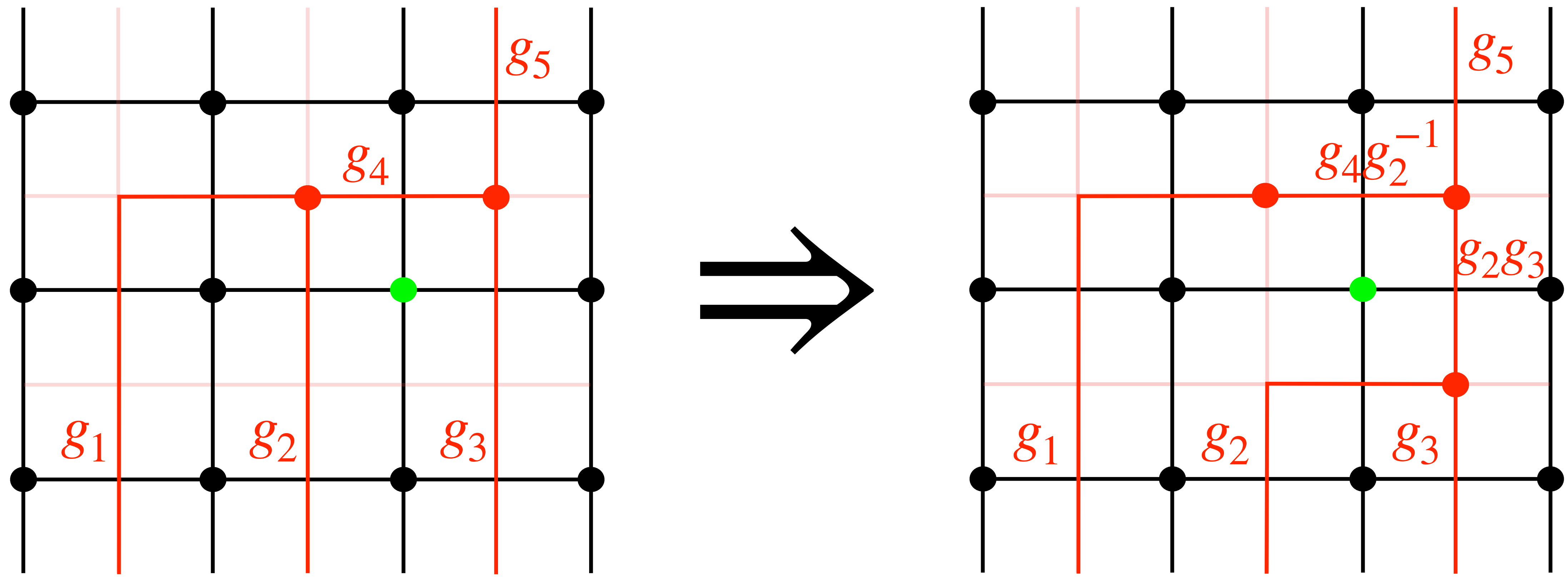
i.e., $g_1 g_2 g_3^{-1} g_4^{-1} = 1$

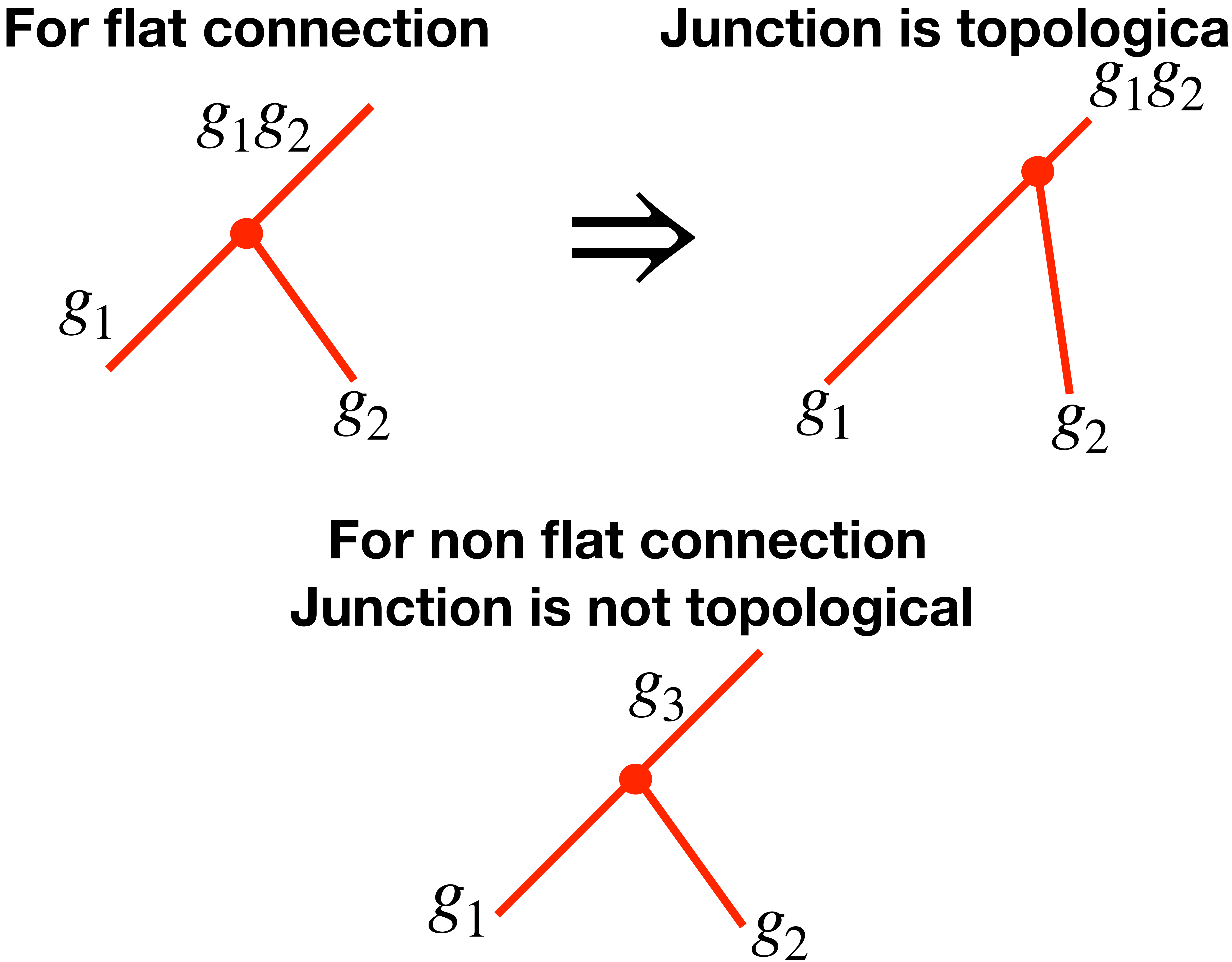
Gauge transformation

Redefinition of ϕ



induces change of network (gauge transformation)




For flat connection
Junction is topological
g1g2
g1
g2
g1g2
g1
g2
For non flat connection
Junction is not topological
g3
g1
g2

’t Hooft anomaly

Partition function $Z[A] = \int D\phi e^{iS[A]}$

under background gauging

is not invariant under gauge transformation

$$Z[A] \to Z[A + d\lambda] = Z[A]e^{i\omega(\lambda,A)} \neq Z[A]$$

The difference is just a phase factor $e^{i\omega(\lambda,A)}$

Projective representation as 't Hooft anomaly

Consider quantum mechanics, whose partition function is $Z = \mathrm{tr} e^{-\beta H}$

$$Z[A] = \mathrm{tr} e^{-\beta H} U_{g_1} U_{g_2} \overset{?}{=} Z[A + d\lambda] = \mathrm{tr} e^{-\beta H} U_{g_1 g_2}$$

If $U_{g_1} U_{g_2} = e^{i\omega(g_1, g_2)} U_{g_1 g_2}$,

$$Z[A + d\lambda] = e^{-i\omega(g_1, g_2)} Z[A]$$ **anomaly**

$U_{g_1}U_{g_2} = e^{i\omega(g_1,g_2)}U_{g_1g_2}$ **needs to satisfy**

associativity: $(U_{g_1}U_{g_2})U_{g_3} = U_{g_1}(U_{g_2}U_{g_3})$

➡ $e^{i\omega(g_1g_2,g_3)+i\omega(g_1,g_2)} = e^{i\omega(g_1,g_2g_3)+i\omega(g_2,g_3)}$

➡ $\delta^{(3)}\omega(g_1, g_2, g_3) := \omega(g_2, g_3) - \omega(g_1g_2, g_3) + \omega(g_1, g_2g_3) - \omega(g_1, g_2) = 0$

Under $U_g \to e^{i\omega(g)}U_g$ ➡ $\omega(g_1, g_2) \to \omega(g_1, g_2) - \delta^{(2)}\omega(g_1, g_2)$

where $\delta^{(2)}\omega(g_2, g_1) := \omega(g_2) - \omega(g_1g_2) + \omega(g_1)$

$\delta^{(3)} \circ \delta^{(2)} = 0$ **is satisfied.**

$\delta^{(3)}\omega(g_1, g_2, g_3) = 0$ **and** $\omega(g_1, g_2) \sim \omega(g_1, g_2) - \delta^{(2)}\omega(g_1, g_2)$

➡ $\omega(g_1, g_2) \in \dfrac{\ker \delta^{(3)}}{\operatorname{im} \delta^{(2)}} =: H^2(G, U(1))$

Projective representation implies nontrivial ground state

Suppose $U_{g_1}U_{g_2} = e^{i\omega(g_1,g_2)}U_{g_1g_2}$

and the ground state $|\Omega\rangle$ **is unique.**

$|\Omega\rangle$ **is an eigenstate of** U_g**,** $U_g|\Omega\rangle = e^{i\omega(g)}|\Omega\rangle$

$$U_{g_1}U_{g_2}|\Omega\rangle = e^{i\omega(g_1)+i\omega(g_2)}|\Omega\rangle$$

$$e^{i\omega(g_1,g_2)}U_{g_2g_1}|\Omega\rangle = e^{i\omega(g_1,g_2)+i\omega(g_1g_2)}|\Omega\rangle$$

➡ $$|\Omega\rangle = e^{i\omega(g_1,g_2)-i\delta^{(2)}\omega(g_1,g_2)}|\Omega\rangle$$

Projective representation means

$\omega(g_1,g_2) - \delta^{(2)}\omega(g_1,g_2)$ **is nontrivial.**

This contradicts the assumption.

More generally, if the theory has an ’t Hooft anomaly, the ground state cannot be trivial.

- Spontaneous symmetry breaking
- Topological order
- CFT
- …..

Ex) U(1) gauge theory

$$S = -\int \frac{1}{2e^2} f \wedge \star f$$

Background gauging

$$S[B_E, B_M] = -\int \frac{1}{2e^2}(f - B_E) \wedge \star\, (f - B_E)$$

$$+\frac{1}{2\pi}\int (f - B_E) \wedge B_M$$

This is not invariant under $B_M \to B_M + d\lambda$

$$S[B_E, B_M] \to S[B_E, B_M] - \frac{1}{2\pi}\int B_E \wedge d\lambda$$

Symmetry protected topological phase

Trivially gapped phase,
but if there is a boundary,
the boundary theory has an anomaly.

Total theory $Z[A]_{\text{total}} = Z[A]_{\text{bulk}} Z[A]_{\text{boundary}}$

is gauge invariant $Z[A + d\lambda]_{\text{total}} = Z[A]_{\text{total}}$

Boundary and bulk theories are not

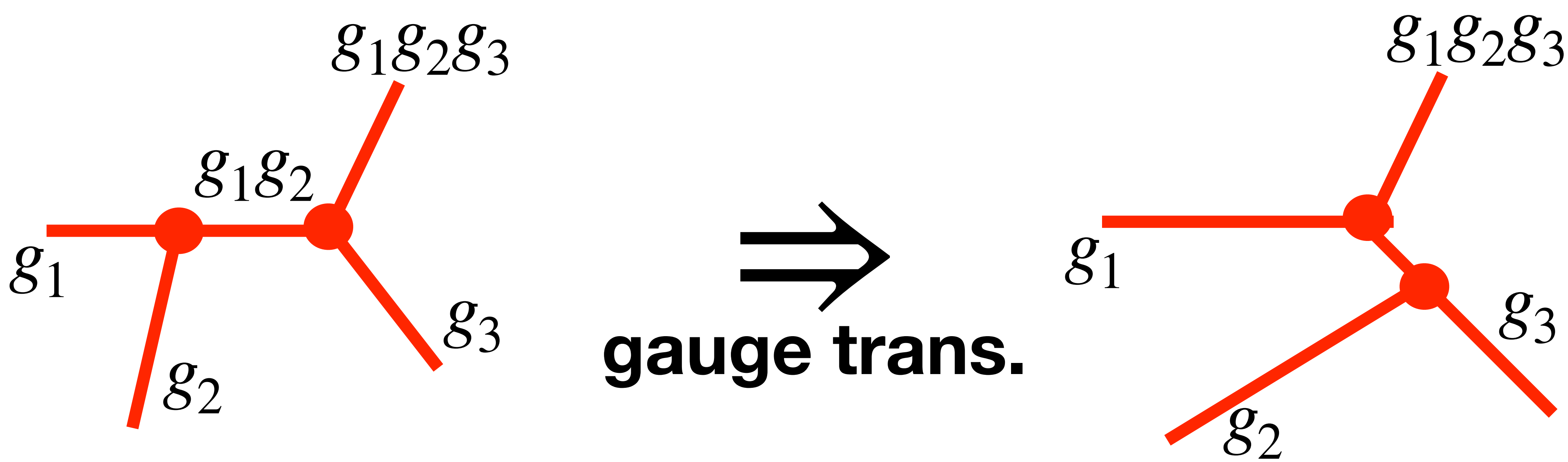
$$Z[A + d\lambda]_{\text{boundary}} = e^{i\omega(A,\lambda)} Z[A]_{\text{boundary}}$$

$$Z[A + d\lambda]_{\text{bulk}} = e^{-i\omega(A,\lambda)} Z[A]_{\text{bulk}}$$

Symmetry protected topological phase

The partition function has a nontrivial phase factor in a background gauging $Z[A] = e^{i\theta(A)}$

Assign a phase on junctions



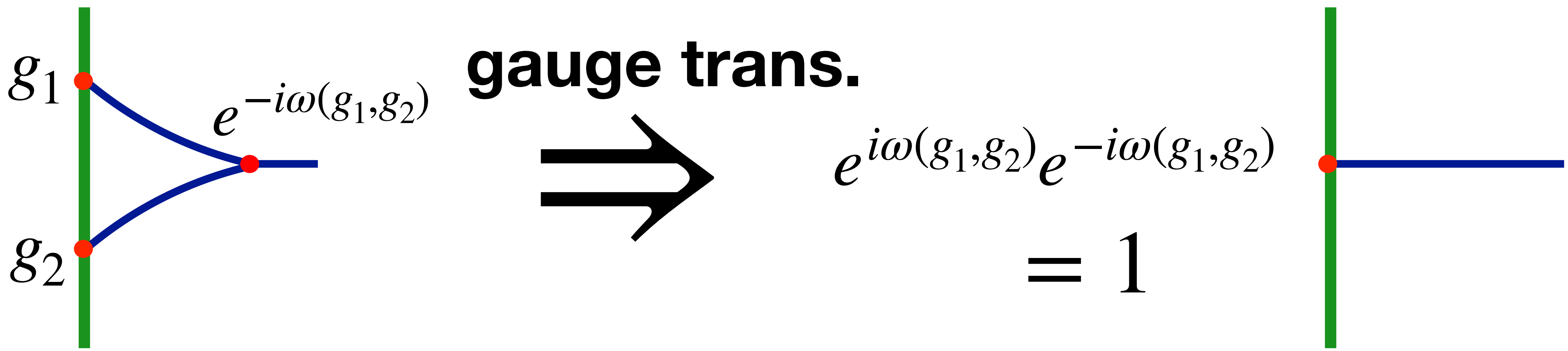
$= e^{-i\omega(g_1,g_2)} e^{-i\omega(g_1 g_2, g_3)}$ $\qquad$ $= e^{-i\omega(g_2,g_3)} e^{-i\omega(g_1, g_2 g_3)}$

$$d^{(3)}\omega(g_1, g_2, g_3) := \omega(g_2, g_3) - \omega(g_1 g_2, g_3) + \omega(g_1, g_2 g_3) - \omega(g_1, g_2) = 0$$

Redefinition of U_g $\quad \omega(g_1, g_2) \to \omega(g_1, g_2) - d^{(2)}\omega(g_1, g_2)$

➡ $\omega \in H^2(G, U(1))$ **The same classification as anomaly in quantum mechanics**

Anomaly inflow



Example U(1) gauge theory

$$Z_{\text{boundary}}[B_E, B_M] = \int \mathscr{D}a e^{iS[a, B_E, B_M]}$$

$$Z_{\text{bulk}}[B_E, B_M] = e^{\frac{i}{2\pi}\int_X dB_E \wedge B_M}$$

Under gauge transformation

$$Z_{\text{boundary}}[B_E, B_M + d\lambda] = Z_{\text{boundary}}[B_E, B_M] e^{\frac{-i}{2\pi}\int_M B_E d\lambda}$$

$$Z_{\text{bulk}}[B_E, B_M + d\lambda] = Z_{\text{bulk}}[B_E, B_M] e^{\frac{i}{2\pi}\int_{\partial X} B_E \wedge d\lambda}$$

$Z_{\text{boundary}}[B_E, B_M] Z_{\text{bulk}}[B_E, B_M]$ **is invariant.**

Summary

Global symmetry:
Topological object
labeled by group element

Useful as well as the usual symmetry

symmetry breaking, ’t Hooft anomaly, etc.

Further directions

Algebra of topological objects labeled by group elements = higher groups

⇒Talk by Yokokura and Tiwari

Algebra of topological objects labeled by something = higher categories