Symmetry protected topological phases and generalized (co)homology theory

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KEK Theory workshop 2020

@Zoom

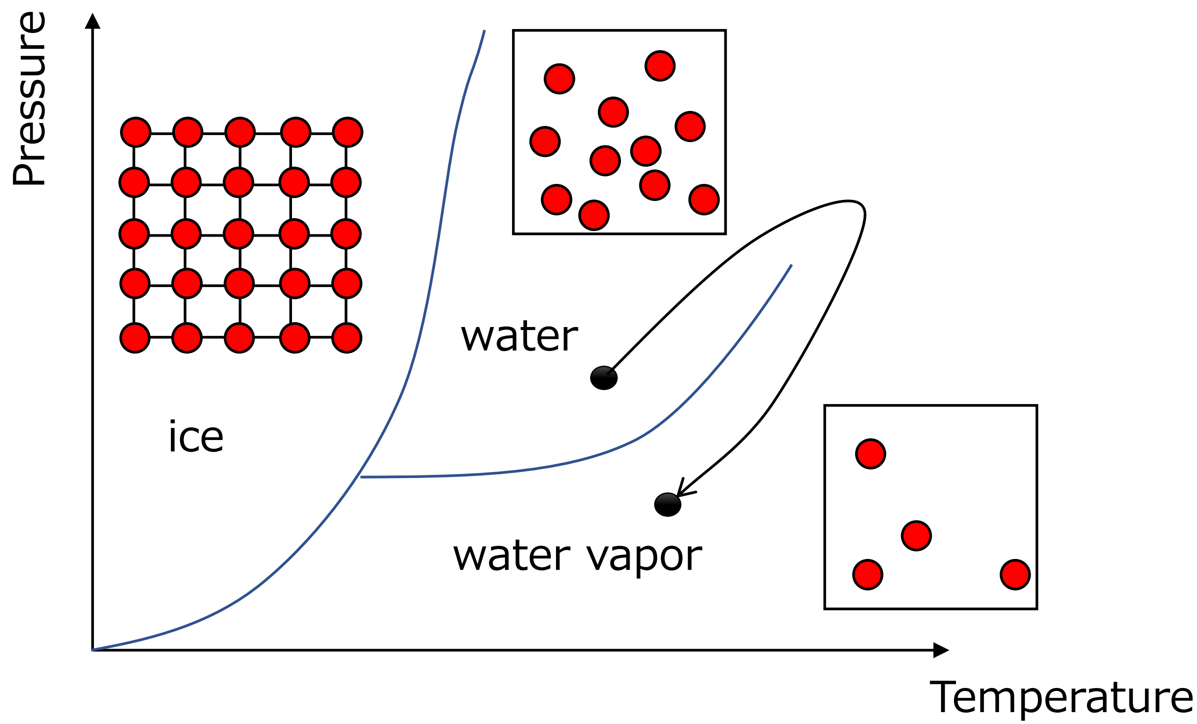
Outline

- Brief review to symmetry protected topological (SPT) phases
  - Classification of SPT phases
  - Bulk-boundary correspondence

- SPT phases as a generalized (co)homology theory

- The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems.

Topological equivalence of phase of matter



- “Topological” equivalence: If there exists a pass connecting two phases A and B without a phase transition, A and B are considered to be in a same phase.
- Ice ≠ water
- Water = water vapor
- SSB of translation symmetry between {ice} and {water, water vapor}

Topological phases of matter

- There may exists a phase distinction without SSB in a certain class of phases of matter.

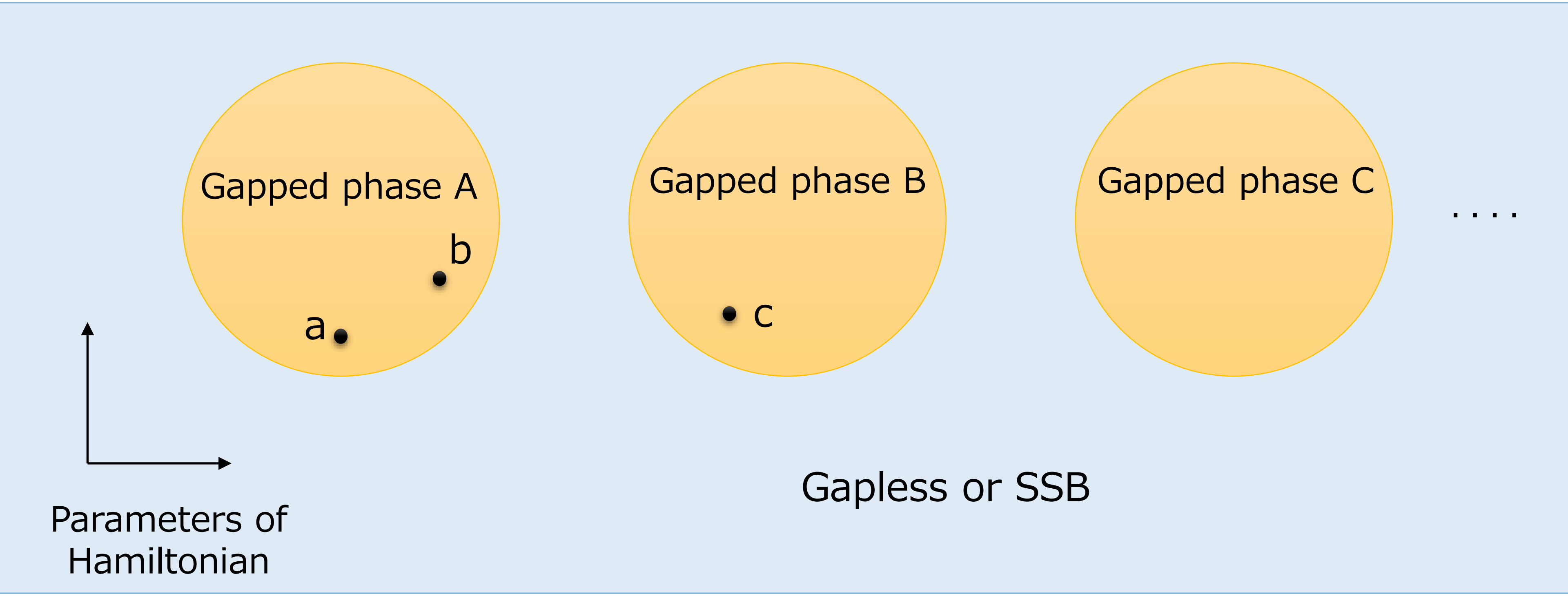
- In the topological phase, we consider the following rule of the game:

  ✓ Zero temperature

  ✓ Gapped (there exists a finite energy gap between the ground and the first excited state.)

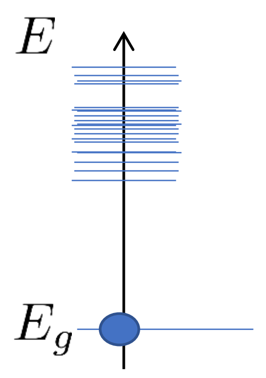
  ✓ With symmetry (Z2 Ising, U(1) particle conservation, time-reversal, crystalline, ⋯)

- A schematic picture of a phase diagram of the topological phase
- A topological phase := an equivalence class under the equivalence relation by the existence of path connecting two points without SSB or gapless phases.

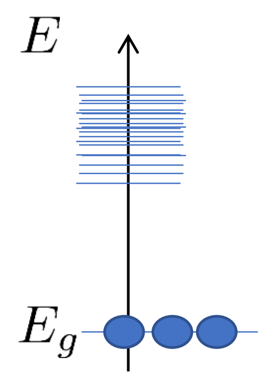


Symmetry Protected Topological phases

- In general, there exists a ground state degeneracy that depends on the global topology of the closed space manifold.
- An SPT phase := a gapped phase that has a unique ground state for any closed space manifold (+α).
  Exs: Haldane chain, topological insulators/superconductors
- A long-range entangled topological phase = a gapped phase that has a ground state degeneracy for a closed space manifold.
  Exs: Toric code, fluctional quantum Hall effect



SPT phase



Long-range entangled topological phase

Classification of SPT phase

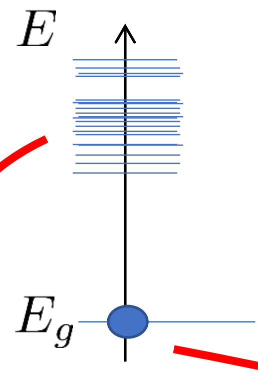
- How to classify SPT phases?
- Recall:

A theory = a set of correlation functions

- In SPT phases, all information is encoded only in the ground state.
  -> No excited states, no scale
    -> Topological field theory (TFT)
- Hilbert space is one-dimensional, therefore, we conclude:

An SPT phase $\sim$ a set of U(1)-valued partition functions

An SPT phase ~ a set of U(1)-valued partition functions



$$Z(M) = \int \prod_{\text{mat. fields}} D\phi_{\text{mat}}\, e^{-S_M^{\text{mat}}[\phi_{\text{mat}}]}$$

$$= \left|Z(M)\right| \times e^{iS_M}$$

Euclidian spacetime manifold

Excited states

Characterizes SPT phases ??

An SPT phase ~ a set of U(1)-valued partition functions

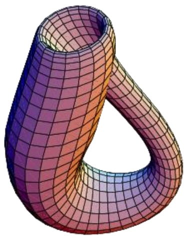
- The partition function over a closed space manifold X and time circle S^1 is always unity:

$$Z(X \times S^1) = \mathrm{Tr}(1) = \langle GS|GS \rangle = 1.$$

- Therefore, to distinguish different SPT phases, we should employ generic closed spacetime manifolds.

  - ◆ Orientation-reversing symmetry → unoriented manifolds

    Exs: Klein bottle, real projective spaces, ⋯



  - ◆ Global G symmetry → background gauge fields A

An SPT phase ~ a set of U(1)-valued partition functions

- In sum,

$$Z(M,A) = \int \prod_{\text{mat. fields}} D\phi e^{-S_M^{\text{mat}}(\phi_{\text{mat.}},A)}$$

$$= |Z(M,A)| \times e^{iS_M(A)}$$

Closed spacetime manifold (M)

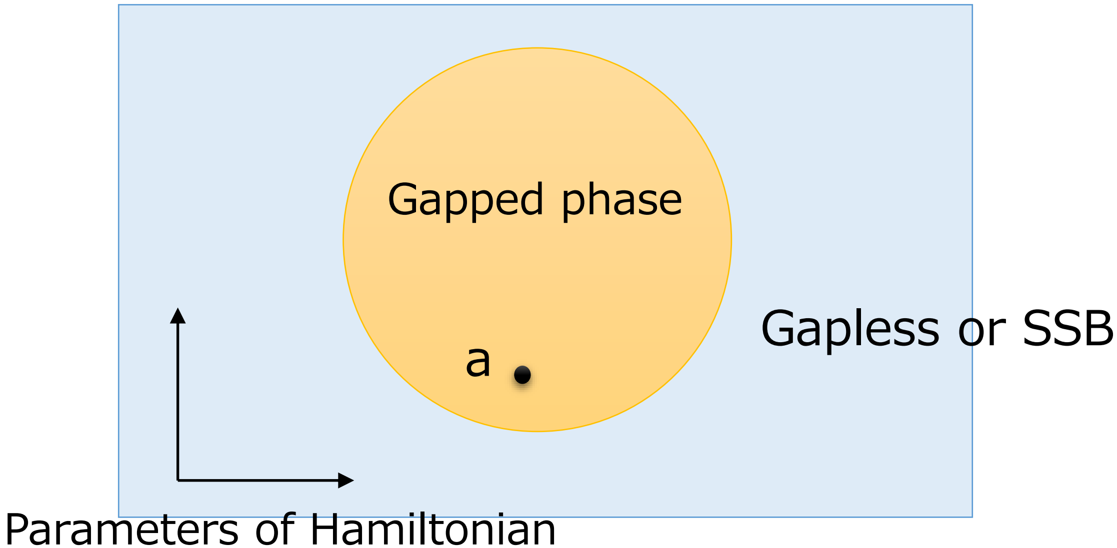
Background G-field (A)

Characterizes an SPT phase ($e^{iS_M(A)}$)

- Comment: For matter degrees of freedom with fermions, spacetime manifold must be equipped with a (variants of) spin structure.

Deformation invariance

- In SPT phases, we are interested in theories which are deformation invariant



- Ex: the (3+1)D theta term of the background U(1)-field

$$\mathrm{Exp}\left(\frac{2\pi i\theta}{4\pi^2}\int_M F^2\right)$$

has a continuous parameter $\theta \in [0,2\pi]$. This is not a partition function of an SPT phase (but a partition function of a gapped phase).

However, if θ is quantized by some symmetry (time-reversal symmetry, for example) this (partially) describes an SPT phase.

- Ex: the (2+1)D Chern-Simons form of the background U(1)-field

$$\mathrm{Exp}\left(\frac{2\pi i k}{4\pi^2}\int_M AdA\right), \quad k \in \mathbb{Z}$$

has a quantized parameter $\mathrm{k} \in Z$. This is a partition function of an SPT phase.

An SPT phase = a set of U(1)-valued deformation invariant partition functions

Classification of SPT phases

[Chen-Gu-Liu-Wen, Levin-Gu, Gu-Wen, Kapustin, Freed-Hopkins, Yonekura, ⋯]

- It was shown that SPT phases are classified by the “Anderson dual” of the “cobordism group”,

$$\Omega^d_{str}(BG) \cong Free\ \Omega^{str}_{d+1}(BG) \times Tor\ \Omega^{str}_d(BG).$$

- Free part (Z) : theta term in (d+1)D → CS term in dD.
- Torsion part (Z_q) : no continuous parameter

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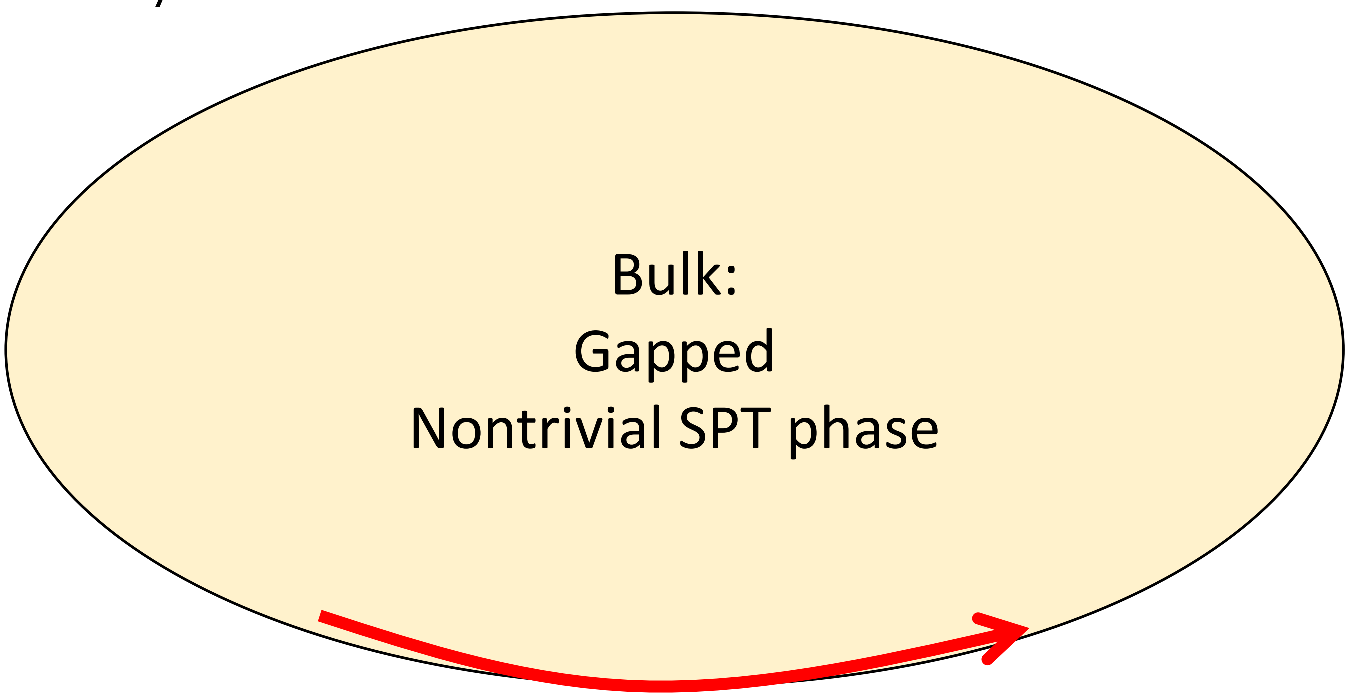
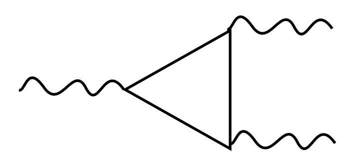
Bulk-boundary correspondence

- For an SPT phase, bulk has no signatures because it is a gapped theory.
- All physical signatures come from the boundary of space manifold.
- A bulk U(1)-valued partition function corresponds to a quantum anomaly of the boundary.



Boundary: low-energy excitation with a quantum anomaly



Ex: Cluster model [Chen-Lu-Vishwanath]

- (1+1)D bosonic SPT phase with $Z_2 \times Z_2$ symmetry
- Matter degrees: spin 1/2 at integer and half integer sites. $\sigma_j^\mu, \tau_{j+\frac{1}{2}}^\mu, j \in Z$.

$$\cdots - \tau_{\frac{1}{2}} - \sigma_1 - \tau_{\frac{3}{2}} - \quad \cdots \quad - \tau_{N-\frac{1}{2}} - \sigma_N - \tau_{N+\frac{1}{2}} - \cdots$$

- Hamiltonian (cluster model [Briegel-Raussendorf])

$$H = -\sum_j A_{j+\frac{1}{2}} - \sum_j B_j := -\sum_{j\in Z} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j\in Z} \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z$$

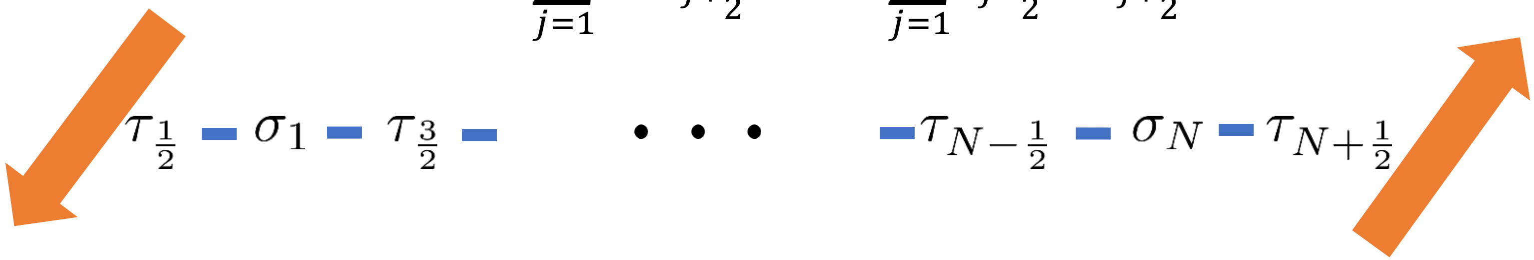
- $Z_2 \times Z_2$ symmetry operators:

$$U_\sigma = \prod_j \sigma_j^x, \qquad U_\tau = \prod_j \tau_{j+\frac{1}{2}}^x$$

- All terms are commuted with each other (frustration-free). The ground state of H is given by $A_{j+\frac{1}{2}} = B_j = 1$. An excited state has at least a finite energy $E = 2 \rightarrow$ gapped.

- Hamiltonian on the open chain

$$H = -\sum_{j=1}^{N-1} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j=1}^{N} \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z$$



- # (dof) = 2N+1, $\#(A_{j+\frac{1}{2}}, B_j)$ = 2N-1

  -> The ground state is 4-fold degenerate.

$$|\Psi(a,b)\rangle = \sum_{DDWs} \left| \left(\tau_{\frac{1}{2}}^x = a\right) - bulk - \left(\tau_{N+\frac{1}{2}}^x = b\right) \right\rangle, \qquad a, b \in \{+,-\}.$$

- This 4-fold degeneracy is not an accident, but is protected by the $Z_2 \times Z_2$ symmetry.

- To see this, let's consider how $Z_2 \times Z_2$ symmetry operators act on the ground states manifold.

$$U_\sigma \Big|_\Psi = \prod_j \sigma_j^x \Big|_\Psi = \prod_j (\tau^z_{j-\frac{1}{2}} \tau^z_{j+\frac{1}{2}}) = \tau^z_{\frac{1}{2}} \otimes \tau^z_{N+\frac{1}{2}} =: U_\sigma^L \otimes U_\sigma^R,$$

$$U_\tau \Big|_\Psi = \prod_j \tau^x_{j+\frac{1}{2}} \Big|_\Psi = \tau^x_{\frac{1}{2}} \left(\prod_j (\sigma_j^z \sigma_{j+1}^z) \right) \tau^x_{N+\frac{1}{2}} = \tau^x_{\frac{1}{2}} \sigma_1^z \otimes \sigma_N^z \tau^x_{N+\frac{1}{2}} =: U_\tau^L \otimes U_\tau^R.$$

- $Z_2 \times Z_2$ symmetry operations split into ones for the left and the right edge spins.

- $Z_2 \times Z_2$ acts on the right spin as a nontrivial projective representation, which can be seen in the algebra

$$U_\sigma^R \, U_\tau^R = -U_\tau^R U_\sigma^R.$$

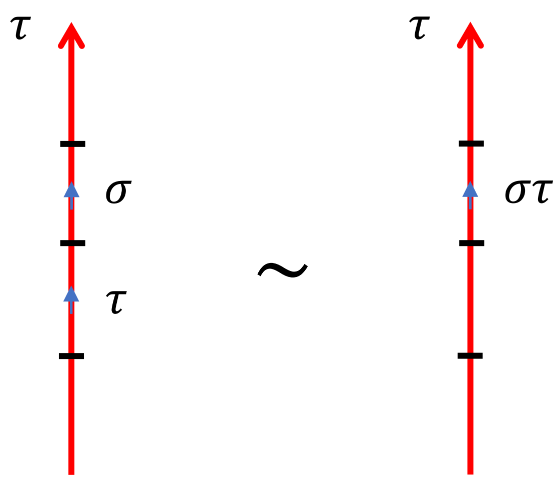
- This is a sort of quantum anomaly.

Why quantum anomaly?

- Let's consider a (0+1)D system with $Z_2 \times Z_2$ symmetry without quantum anomaly.

- An anomaly-free system is invariant under the gauge transformation of background field, implying the group low of the *G*-actions is preserved.

$$U_\sigma U_\tau = U_{\sigma\tau} = U_\tau U_\sigma$$



- Therefore, the breaking of group structure $U_\sigma^R \, U_\tau^R = -U_\tau^R U_\sigma^R$ signals a quantum anomaly.

Edge perspective: projective representations

- Let G be a finite group. A set of matrices $\{D_g\}_{g\in G}$ is called a projective representation when it is a group representation up to a U(1) phase

$$D_g D_h = \omega_{g,h}\, D_{gh}, \qquad \omega_{g,h} \in U(1).$$

- The associativity $(D_g D_h) D_k = D_g (D_h D_k)$ yields the 2-cocyle condition

$$\omega_{g,h}\omega_{gh,k} = \omega_{g,hk}\,\omega_{h,k}.$$

- A redefinition $D_g \mapsto \alpha_g\, D_g, \alpha_g \in U(1)$ yields the equivalence relation (2nd coboundary)

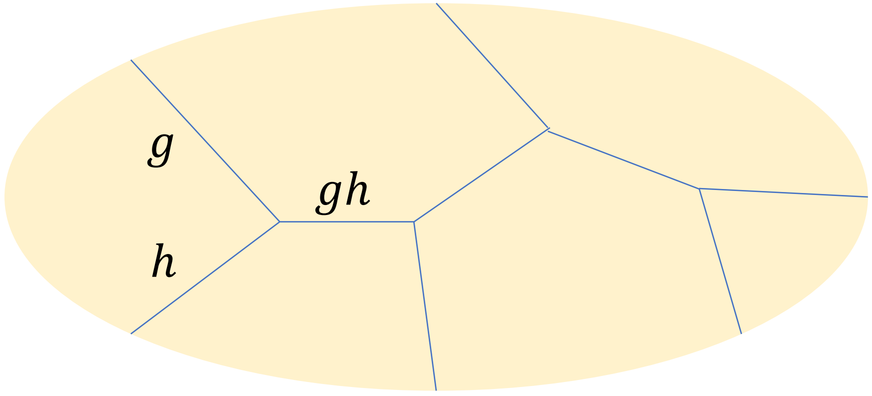
$$\omega_{g,h} \sim \omega_{g,h}\, \alpha_h \alpha_{gh}^{-1} \alpha_g$$

- The factor system $\omega_{g,h}$ is classified by the 2nd group cohomology

$$H^2\big(G, U(1)\big) = Z^2(G, U(1))/B^2(G, U(1)).$$

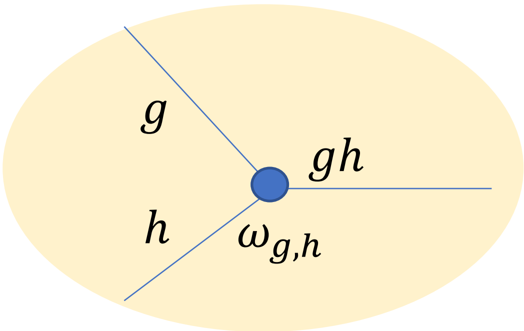
Bulk perspective: (1+1)D Dijkgraaf-Witten theory

- Let's consider the relationship between edge anomaly and the bulk U(1)-valued partition function.
- A G-field A over a spacetime manifold M is a symmetry defect network over M.
- When a matter field passes a defect line labeled by $g \in G$, the matter field is charged by g.

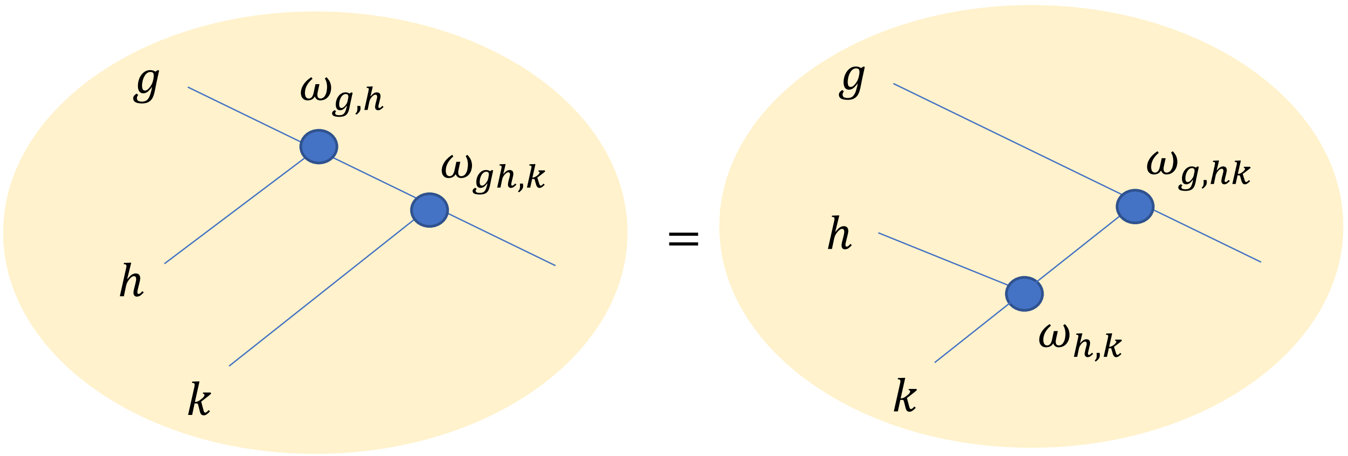


- The ansatz for U(1)-valued topological action:

$$e^{iS_M(A)} = \prod_{\text{junctions}} \omega_{g,h}, \quad \omega_{g,h} \in U(1).$$



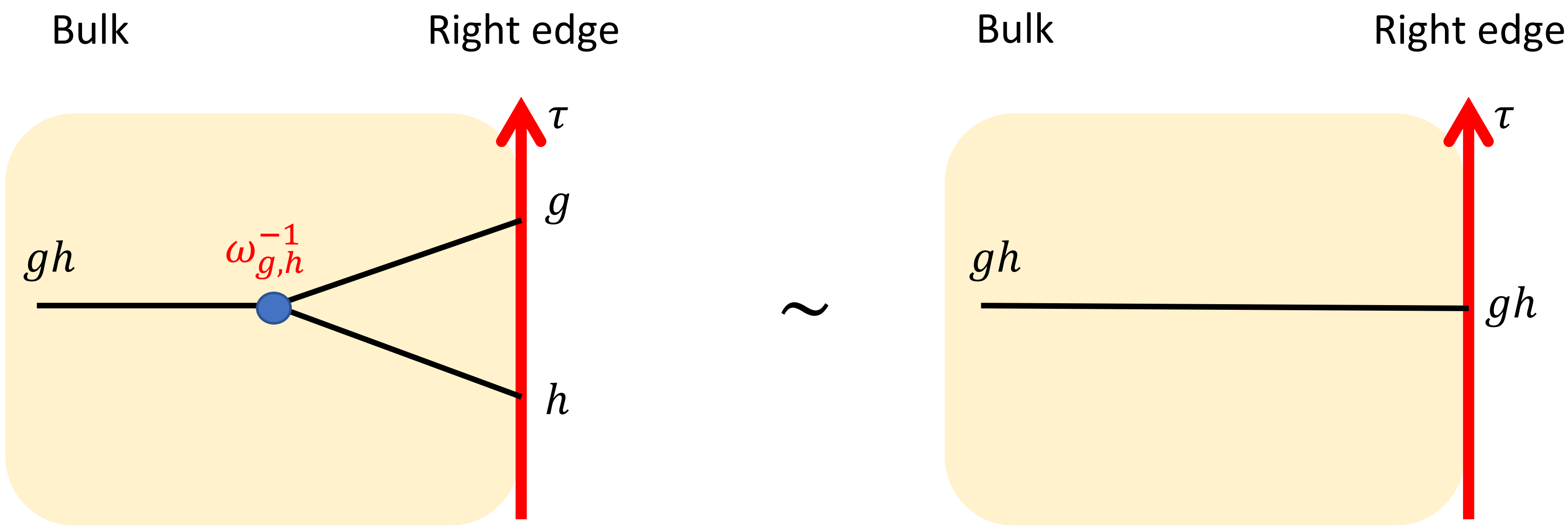
- Gauge invariance requires the 2-cocycle condition on $\omega_{g,h}$.



- We get the (1+1)D Dijkgraaf-Witten topological action labeled by a group cocycle $\omega_{g,h}$.

Anomaly cancellation

- The total system composed of the bulk and the boundary is anomaly free, namely, invariant under gauge transformations.



$U_g^R U_h^R = \omega_{g,h} U_{gh}^R$ $\qquad$ U_{gh}^R

Bulk-boundary correspondence

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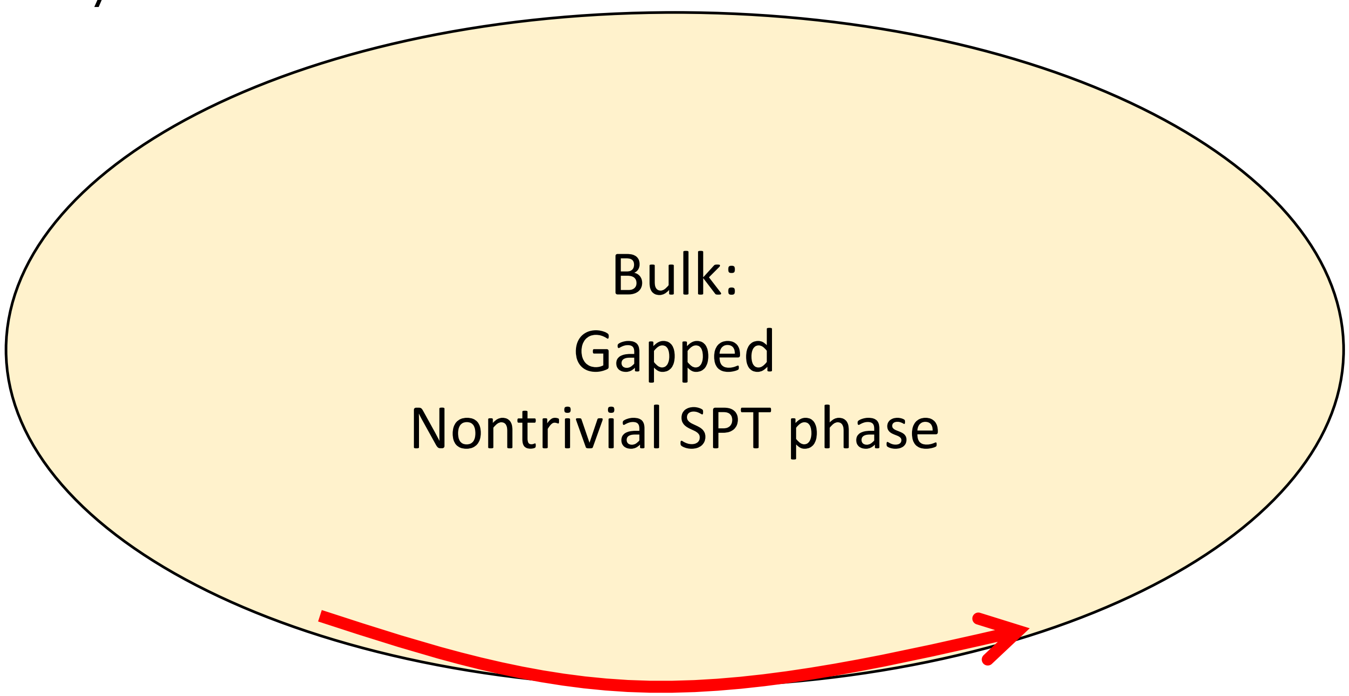


Boundary: low-energy excitation with a quantum anomaly

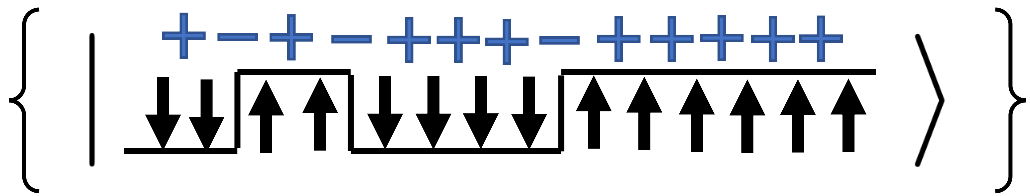
Short summary

- SPT phases are classified by U(1)-valued partition function of which coefficients are quantized.

- A characteristic of SPT phase is the bulk-boundary correspondence: There is one-to-one correspondence between an SPT phase of bulk and a quantum anomaly of boundary.

- Let us write the ground state with the bases of $\sigma_j^z = \{\uparrow, \downarrow\}, \tau^x_{j+\frac{1}{2}} = \{+. -\}$.

- 1st terms -> $\sigma_j^z \sigma_{j+1}^z = \tau^x_{j+\frac{1}{2}}$ -> decorated domain walls (DDWs)



- 2nd terms $\sum_{j \in Z} \tau^z_{j-\frac{1}{2}} \sigma_j^x \tau^z_{j+\frac{1}{2}}$ fluctuate the decorated domain walls

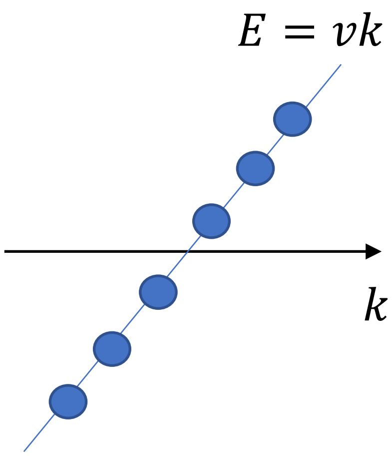
  -> The ground state is the equal-weight superposition of the decorated domain walls.

$$|\Psi\rangle = \sum_{DDWs} |\, DDW \,\rangle$$

(2+1)d example: the integer quantum Hall state

- Matter: Dirac fermion with U(1) symmetry.
- Bulk U(1)-valued partition function: Chern-Simons form $\exp \frac{i}{4\pi} \int_M AdA$.
- Boundary: chiral Dirac fermion

$$\hat{H}_{bdy} = \sum_{k \in Z + \frac{\theta}{2\pi}} vk\, \psi_k^\dagger \psi_k$$



- The partition function is not invariant under the large gauge transformation $\theta \mapsto \theta + 2\pi$.
- The anomaly on the boundary is cancelled by the bulk CS action.

Outline

Part 2 SPT phases as a generalized (co)homology theory.

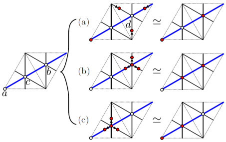
Part 3 The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems. (if I have much time)

The following talk is based on KS-Xiong-Gomi, arXiv:1810.00801.

Related works:
Teo-Kane, Kitaev, Xiang, Gaiotto-Johnson-Freyd, Song-Huang-Fu-Hermele, Po-Watanabe-Jian-Zaletel, Thorngren-Else, Shenghan-Meng-Yang, Song-Fang-Yang, Okuma-Sato-KS, Freed-Hopkins, ...

Literature

- Kitaev proposed that SPT phases are described by a generalized (co)homology theory. [11]
- Gaiotto and Johnson-Freyd studied this proposal in the perspective of field theory [18].
- In studying crystalline SPT phase in condensed matter physics, two similar strategies appeared.
  - "Dimensional reduction" to classify SPT phases with crystalline symmetry [Song-Huang-Fu-Hermele 16],
  - "Lattice homotopy" to classify LSM-type theorems [Po-Watanabe-Jian-Zaletel 17].
- These two procedures would be summarized as "trivializing something nontrivial living in low-dimensional spaces by using something nontrivial living in higher-dimensional spaces. (The image from [Huang-Song-Huang-Hermele, 17])



- This reminds us the Atiyah-Hirzebruch spectral sequence (AHSS) of the generalized homology.
- We reconstruct these studies in terms of the AHSS of the generalized homology theory. [KS-Xiong-Gomi]

Why generalized (co)homology?

Two important "physical" observations (not mathematically rigorous):

(1) The space of short-range entangled (SRE) states (= unique gapped ground states) forms an Ω-spectrum of a generalized (co)homology theory. [Kitaev 11,13,15]

(2) The bulk-boundary correspondence is regarded as the boundary map of a generalized homology theory.

SRE states and an Ω-spectrum [Kitaev]

- Let F_n be the "space of SRE states in n-spatial dimensions".
- F_n is a based topological space. The trivial tensor product state can be regarded as a base point $* = |0\rangle \in F_n$.
- Kitaev proposed that $\{F_n\}_{n \in Z}$ forms an Ω-spectrum, i.e., F_n is homotopically equivalent to the loop space ΩF_{n+1},

$$F_n \sim \Omega F_{n+1},$$

  where $\Omega X = \{\ell : [0,1] \to X | \ell(0) = \ell(1) = *\}$ is the based loop space of X.
- As a matter of mathematical fact, given an Ω-spectrum $\{F_n\}_{n \in \mathbb{Z}}$, one can construct generalized cohomology and homology theories.

$$h^n(X, Y) = [X/Y, F_n],$$
$$h_n(X, Y) = \mathrm{colim}_{k \to \infty}[S^{n+k}, (X/Y) \wedge F_k].$$

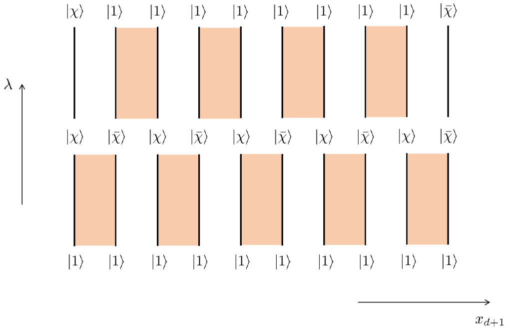
- cf. $F(Y) =$"interacting Hamiltonians over Y". $F_n := F(D^n, \partial D^n)$. [Kitaev 11]

Discussion of the homotoly equivalence $F_n \to \Omega F_{n+1}$ [Kitaev]

- A characteristic of SRE state is the existence of its inverse

$$|\chi\rangle \otimes |\bar{\chi}\rangle \sim |1\rangle \otimes |1\rangle \,,$$

  where $|\chi\rangle \in F_n$ and $|1\rangle = * \in F_n$ is a trivial tensor product state. ("invertible state")
- For a given n-dim. SRE state $|\chi\rangle \in F_n$, one can canonically construct an adiabatic pumping process that pumps the SRE state $|\chi\rangle$ from the right to the left boundaries.



$$F_n \to \Omega F_{n+1} = \{|\psi(\lambda)\rangle : [0,1] \to F_n \,|\, |\psi(0)\rangle = |\psi(1)\rangle = |\mathrm{triv}\rangle\}$$

Some useful mathematical facts and physical interpretation

- By design, the classification of n-dim. SPT phases is given by the disconnected parts of F_n,

$$\pi_0(F_n) = [pt, F_n] = h^n(pt).$$

- From the Poincaré duality and the suspension isomorphism,

$$h^n(pt) = h_{-n}(pt) = h_0(D^n, \partial D^n).$$

- $h_0(D^n, \partial D^n)$ can be identified with the classification of SPT phases over D^n relative to its boundary ∂D^n.

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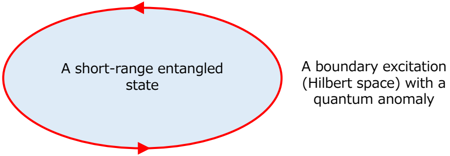
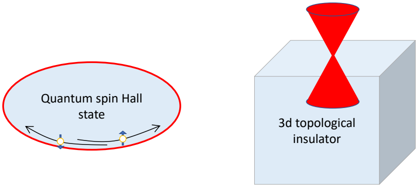
The bulk-boundary correspondence



- Ex: Haldane chain protected by either TRS or $\mathbb{Z}_2 \times \mathbb{Z}_2$ onsite symm.

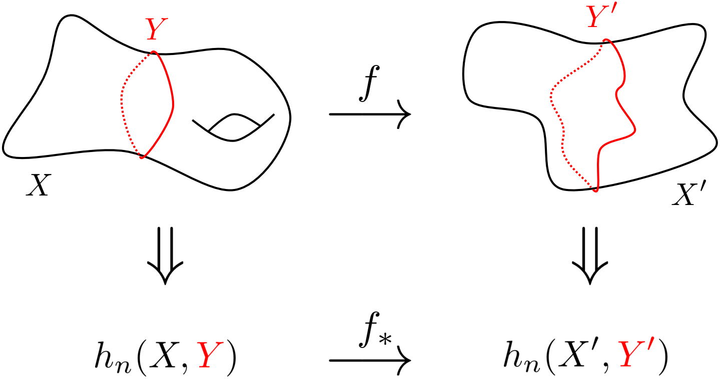
- Ex: $2d$ and $3d$ topological insulator protected by TRS.



- The bulk-boundary correspondence reminds us the boundary map of homology $\partial : X \to \partial X$.

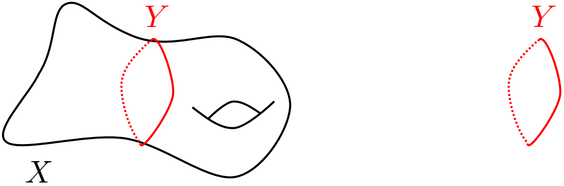
The axioms of generalized homology theories

- A generalized homology theory $h_n(X, Y)$, $Y \subset X$, $n \in \mathbb{Z}$, is a covariant functor from topological spaces to ablian groups.
- For a given map $f : (X, Y) \to (X', Y')$, we have a homomorphism $f_* : h_n(X, Y) \to h_n(X', Y')$ with the same direction.



- Equipped with the boundary map

$$\partial : h_n(X, Y) \to h_{n-1}(Y).$$



$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$h_n(X, Y) \xrightarrow{\partial} h_{n-1}(Y)$$

- Axioms of generalized homology theory:
  - (homotopy)
    If $f, f' : X \to X'$ are homotopic, then $f_* = f'_*$.
  - (excision)
    For $A, B \subset X$, the inclusion $A \to A \cup B$ induces an isomorphism $h_n(A, A \cap B) \to h_n(A \cup B, B)$.
  - (additivity)
    $h_n(\sqcup_\lambda X_\lambda, \sqcup_\lambda Y_\lambda) = \sqcup_\lambda h_n(X_\lambda, Y_\lambda)$.
  - (exactness)
    For $Y \subset X$, there is a long exact sequence

$$\cdots \to h_n(Y) \to h_n(X) \to h_n(X, Y) \xrightarrow{\partial} h_{n-1}(Y) \to \cdots$$

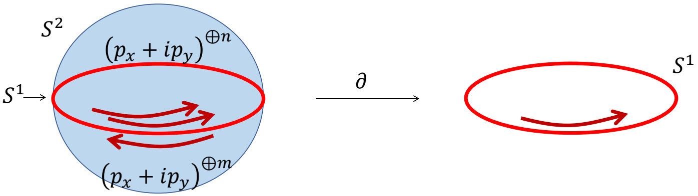
- What is homology group $h_n(X, Y)$ for SPT phases?

From SPT phases to a generalized homology theory

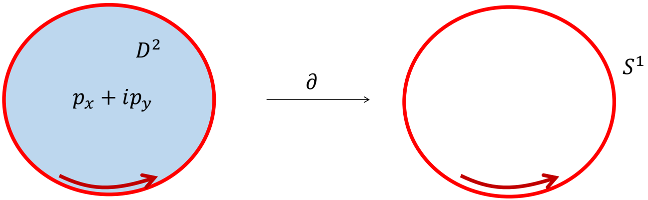
- $h_0(X, Y) :=$ the abelian group of SPT phases over a real space X which may have an quantum anomaly over a real space $Y \subset X$.
- We define the boundary map $\partial : h_0(X, Y) \to h_{-1}(Y)$ as the bulk-boundary correspondence.
- This implies $h_{-1}(Y)$ should be regarded as the abelian group of quantum anomaly over a real space Y.

Ex: Superconductors over $X = S^2$ that may have an anomalous edge state over the equator $Y = S^1$. We have

$$h_0(S^2, S^1) = \mathbb{Z} \times \mathbb{Z}, \qquad h_{-1}(S^1) = \mathbb{Z},$$
$$\partial : h_0(S^2, S^1) \to h_{-1}(S^1), \quad (n, m) \mapsto n - m.$$



- The ordinary bulk-boundary correspondence is the special case of the boundary map ∂ where $X = D^n$ and $Y = \partial D^n$.

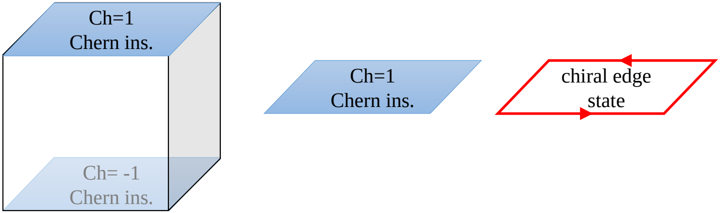


For generic $n \in \mathbb{Z}$

- The degree $n \in \mathbb{Z}$ of the generalized homology theory $h_n(X, Y)$ can be understood as a kind of a "degree of SPT phenomena".
- The proper meaning of the $(n-1)$-th homology is obtained by considering what the physical phenomenon living on the boundary of the n-th homology is.

$$\xrightarrow{\partial} \quad n = 1 \quad \xrightarrow{\partial} \quad n = 0 \quad \xrightarrow{\partial} \quad n = -1 \quad \xrightarrow{\partial}$$

Adiabatic pump — SPT phase — Anomaly



$\vdots$

- $h_1(X, Y) :=$ the abelian group of adiabatic pumps over a real space X which may create a SRE state on $Y \subset X$.
- $h_0(X, Y) :=$ the abelian group of SPT phases over a real space X which may have anomalous excitation on $Y \subset X$.
- $h_{-1}(X, Y) :=$ the abelian group of anomalous theories over a real space X which may have a “source or sink” of an anomalous excitation on $Y \subset X$.

$\vdots$

“Physical definition” of $h_n(X, Y)$ v.s. the axioms

Let's consider if the above identification of the group $h_n(X, Y)$ with a physical phenomenon related to SPT phases satisfies the axioms.

- ✓ A covariant functor (Because of the real-space picture)
- ✓ (homotopy)
  If $f, f' : X \to X'$ are homotopic, then $f_* = f'_*$.
- ✓ (excision)
  For $A, B \subset X$, the inclusion $A \to A \cup B$ induces an isomorphism $h_n(A, A \cap B) \to h_n(A \cup B, B)$.
- ✓ (additivity)
  $h_n(\sqcup_\lambda X_\lambda, \sqcup_\lambda Y_\lambda) = \sqcup_\lambda h_n(X_\lambda, Y_\lambda)$.
- ✓ (exactness)
  For $Y \subset X$, there is a long exact sequence

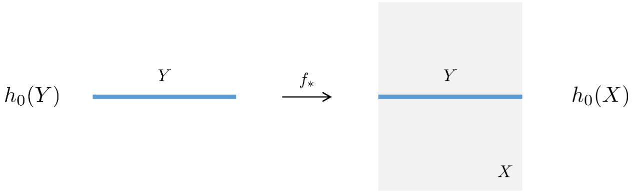
$$\cdots \to h_n(Y) \to h_n(X) \to h_n(X, Y) \xrightarrow{\partial} h_{n-1}(Y) \to \cdots$$

... It looks OK.

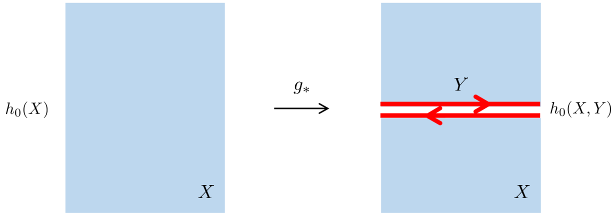
Exactness

$$\cdots \xrightarrow{\partial^1} h_0(Y) \xrightarrow{f_*^0} h_0(X) \xrightarrow{g_*^0} h_0(X,Y) \xrightarrow{\partial^0} h_{-1}(Y) \xrightarrow{f_*^{-1}} \cdots$$

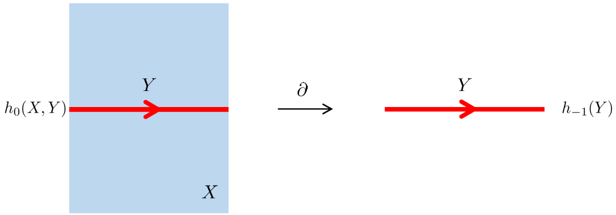
- f_* and g_* are induced homomorphisms of inclusions $f : X \to Y$ and $g : (X, \emptyset) \to (X,Y)$, respectively.
- f_*^0 is regarded as embedding an SPT phase over Y in X.



- g_*^0 is regarded as cutting out Y from X, which leads to anomalous states over Y from an SPT phase over X.



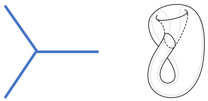
- ∂^0 is the bulk-boundary correspondence.



- From these physical interpretations, we can see the long exact sequence is compatible with properties of the SPT phases.

How useful is it?

- So far, I have discussed the abstract mathematical structure of SPT phases.
- There are several practical merits to study SPT phases.
- The real space X can be an arbitrary real space, which can not be a manifold (a manifold is locally Euclidean). For example, we can ask what is the classification of SPT phases, adiabatic pumps, anomalies over a trijunction, the Klein bottle, ...



- Since the generalized homology description is based on the real space on which SPT phases defined, it is straightforward to implement spatial symmetry (point group and space group).

  $\Rightarrow$ Equivariant homology $h_n^G(X, Y)$.

  In particular, the Atiyah-Hirzebruch Spectral Sequence (AHSS), a spectral sequence of generalized (co)homology theories, gives us a systematic way to thinking the interplay of crystalline symmetry and SPT phases.

Summary for part 2

- Invertibility of SRE states
  $\Rightarrow$ an Ω-spectrum
  $\Rightarrow$ a generalized (co)homology theory.
- The mathematical structure behind SPT phenomena such as SPT phases, anomalous excitations, adiabatic pumps, can be understood in the framework of the generalized homology theory.
- Open questions:
  - What is the generalized (co)homology description of the bulk-defect correspondence?
    $\rightarrow$ Probably, the KK-theory, which is like a combination of the homology and the cohomology, does work.
    Real space : homological
    Parameter space: cohomological
  - Can we find good physical interpretation for $h_{-2}(X, Y), h_{-3}(X, Y), \dots$?
- Applying the Atiyah-Hirzebruch spectral sequence, which is a well-developed machinery in generalized (co)homology theories, to crystalline SPT phases gives us the comprehensive understanding of higher-order SPT phases and LSM theorems. (Part 3)

Outline

Part 2 SPT phases as a generalized homology theory.

Part 3 The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems. (if I have much time)

Atiyah-Hirzebruch Spectral Sequence (AHSS)

- The AHSS [Atiyah-Hirzebruch '61] is a spectral sequence to compute a generalized (co)homology theory h_*.
- This is the mathematical structure behind the "dimensional reduction" [Song-Huang-Fu-Hermele 16] and the "lattice homotopy" [Po-Watanabe-Jian-Zaletel 17], but the AHSS goes beyond and complete their strategy. [KS-Xiong-Gomi, Song-Fang-Qi, Jiang-Cheng-Qi, Else-Thorngren]
- In general, a spectral sequence starts from the E^1-page, which is a something computable.
- We compute the nth differential $(n = 1, 2, \ldots,)$

$$d^n_{p,q} : E^n_{p,q} \to E^n_{p-n,q+n-1}, \qquad d^n \circ d^n = 0.$$

- The next page is defined as the homology of d^n,

$$E^{n+1}_{p,q} = \text{Ker } d^n_{p,q} / \text{Im } d^n_{p+n,q-n+1}.$$

- Assume that this iteration converges at some E^r-page.

$$E^1 \Rightarrow E^2 \Rightarrow \cdots E^r = E^{r+1} = \cdots =: E^\infty.$$

- The E^∞-page approximates the homology theory $h^*(X, Y)$. (see below)

- The starting point of the AHSS is to give a filtration of the space X,

$$X_0 \subset X_1 \subset \cdots X.$$

- A useful filtration is a cell-decomposition

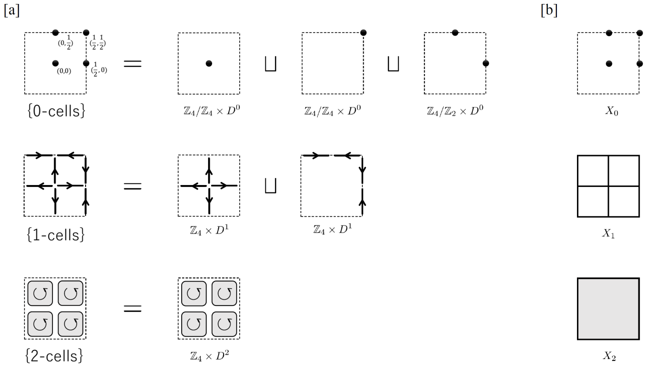
$$X = \{0\text{-cells}\} \sqcup \{1\text{-cells}\} \sqcup \{2\text{-cells}\} \sqcup \cdots$$

  with the following property: On each p-cell D^p the crystalline symmetry G behaves as an onsite symmetry of the little group G_{D^p} over the p-cell D^p., which we call a *uniform cell decomposition* (only for this slide).
- The p-skeleton X_p is defined by

$$X_0 = \{0\text{-cells}\}, \qquad X_p = X_{p-1} \cup \{p\text{-cells}\}.$$

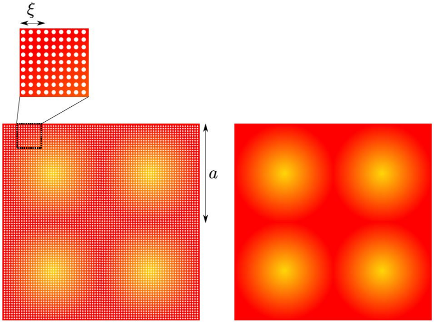
A uniform cell decomposition

Ex: 2d real space with C_4 rotation $\ltimes \mathbb{Z}^2$ translation symmetry.



Topological Crystalline Liquid [Thorngren-Else 16]

- Please keep in your mind the following picture:
  The spatial scale a of crystalline symmetry is much larger than the scale ξ of microscopic degrees of freedom.
- Of course, this is not the case in cond-mat problems, however, it can be used to classify SPT phases because the effective theory would be "topological". (Under debate.)



(a) Smooth state (b) Topological limit

FIG. 1. (a) In a smooth state, the lattice spacing and the correlation length ξ are much less than the unit cell size a and the radius of spatial variation. (b) The topological response of a crystalline topological liquid is captured by a spatially-dependent TQFT that captures the spatial dependence within each unit cell but "forgets" about the lattice.

E^1-page

- The E^1-page is defined as
$$E^1_{p,-q} := h^G_{p-q}(X_p, X_{p-1}),$$
the $(p-q)$-th homology over X_p relative to X_{p-1}.
- For instance, $E^1_{p,-p} = h^G_0(X_p, X_{p-1})$ is the abelian group of SPT phases over X_p which can be anomalous over X_{p-1}.
- For a uniform cell decomposition, we have equalities
$$\begin{aligned} E^1_{p,-q} &\cong \prod_{j \in p\text{-cells}} h^{G_{D^p_j}}_{p-q}(D^p_j, \partial D^p_j) \cong \prod_{j \in p\text{-cells}} \tilde{h}^{G_{D^p_j}}_{p-q}(D^p_j / \partial D^p_j (= S^p)) \\ &\cong \prod_{j \in p\text{-cells}} h^{G_{D^p_j}}_{-q}(pt) \qquad \text{(suspension iso.)} \\ &\cong \prod_{j \in p\text{-cells}} h^{G_{D^p_j}}_0(D^q, \partial D^q) \end{aligned}$$
- Therefore, $E^1_{p,-q}$ is the ablian group of q-dim. SPT phases (we denote them by SPT^q) with the onsite $G_{D^p_j}$ symmetry.

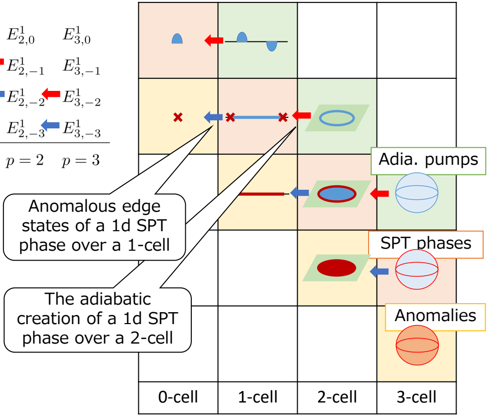
The first differential d^1

- The E^1-page hosts the "local information" of SPT phenomena.
- We should properly glue the local information together, which is partly done by the first differential $d^1_{p,-q} : E^1_{p,-q} \to E^1_{p-1,-q}$.
- The first differential d^1 can be "physically" understood and is computable. (cf. [Song-Huang-Fu-Hermele 16, Po-Watanabe-Jian-Zaletel 17])

SPT^0	$q=0$	$E^1_{0,0}$	$E^1_{1,0}$	$E^1_{2,0}$	$E^1_{3,0}$
SPT^1	$q=1$	$E^1_{0,-1}$	$E^1_{1,-1}$	$E^1_{2,-1}$	$E^1_{3,-1}$
SPT^2	$q=2$	$E^1_{0,-2}$	$E^1_{1,-2}$	$E^1_{2,-2}$	$E^1_{3,-2}$
SPT^3	$q=3$	$E^1_{0,-3}$	$E^1_{1,-3}$	$E^1_{2,-3}$	$E^1_{3,-3}$
	$E^1_{p,-q}$	$p=0$	$p=1$	$p=2$	$p=3$

$$d^1_{p,-q} : E^1_{p,-q} \to E^1_{p-1,-q}$$

Homology



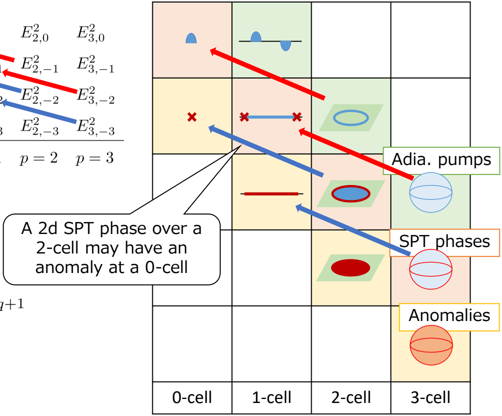
The second differential d^2

- The E^2-page hosts the "local information" of SPT phenomena which are glued together over the 1-skeleton X_1.
- We should further compute the compatibility over 2-cells, which is represented by the second differential $d^2_{p,-q} : E^2_{p,-q} \to E^2_{p-2,-q+1}$.
- The second differential d^2 is also "physically" understood and computed.

SPT^0	$q=0$	$E^2_{0,0}$	$E^2_{1,0}$	$E^2_{2,0}$	$E^2_{3,0}$
SPT^1	$q=1$	$E^2_{0,-1}$	$E^2_{1,-1}$	$E^2_{2,-1}$	$E^2_{3,-1}$
SPT^2	$q=2$	$E^2_{0,-2}$	$E^2_{1,-2}$	$E^2_{2,-2}$	$E^2_{3,-2}$
SPT^3	$q=3$	$E^2_{0,-3}$	$E^2_{1,-3}$	$E^2_{2,-3}$	$E^2_{3,-3}$
	$E^2_{p,-q}$	$p=0$	$p=1$	$p=2$	$p=3$

$$d^2_{p,-q} : E^2_{p,-q} \to E^2_{p-2,-q+1}$$

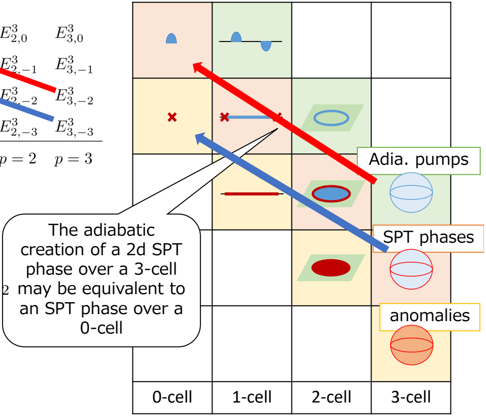
⇩ Homology



The third differential d^3

- In the same way, we have the third differential $d^3_{p,-q} : E^3_{p,-q} \to E^3_{p-3,-q+2}$.
- The third differential d^3 is also "physically" understood and computed.

SPT^0	$q=0$	$E^3_{0,0}$	$E^3_{1,0}$	$E^3_{2,0}$	$E^3_{3,0}$
SPT^1	$q=1$	$E^3_{0,-1}$	$E^3_{1,-1}$	$E^3_{2,-1}$	$E^3_{3,-1}$
SPT^2	$q=2$	$E^3_{0,-2}$	$E^3_{1,-2}$	$E^3_{2,-2}$	$E^3_{3,-2}$
SPT^3	$q=3$	$E^3_{0,-3}$	$E^3_{1,-3}$	$E^3_{2,-3}$	$E^3_{3,-3}$
	$E^3_{p,-q}$	$p=0$	$p=1$	$p=2$	$p=3$



$d^3_{p,-q} : E^3_{p,-q} \to E^3_{p-3,-q+2}$

$\Downarrow$ Homology

E^∞-page

Filtration of the homology group

- E^{∞}-page itself does not provides the classification of SPT phenomena.
- Introduce the following subgroups of $h_n^G(X, Y)$,

$$F_p h_n := \mathrm{Im}\ [h_n^G(X_p, X_p \cap Y) \to h_n^G(X, Y)], \qquad p = 0, 1, \dots.$$

- This has the clear physical meaning. For instance, $F_p h_0$ is the classification of SPT phases over the p-skeleton X_p which persists after being embedded in the whole space X.
- We have a filtration of the homology group

$$0 \subset F_0 h_n \subset F_1 h_n \subset \cdots \subset F_d h_n = h_n^G(X, Y),$$

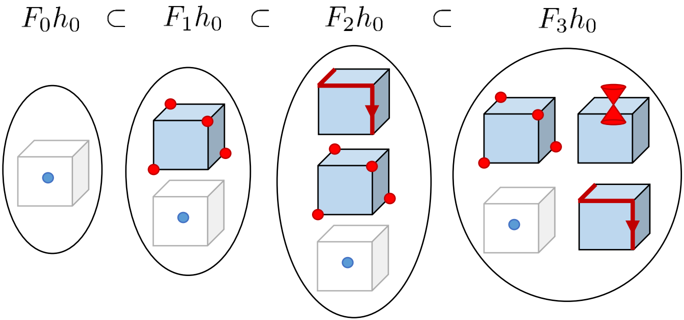
  where d is the space dimension of X.
- The following relation connects the E^{∞}-page and the homology group.

$$F_p h_n / F_{p-1} h_n \cong E^{\infty}_{p,n-p}.$$

- The E^{∞}-page has good physical meanings.

Higher-order SPT phases

- $E^{\infty}_{p,-p}$: The classification of $(d-p+1)$th-order SPT phases. (cf. Huang-Song-Huang-Hermele)
- Ex: 3d with point group symmetry (without translation symmetry):



- This unifies the terminology of "strong" and "weak" SPT phases and higher-order SPT phases.

Ex: the classification of higher-order TIs with magnetic point group symmetry via the AHSS [Okuma-Sato-KS, cf. Cornfeld-Chapman, KS]

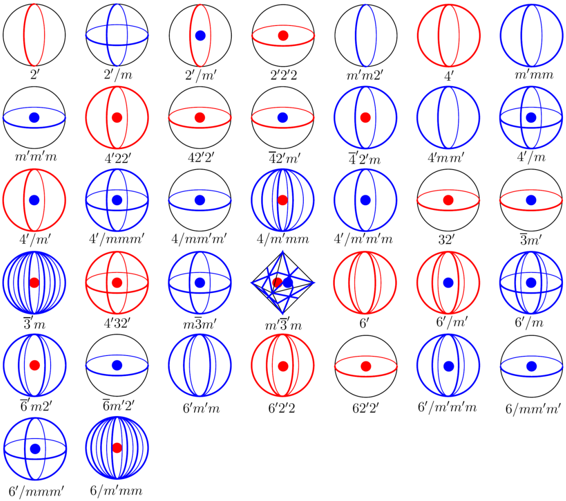
G	$2'$	$2'/m$	$2'/m'$	$2'2'2$	$m'm2'$	$4'$	$m'mm$	$m'm'm$	$4'22'$	$42'2'$	$\overline{4}2'm'$	$\overline{4}'2'm$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^2$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

2nd-order TI

G	$4'mm'$	$4'/m$	$4'/m'$	$4'/mmm'$	$4/mm'm'$	$4/m'mm$	$4'/m'm'm$	$32'$	$\overline{3}m'$	$\overline{3}'m$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$E_{0,0}^{\infty}$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}^3$	$\mathbb{Z}_2$
$E_{2,-2}^{\infty}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

G	$4'32'$	$m\overline{3}m'$	$m'\overline{3}'m$	$6'$	$6'/m'$	$6'/m$	$\overline{6}'m2'$	$\overline{6}m'2'$	$6'm'm$	$6'2'2$	$62'2'$
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}_2^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^6$
$E_{0,0}^{\infty}$	$\mathbb{Z}_2^3$	$\mathbb{Z}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$
$E_{2,-2}^{\infty}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

G	$6'/m'm'm$	$6/mm'm'$	$6'/mmm'$	$6/m'mm$	Others[†]
$K_0^G(\mathbb{E}^3)$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$	0
$E_{0,0}^{\infty}$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	0
$E_{2,-2}^{\infty}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0


$2'$
$2'/m$
$2'/m'$
$2'2'2$
$m'm2'$
$4'$
$m'mm$
$m'm'm$
$4'22'$
$42'2'$
$\overline{4}2'm'$
$\overline{4}'2'm$
$4'mm'$
$4'/m$
$4'/m'$
$4'/mmm'$
$4/mm'm'$
$4/m'mm$
$4'/m'm'm$
$32'$
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$m\overline{3}m'$
$m'\overline{3}'m$
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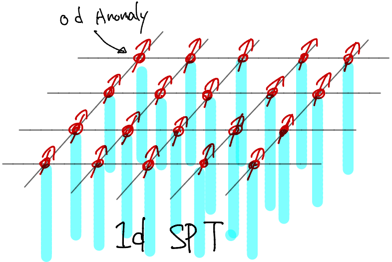
$6'/m'$
$6'/m$
$\overline{6}'m2'$
$\overline{6}m'2'$
$6'm'm$
$6'2'2$
$62'2'$
$6'/m'm'm$
$6/mm'm'$
$6'/mmm'$
$6/m'mm$

LSM type theorems

- LSM-type theorems forbid the system with a sort of dof having a unique symmetric gapped ground state in the presence of translation symmetry and others. [Chen-Gu-Wen 11, Watanabe-Po-Vishwanath-Zaletel 15]
- The group $E^{\infty}_{0,-1}$ is the classification of the LSM theorem with crystalline G symmetry. (cf. Po-Watanabe-Jian-Zaletel 17,)
- See [KS-Xiong-Gomi 18, Else-Thorngren 19, Jiang-Cheng-Qi 19] for the detail.
- " A LSM theorem as a boundary of an SPT phase" [Metlitski-Thorngren, ...]
- Using the AHSS, one can systematically classify the LSM-type theorems for a given space group and onsite symmetry. Many symmetry classes are remain unclassified.



Summary for part 3

- The AHSS gives us a useful tool to study the SPT phases and LSM theorems with crystalline symmetry with respect to high-symmetry regions in the real space.
- The differentials of the AHSS can be physically understood, thus they are computable from physical arguments. See KS-Xiong-Gomi for various worked examples of higher-differentials.
- The E^{∞}-page itself has a physical meaning. It represents the classification of higher-order SPT phases, anomalies, and adiabatic pumps. In particular, $E^{\infty}_{-1,0}$ is the classification of LSM theorems.