Higher groups and topological phases of matter

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References:

- **JHEP 2018 (10), 49**
- **JHEP 2019 (5), 64**

Marie Skłodowska-Curie Actions

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Plan

- Higher symmetry structures: topological defects and gauging.
- Anomalies and symmetry protected topological phases.
- Generalized topological gauge theories.
- Outlook and open directions

- Global symmetries = Collection of topological defects i.e symmetry operators.

- 0-form symmetry operators are:

1. Co-dimension-1 in spacetime.
2. Topological (commute with energy-momentum tensor).
3. Invertible
4. Form a fusion category

Gaiotto-Kapustin-Seiberg-Willett

q-form Global Symmetries

- **Consider a d+1 manifold M.**

- **Operators:** $\mathscr{U}_{\mathrm{g}}(\Sigma_{d-q})$.

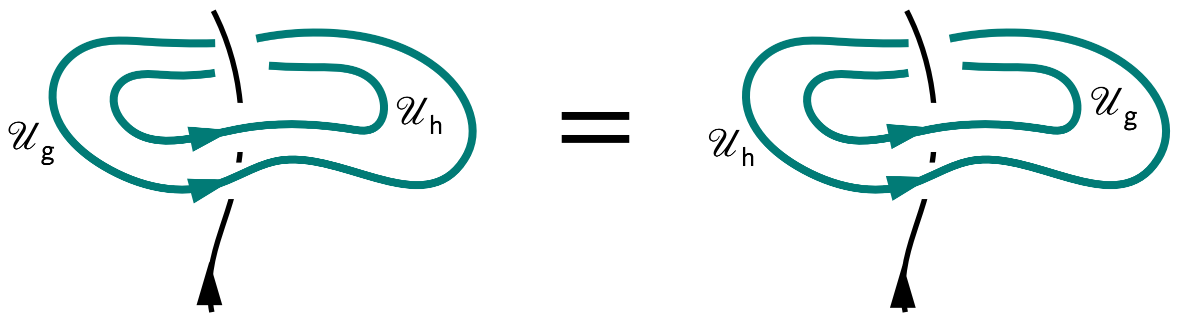
  1. **Co-dimension-*q* in spacetime.**
  2. **Topological (commute with energy-momentum tensor).**
  3. **Invertible** $\mathscr{U}_{\mathrm{g}}(\Sigma) \circ \mathscr{U}_{\mathrm{g}^{-1}}(\Sigma) \simeq 1$ **.**
  4. **Form a group-like fusion category** $\mathrm{g} \in \mathrm{G}$**.**

- **Charges** $\mathscr{O}_\alpha(X_q)$ **.**

  - **Labelled by** $\alpha \in \mathbf{Rep}(\mathscr{C})$ **.**

$$\mathscr{U}_{\mathrm{g}}(\Sigma)\mathscr{O}_\alpha(X)\mathscr{U}_{\mathrm{g}}^{-1}(X) = \alpha(\mathrm{g})^{\mathbf{Link}(\Sigma,X)}\mathscr{O}_\alpha(X)$$

  - **When** $q>0$, G **is abelian.**



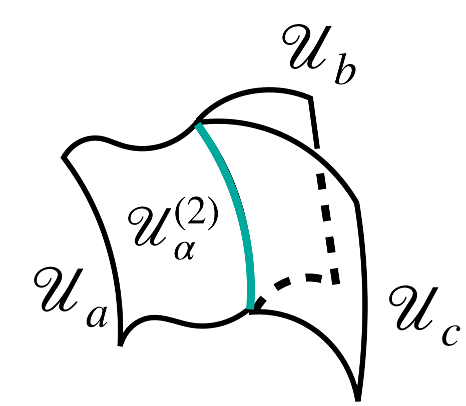
Generalized global symmetry structures

n-Fusion category symmetry

- 0-form global symmetries
- q-form global symmetries
- 2-group global symmetries etc.
- Non-Invertible symmetry structures

<u>n-fusion category $\mathscr{C}$:</u>

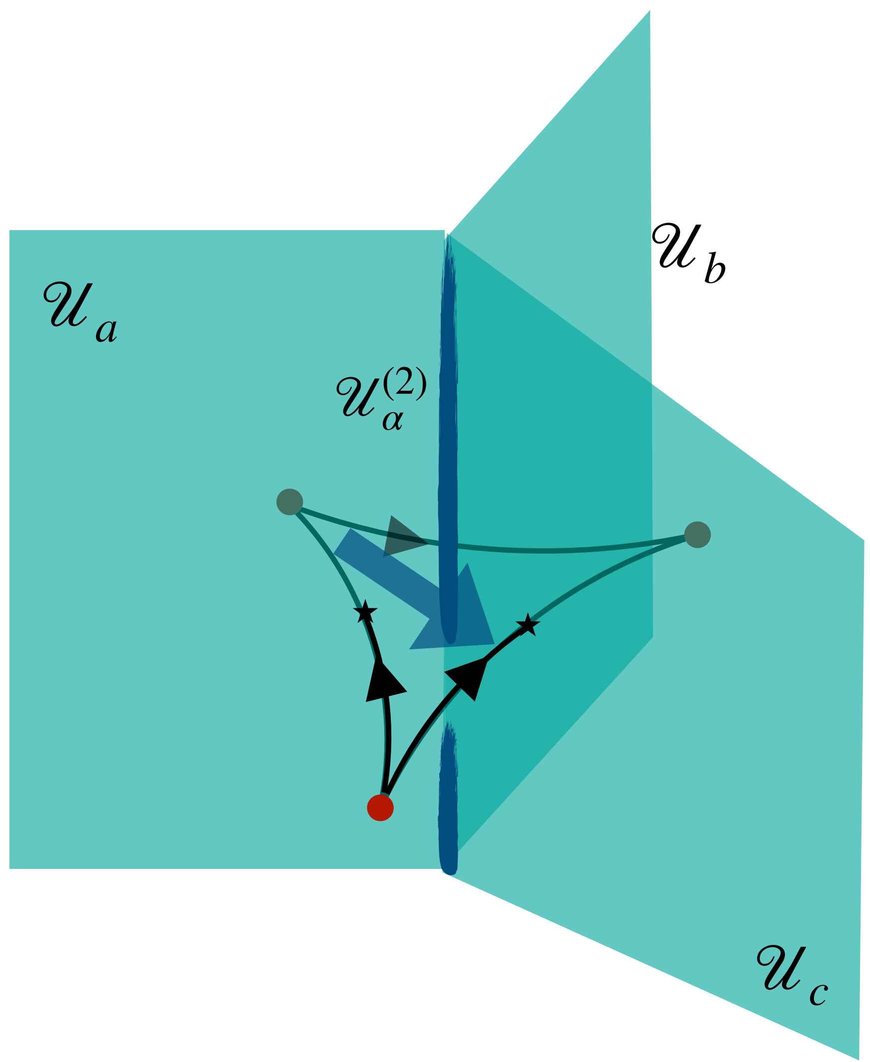
- Objects: codimension-0 quantum system,
- 1-morphisms: codimension-1 topological operators,
- 2-morphisms: codimension-2 topological operator,
- ……
- ……
- q-morphism: codimension-q operator.



$\alpha \in \textbf{hom}(a \otimes b, c)$

Algebraic structures from Topological invariance

- **Gauging the symmetry structure $\mathscr{C}$:**
- **Background gauge field is a map/functor** $\mathscr{A}$: $\textbf{Path}_n(M) \longrightarrow \mathscr{C}$



$\alpha \in \textbf{hom}(a \otimes b, c)$

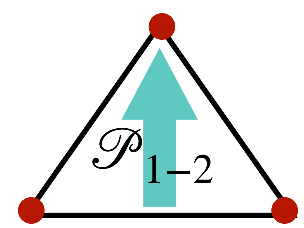
Carter, Kauffmann & Saito, Baez-Lauda, Schrieber

- **Gauge transformations as Pachner moves**

Local moves on a triangulation,
``lattice deformations”

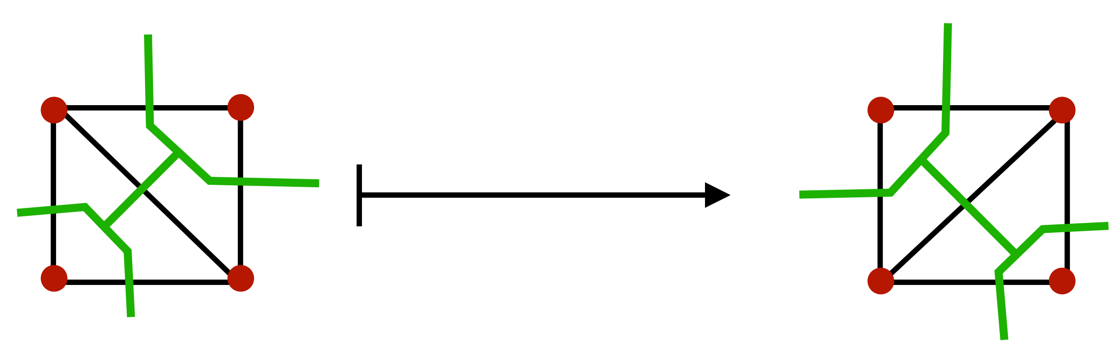
- **’t-Hooft Anomaly =** **Gauged partition function not invariant under Pachner moves.**

- 1d: $\mathscr{P}_{1-2}$:

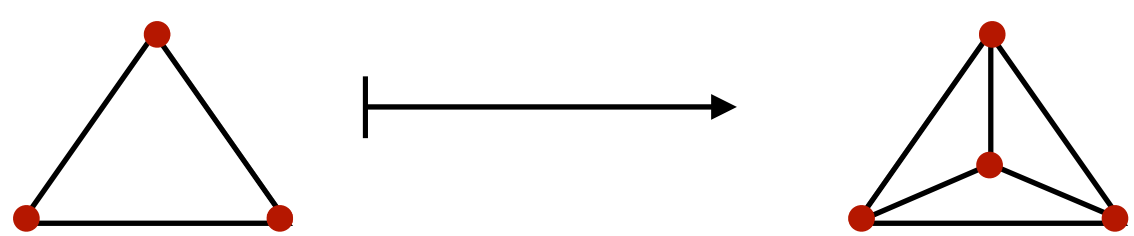


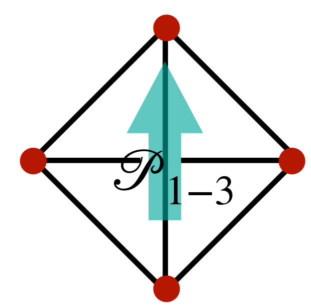
Associative algebra

- 2d: $\mathscr{P}_{2-2}$:



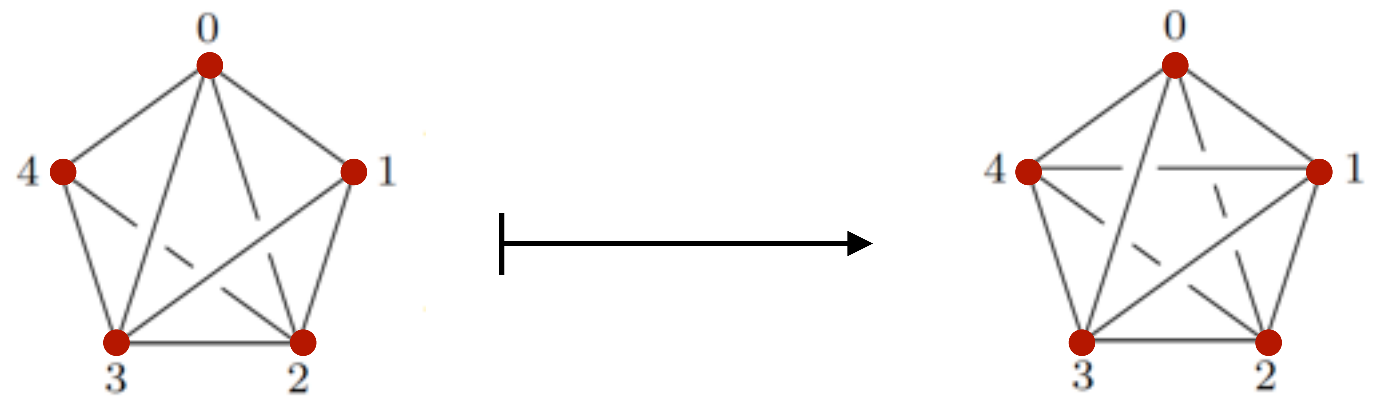
$\mathscr{P}_{1-3}$:

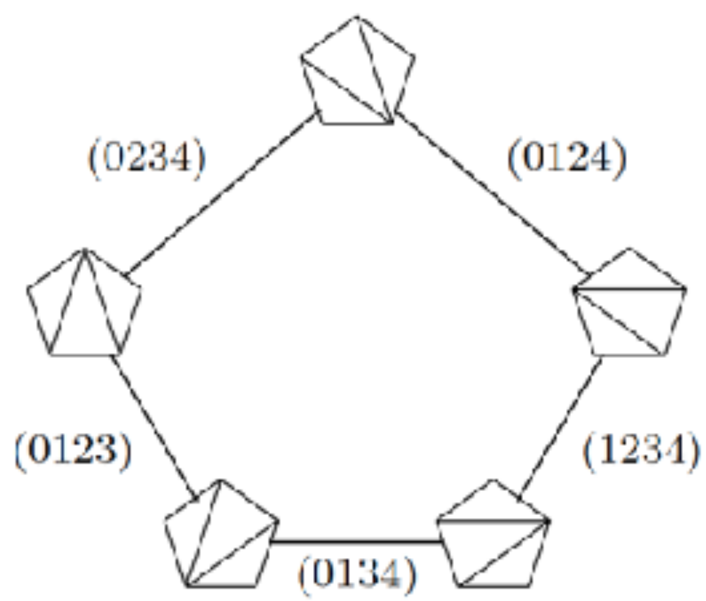




- 3d:

$\mathscr{P}_{2-3}$:

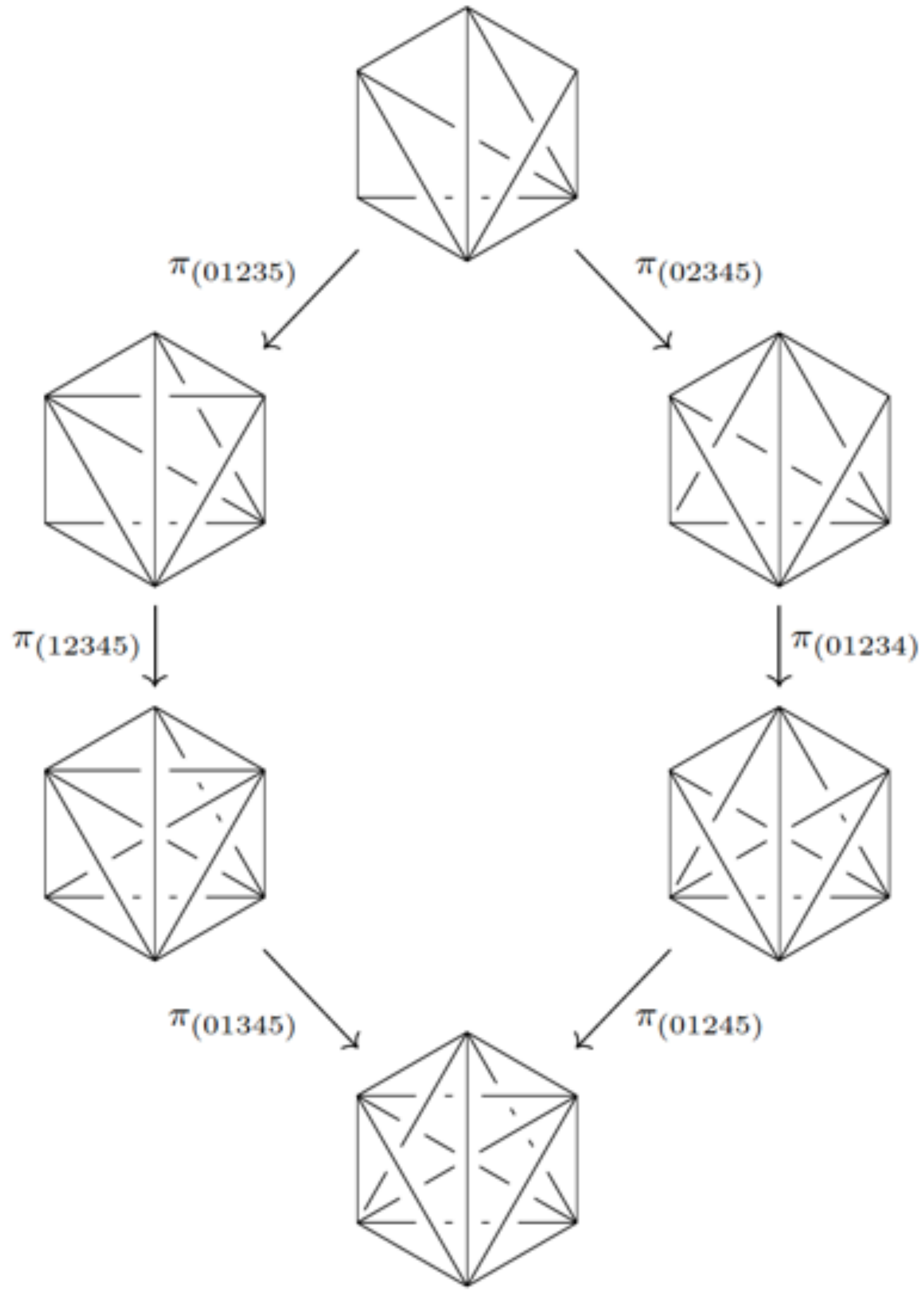




$=: \mathcal{K}_4(01234)$

Associativity is weakened/categorified. Hopf algebra

- 4d:



Hopf category

Carter, Kauffmann & Saito

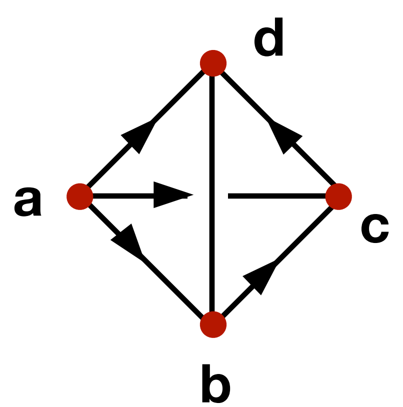
Non-trivial 2-morphisms and categorified groups

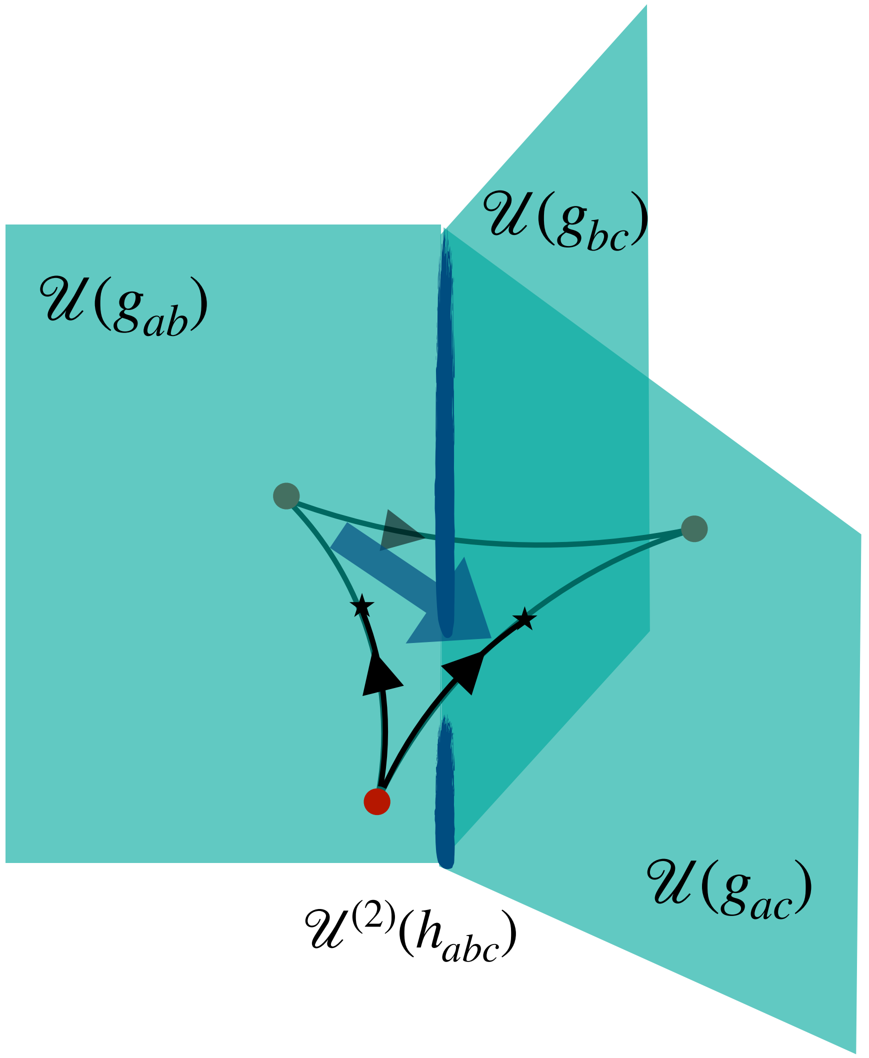
- **Topological defects and gauge bundle**

$$dg = t(h) \quad \text{i.e.} \quad g_{ab} + g_{bc} - g_{ca} = t(h_{abc})$$

$$d_{g\triangleright} h = 0 \quad \text{i.e.} \quad g_{ab} \triangleright h_{bcd} - h_{acd} + h_{abd} - h_{abc} = 0$$

"Strict 2-group bundle"





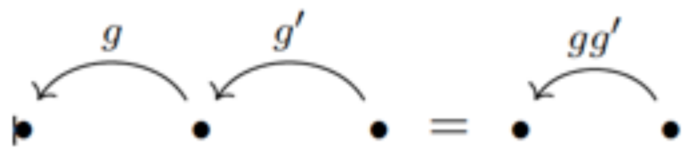
- **Topological gauge theory for a strict 2-group**

$$I_{\mathbb{G}}(M) = \frac{\#(\mathbb{G}\text{-configurations on } \mathcal{M})}{|\Gamma_1|^{b_0} |\Gamma_2|^{b_1 - b_0}}$$

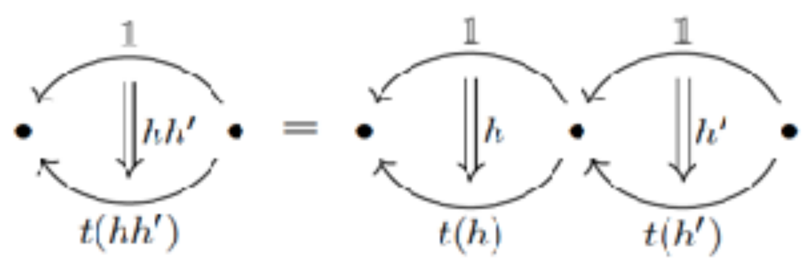
Yetter '93

Categorified groups

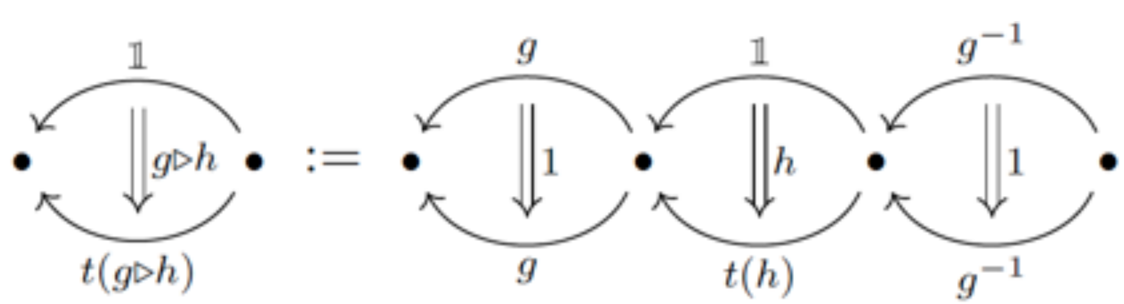
- **Strict 2-group:** A weak 2-groupoid with a single object.
  - The 1-morphisms form a group Γ_1.



  - The 2-morphisms from the identity 1-morphism form a group Γ_2.



  - An action $\triangleright : \Gamma_1 \times \Gamma_2 \to \Gamma_2$



  - Satisfies the identity $t(h) \triangleright h' = hh'h^{-1}$.

Crossed module

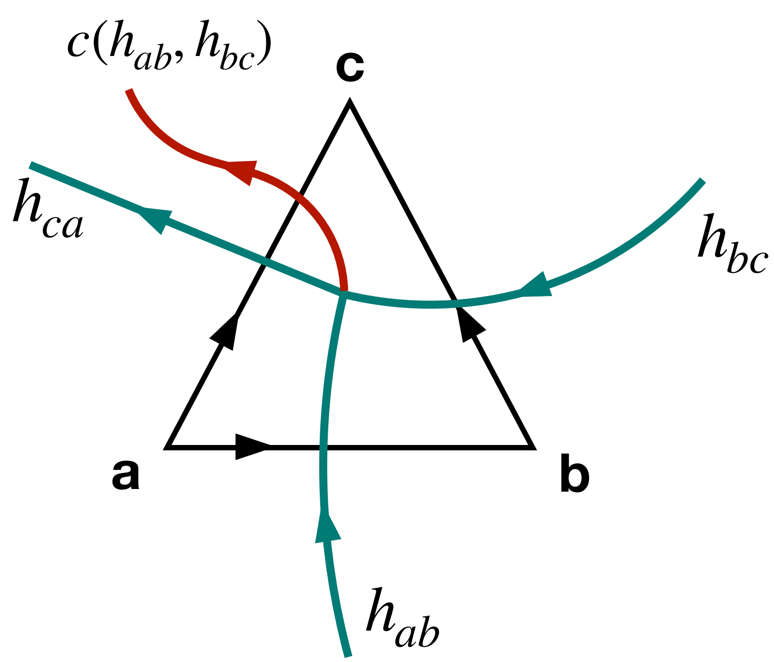
Baez Lauda

Group extensions and weak 2-groups

- Usual group extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

labelled by $[c] \in H^2(H, N)$.



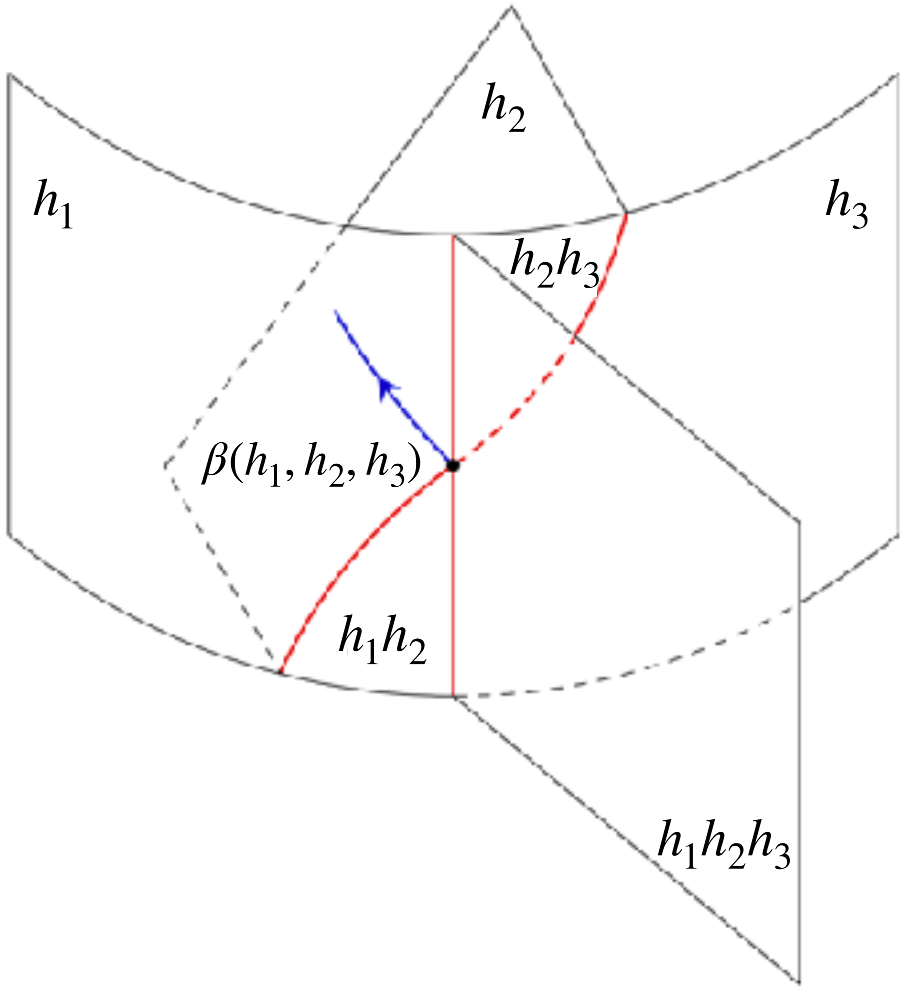
- Gauge bundle

$$dh = 0$$

$$dn = c(h)$$

- Weak 2-groups $1 \longrightarrow N_1 \longrightarrow \mathbb{G} \longrightarrow H_0 \longrightarrow 1$

labelled by $[\beta] \in H^3(H, N)$, the ``Postnikov class".



- Gauge bundle

$$dA = 0$$

$$dB = \beta(A)$$

- Gauge transformations:

$$A \sim A + d\lambda_A^{(0)}$$

$$B \sim B + d\lambda_B^{(1)} + \zeta(A, \lambda_A^{(0)})$$

where, $\beta(A + d\lambda_A^{(0)}) - \beta(A) = d\zeta(A, \lambda_A^{(0)})$

Thorngren-Kapustin, Cordova-Benini-Hsin

An example: 2+1d model with 2-group global symmetry

- Consider a triangulated 3 manifold M with the action:

$$S = \int \left\{ \frac{1}{2\pi}\delta_{IJ} d\phi^I \cup da^J + \frac{\lambda_{IJK}}{4\pi^2} d\phi^I \cup d\phi^J \cup d\phi^K - V(\phi^I, a^J) \right\}$$

$\phi^I \in C^0(M, \mathbb{Z}_n)$ & $a^I \in C^1(M, \mathbb{Z}_n)$

- Global $\mathbb{Z}_n$ 0-form symmetries $\phi^I \longmapsto \phi^I + \Lambda^I$ where $\Lambda^I \in H^0(M, \mathbb{Z}_n)$.

- Gauging the global symmetry

$$A^I \longmapsto A^I + d\lambda^{(0),I}$$

$$\phi^I \longmapsto \phi^I + \lambda^{(0),I}$$

$$d\phi^I \longmapsto d\phi^I - A^I$$

- The gauged action is anomalous with anomaly theory

$$S_{anom}[A^I] = \frac{\lambda_{IJK}}{4\pi^2} \int_{N_4} A^I \cup A^J \cup \delta A^K$$

AT-Shiozaki-Chen-Ryu, Kobayashi-Shiozaki-Ryu, AT-Moradi-Moosavian

- The gauged action is anomalous with anomaly theory

$$S_{anom}[A^I] = \frac{\lambda_{IJK}}{4\pi^2}\int_{N_4} A^I \cup A^J \cup \delta A^K$$

- Let $\lambda_{123} = \lambda$ and 0 otherwise, i.e. $G = \mathbb{Z}_n^3$

$$S_{anom}[A^I] = \frac{\lambda}{4\pi^2}\int_{N_4} A^1 \cup A^2 \cup \delta A^3$$

- Anomaly is trivialised by restricting to $\mathbb{Z}_n \subset \mathbb{Z}_n^3$. Therefore, we can gauge $\mathbb{Z}_n \subset \mathbb{Z}_n^3$.

- The gauged theory has a dual 1-form symmetry $\mathbb{Z}_{n,1}$ and $\mathbb{Z}_n^2$ 0-form symmetry.

- The Postnikov class (`group extension") is given by $\beta(A_2, A_3) = \frac{\lambda}{2\pi} A_2 \cup dA_3$.

- Can be further generalized to include 2-group with response to orientation bundle.

$$d\phi \longmapsto d_{w_1}\phi$$

- Connection to Lieb-Schultz-Mattis constraints.

Tachikawa '17, AT-Moradi-Moosavian

Topological phases protected by weak 2-group global symmetry

- **Gapped with unique ground state.**
- **Short-range entangled.**
- **Trivial in the absence of symmetry.**
- **Surface/edge with 't-Hooft anomaly.**

- Topological response action is labelled by an element $H^{d+1}(\mathbb{G}, \mathbb{R}/2\pi\mathbb{Z})$.

- In 3+1 dimensions, general response action takes the form

$$S[A, B] = \int \left\{ \omega(A) + \xi(A) \cup B + q_*(\mathfrak{P}B) \right\}$$

0-form bosonic SPT

$\omega \in H^4(H_0, \mathbb{R}/2\pi\mathbb{Z})$

Mixed SPT

$\xi \in H^2(H_0, \hat{N}_1)$

1-form bosonic SPT

$q_*(\mathfrak{P}B) \in H^4(N_1, \mathbb{R}/2\pi\mathbb{Z})$

Thorngren-Kapustin

Surface anomalies for topological phases protected by weak 2-group global symmetry

- **Mixed SPT response:** consider a 2+1d theory with global symmetry G such that

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

labelled by $[\xi] \in H^2(H, N)$.

- Upon gauging N, we get $Z_{gauged}[M] \propto \sum_{n \in H^1(M,N)} Z[M, n]$.

- The gauged theory has a 0-form global symmetry corresponding to H and 1-form global symmetry corresponding to $\hat{N}$. We gauge these by turning on background fields h and $\hat{n}$ respectively.

$$Z_{gauged}[M, \hat{n}, h] \propto \sum_{n \in H^1(M,N)} Z[M, n, h] e^{i \int_M n \cup \hat{n}}$$

- Under a gauge transformation $\hat{n} \longmapsto \hat{n} + d\hat{\lambda}_n$

Mixed anomaly

$$Z_{gauged}[M, \hat{n}, h] \longmapsto Z_{gauged}[M, \hat{n}, h] \exp\left\{ i \int_M \hat{\lambda}_n \cup \xi(h) \right\} \longrightarrow S_{anom} = \int n \cup \xi(h)$$

Wang-Wen-Witten, Cordova, Tachikawa, AT-Moradi-Moosavian

1-form SPT response

- 2+1d Abelian abelian discrete gauge theory has a 1-form global symmetry group.
- Generators correspond to abelian anyons. E.g. $\mathbb{Z}_n$ quantum double

$$S = \frac{n}{2\pi}\int_M a^1 \cup da^2 + \dots$$

- 1-form global symmetries $a^I \longmapsto a^I + \Lambda^{(1),I}$ where $\Lambda^{(1),I} \in H^1(M, \mathbb{Z}_n)$

- We can turn on background 2-form gauge fields B^I. The gauged action is no longer , i.e. it suffers from an 't-Hooft anomaly.

$$S[B^I + d\lambda^{(1),I}] = S[B^I] + \frac{n}{2\pi}\int_M \left\{B^1 \cup d\lambda^{(1),2} + d\lambda^{(1),1} \cup B_2 + d\lambda^{(1),1} \cup dd\lambda^{(1),2}\right\}$$

- Anomaly can be cancelled by a bulk SPT with a topological response action

$$S_{anom} = \frac{n}{2\pi}\int_{N_4} B_1 \cup B_2$$

- Consequence: both the `e' and `m' Wilson lines can condense on the surface of a 2-form topological gauge theory.

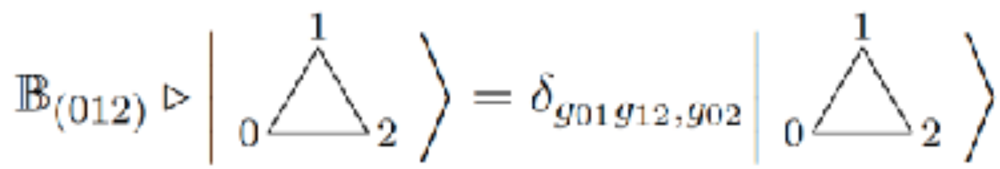
2d Topological gauge theories

- **Topological models from** $\mathscr{C} = \textbf{Vec}(\mathrm{G}, \alpha)$**:** $\alpha \in H^3(G, U(1))$

- **Dijkgraaf-Witten partition function:** $\mathcal{Z}^G_\alpha(\mathcal{M}) = \frac{1}{|G|^{|\triangle_0|}} \sum_g \prod_{\triangle_3} \alpha^{\epsilon(\triangle_3)}(g)$

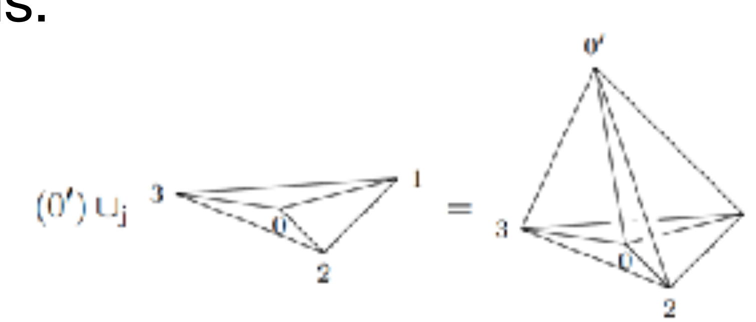
- **Twisted quantum double models:** $\mathbb{H} = -\sum_{\triangle_0} \mathbb{A}_{(\triangle_0)} - \sum_{\triangle_2} \mathbb{B}_{(\triangle_2)}$

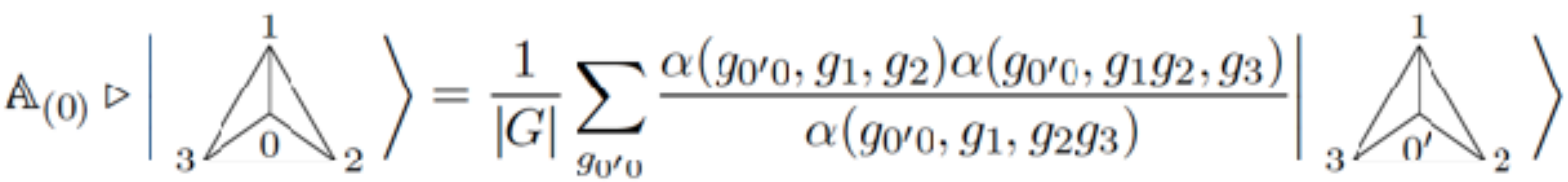
  - Imposing flatness (zero-flux)



  - Implementing gauge transformations:

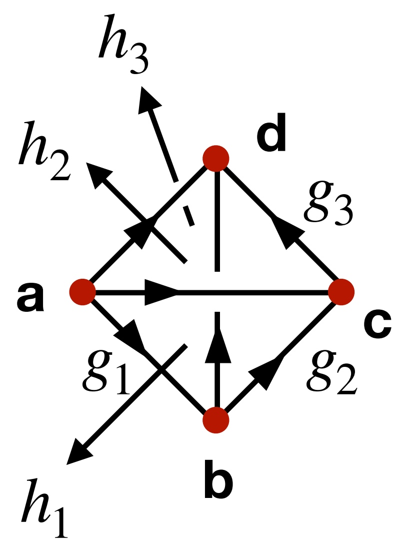
$\mathbb{A}_v = \frac{1}{|G|} \sum_{g_{v'v}} \mathcal{Z}^G_\alpha[v' \cup_j \mathrm{cl}(v)]$ where





  - Can be generalized to higher dimensions.

Generalized topological gauge theories



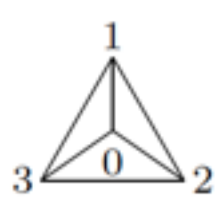
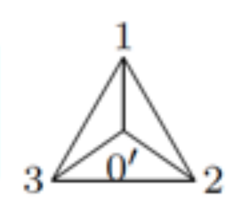
$$\longmapsto \quad \omega_3(g_2, g_3, g_4 | h_1, h_2, h_3)$$

- Implementing gauge transformations:

$$\omega_3(\triangle_3) \longrightarrow \omega_3(\triangle_3)\mathfrak{G}[g,h,k,\lambda]$$

$$\begin{aligned}\mathfrak{G}[g,h,k,\lambda] = &\ \omega_3(k, g_1, g_2| -\lambda_{01}, h_1 - \lambda_{02} + \lambda_{01} + \zeta_{012}, -\lambda_{02}) \\ &\times \omega_3(k, g_1 g_2, g_3| -\lambda_{02}, h_3 - \lambda_{03} + \lambda_{02} + \zeta_{023}, -\lambda_{03}) \\ &\times \omega_3(k, g_1, g_2 g_3| -\lambda_{01}, h_2 - \lambda_{03} + \lambda_{01} + \zeta_{013}, -\lambda_{03})^{-1}\end{aligned}$$

- Exactly solvable Hamiltonian $\quad \mathbb{H} = -\sum_{\triangle_0} \mathbb{A}_{(\triangle_0)} - \sum_{\triangle_2} \mathbb{B}_{(\triangle_2)}$

$\mathbb{A}_{(0)} \triangleright \Big|$  $\Big\rangle = \frac{1}{|G||H|^3} \sum_{g_{0'0}} \sum_{\{h_{0'0i}\}_{i=1}^3} \mathfrak{G}[g,h,k,\lambda] \Big|$  $\Big\rangle$

Outlook and open directions:-

- Hamiltonian understanding of 2-group anomalies.
- Fermionic 2-group structures.
- Applications in practical models,..2D quantum magnetism?
- n-Fusion category symmetry, non-invertible symmetries.
- Representation theory for n-Fusion category = Study of charges in generalized models.