Resummation of Perturbative Series & Resurgence in Quantum Field Theory

Day1: Basics

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8th, July 2020

KEK-PH lecture series

I am asked to give 2 hours lecture on

formal aspects of resurgence

Resurgence

||

Technique to resum non-convergent series

ubiquitous!

∃ Many possible applications in various contexts

Different expansions have different stories…

Physical setup:

Field Theory, String theory, Statistical system, etc… ?

Expansion parameters:

Coupling constant, N, N_f, α', time, T, μ, ϵ ,etc… ?

around where?

0, ∞, or finite… ?

Technical setup:

(path) integral or differential/difference eq… ?

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around…?

0, ∞, or finite point… ?

Focus:

Weak coupling expansion in Quantum Field Theory

Technical setup:

(path) integral or differential/difference eq… ?

Contents of day 1: Basics

0. Prologue

1. Expectations on weak coupling perturbative series in QFT

2. What is resurgence?

3. Summary of day 1

4. Preview of day 2 (Application to QFT)

1. Expectations on weak coupling perturbative series in QFT

- Perturbative series in typical QFT
- Borel resummation
- Borel summability in QFT?

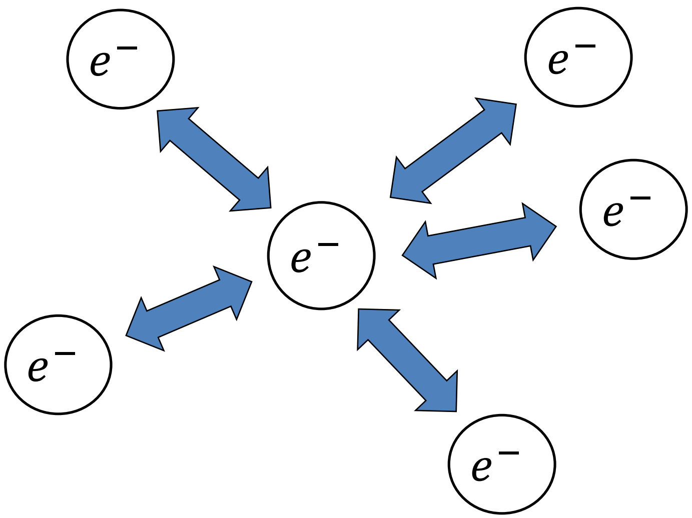
Perturbative expansion in QFT

- Typically non-convergent [Dyson ’52]
- Naïve sum of all-orders → divergent

Why perturbative series is not convergent

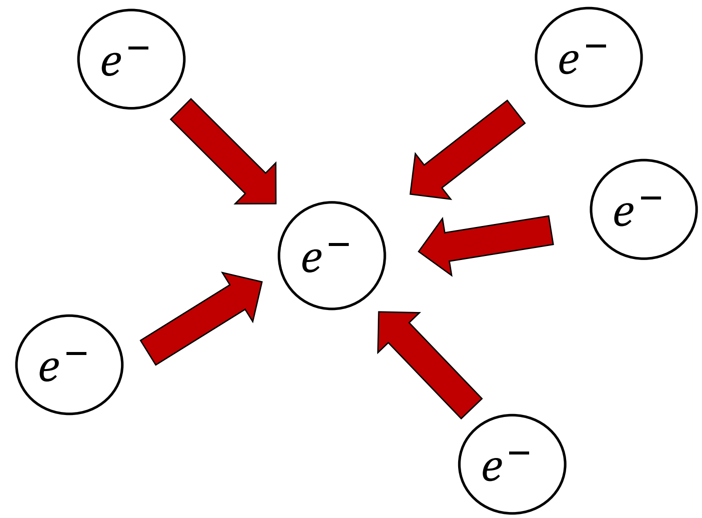
～Dyson's original argument (very rough)～ [Dyson '52]

World w/ $e^2 > 0$



repulsive

World w/ $e^2 < 0$



attractive, prefer to be dense

looks qualitatively different $\Rightarrow$ non-analytic?

Why perturbative series is not convergent

～technical reasons～

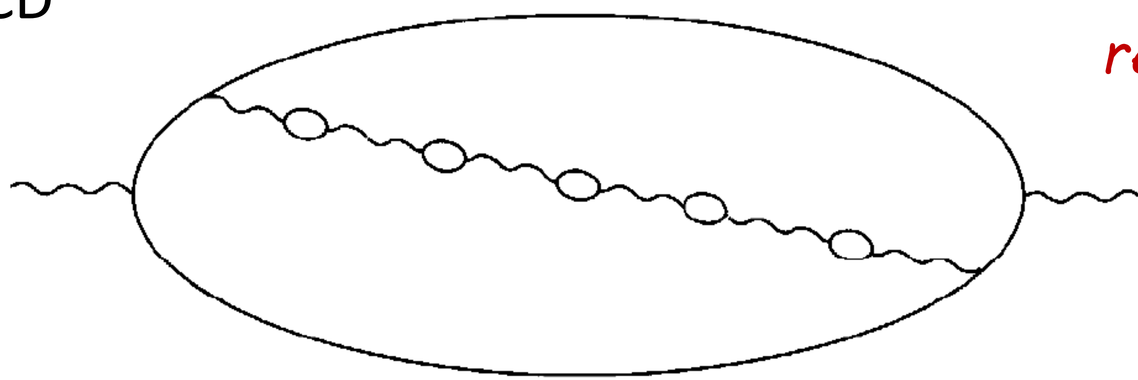
① (# of n-loop Feynmann diagrams) ～ n!

proliferation

② $^{\exists}$Feynmann diagrams contributing by ～n!

Ex.) QCD

renormalon



[Fig.20.3 in Weinberg's book,
cf. Takaura-san's lectures]

Best way by Naïve sum = Truncation

N-th order approximation of a function $P(g)$:

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell$$

"error" of the approximation:

$$\delta_N(g) \equiv P_{N+1}(g) - P_N(g) = c_{N+1} g^{N+1}$$

Optimized order N_*: (given g)

$$\frac{\partial}{\partial N}\delta_N(g)\Big|_{N=N_*} = 0 \quad \xrightarrow{N \gg 1} \quad \frac{\partial}{\partial N}(\log c_N + N \log g)\Big|_{N=N_*} = 0$$

Best way by Naïve sum = Truncation (Cont'd)

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell \quad \xrightarrow{\text{optimize}} \quad \frac{\partial}{\partial N}(\log c_N + N \log g)_N \Big|_{N=N_*} = 0$$

In QFT, typically

$$c_\ell \sim \ell! \, A^\ell \quad (\ell \gg 1)$$

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Then,

$$0 = \frac{\partial}{\partial N}\left(N \log N - N + N \log(\mathrm{Ag})\right)\Big|_{N=N_*} \quad \Rightarrow \quad N_* = \frac{1}{Ag}$$

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optimize

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Error of the truncation:

$$\delta_{N_*}(g) = c_{N_*+1} g^{N_*+1} \sim e^{-N_*} = e^{-\frac{1}{Ag}}$$

Non-perturbative effect

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Non-perturbative effect

Is there a good way to resum perturbative series?

General questions in this lecture

- What does perturbative series actually know?
- Is there a way to obtain exact answer from information on perturbative expansion?
- If yes, how?

More precise (but still imprecise) question

Perturbative series around saddle points:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell + \sum_{I \in \text{saddles}} e^{-S_I(g)} \sum_{\ell=0}^{\infty} c_\ell^{(I)} g^\ell$$

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Can we get the exact result by using the coefficients?

= What is a correct way to resum the perturbative series?

(∼continuum definition of QFT?)

This lecture (day 2) = To give a partial answer

A standard resummation

Borel transformation:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell} \Rightarrow \boxed{\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}}$$

Borel resummation (along θ):

$$\boxed{S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\; e^{-\frac{t}{g}}\; \mathcal{B}\mathcal{O}(t)}$$

(usually, $\theta = \arg(g) = 0$)

Why Borel resummation may be nice

(Let's take $\theta = \arg(g)$)

$$S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \qquad \mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

① Reproduce original perturbative series:

$$S_\theta \mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} \int_0^{e^{i\theta}\infty} dt\ t^{a+\ell-1} e^{-\frac{t}{g}} = \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell}$$

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② Finite for any g if

1. Borel trans. is convergent
2. Its analytic continuation does not have singularities along the contour
3. The integration is finite

"Borel summable (along θ)"

related to exact result?

Some simple examples

1. Analytic function

$$\mathcal{O}(g) = \sum_{\ell} c_\ell g^\ell$$ convergent inside radius of convergence

= (Borel resummation)

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2. Incomplete gamma function

$$\mathcal{O}(g) = \frac{1}{g} e^{\frac{1}{g}} \Gamma\left(0, \frac{1}{g}\right) \sim \sum_{\ell} \ell! (-g)^\ell$$

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⇨ $$\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0} (-t)^\ell = \frac{1}{1+t}$$ Borel summable along $\boldsymbol{R}_+$

$$S_0 \mathcal{O}(g) = \frac{1}{g} \int_0^\infty dt\, e^{-\frac{t}{g}} \mathcal{B}\mathcal{O}(t) = \frac{1}{g} \int_0^\infty dt\, \frac{e^{-\frac{t}{g}}}{1+t} = \mathcal{O}(g)$$

Expectations in typical QFT

['t Hooft '79]

Non-Borel summable due to singularities along $\boldsymbol{R}_+$

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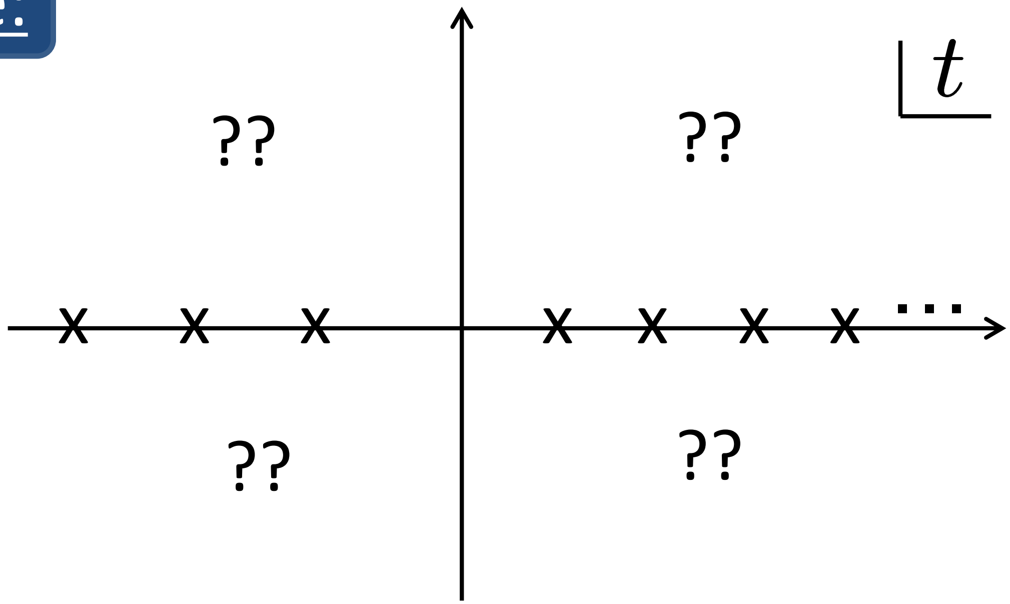
Borel plane:



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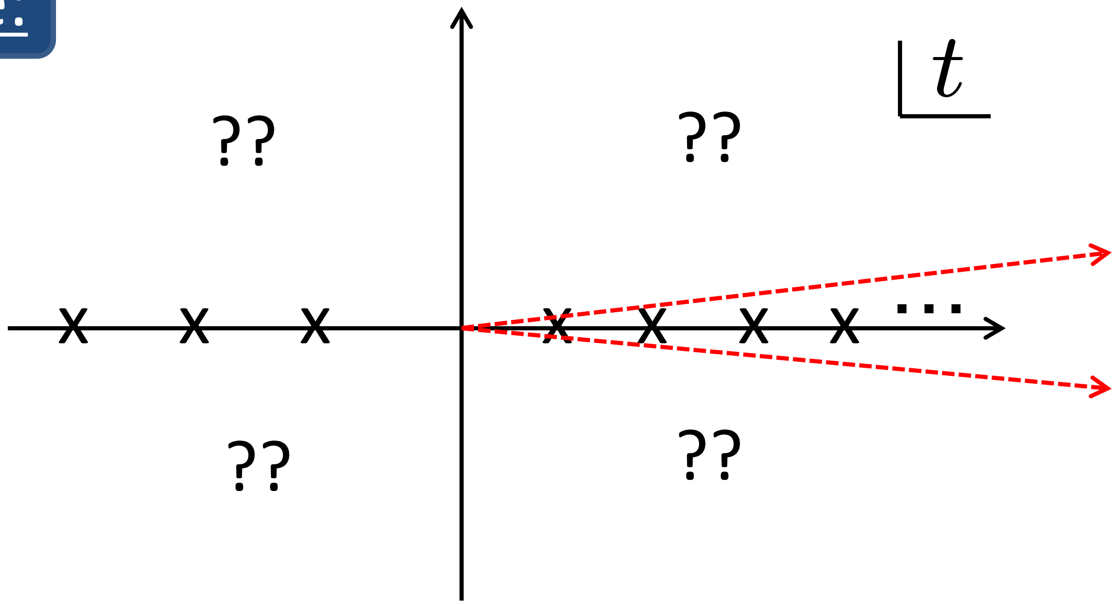

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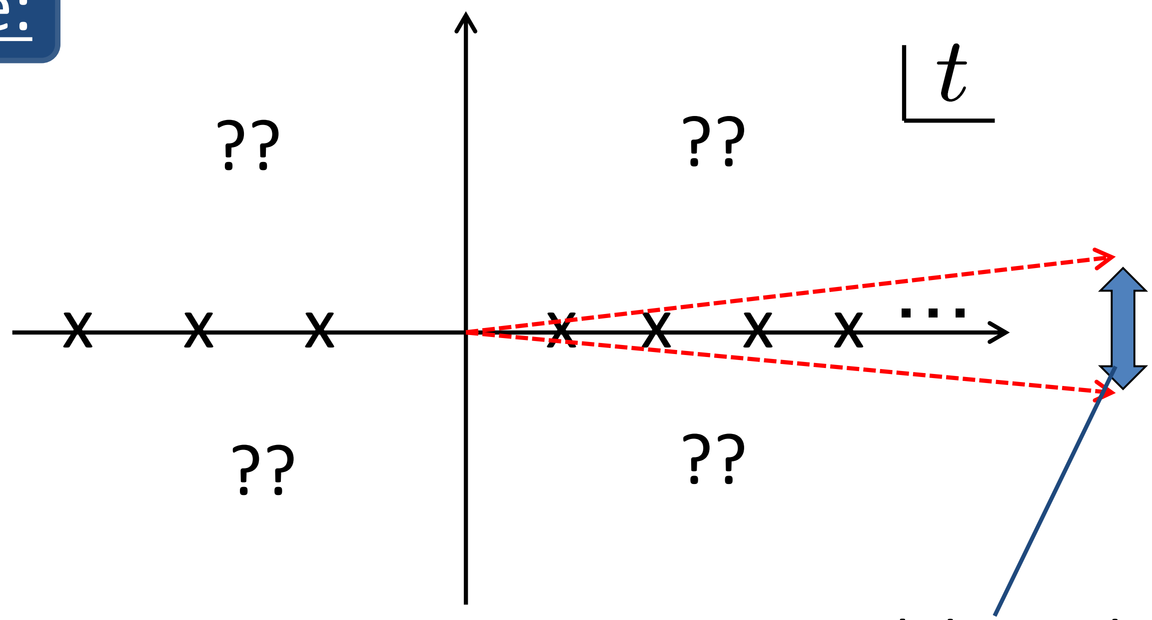

Integral depends on a way to avoid singularities

Expectations in typical QFT

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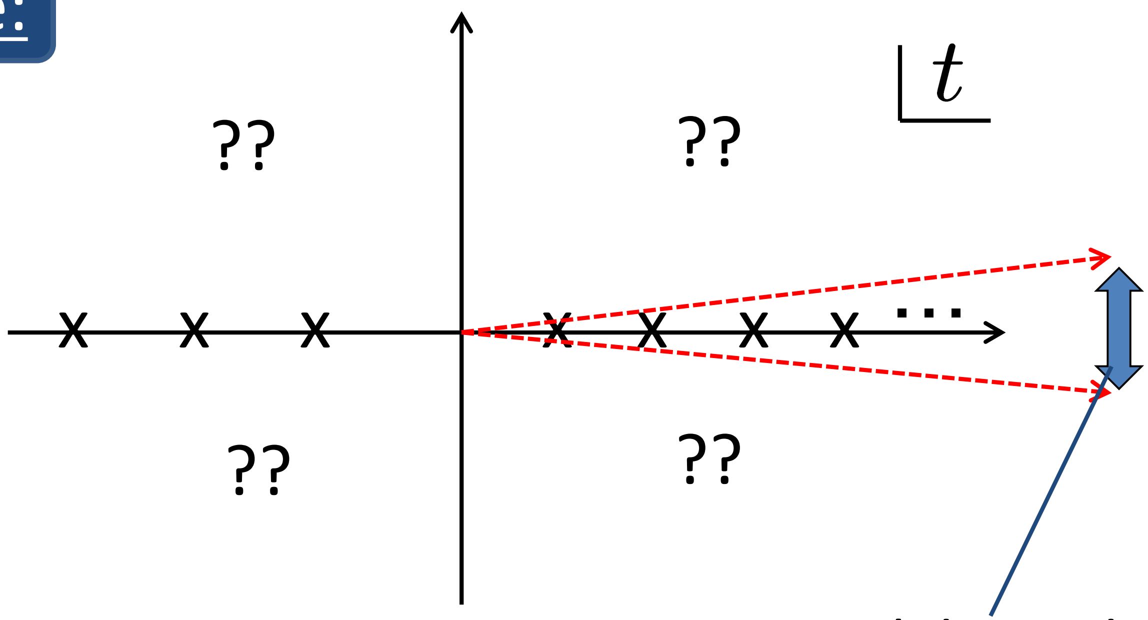

$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \Rightarrow (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Non-perturbative effect?

Interpretation of Borel singularities

$$Z(g) = \int D\Phi e^{-\frac{1}{g}S[\Phi]} \simeq \sum_{\ell} c_\ell g^\ell$$

[Lipatov '77]

Large order coefficient:

$$c_\ell = \frac{1}{2\pi i}\oint \frac{dg}{g^{\ell+1}} Z(g) = \frac{1}{2\pi i}\oint dg \int D\phi e^{-\frac{1}{g}S[\phi]-(\ell+1)\ln g} \qquad (\ell \to \infty)$$

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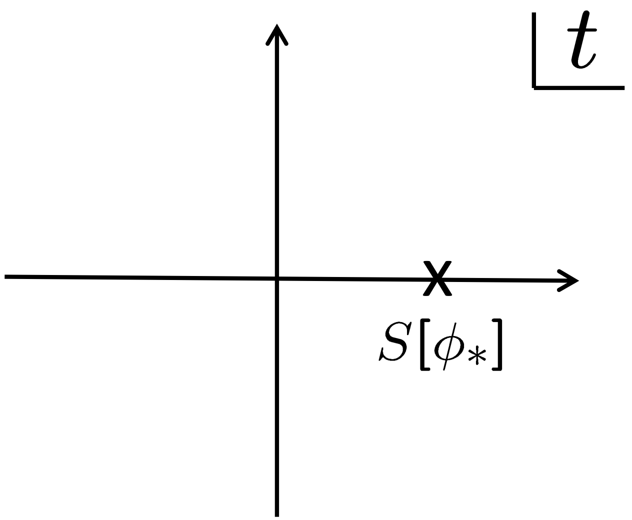
$$= e^{(\ell+1)\ln(\ell+1)-(\ell+1)} \left(S[\phi_*]\right)^{-(\ell+1)} \simeq \ell! \left(S[\phi_*]\right)^{-(\ell+1)}$$

$$\mathcal{B}Z(t) \simeq \sum_{\ell} \left(S[\phi_*]\right)^{-\ell} = \frac{1}{1-\frac{t}{S[\phi_*]}}$$



Nontrivial saddle point gives Borel singularities

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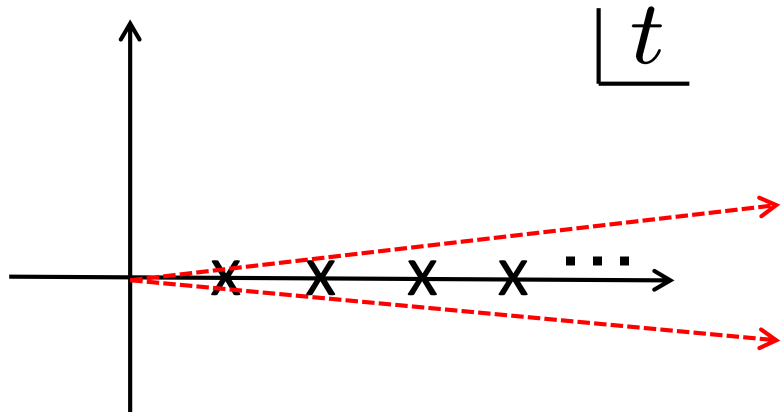
Resurgence



$$(\text{Ambiguities}) \sim (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Idea of resurgence: (explicit examples in next slides)

This is precisely canceled by ambiguities of perturbative series around other saddle points ($\sim$ non-pert. sector):

(perturbative ambiguity) = $-$(non-perturbative ambiguity)

⇒ (unambiguous answer)

Ex.1: Stirling's formula v.s. Exact gamma function

$$\log n! \sim n \log n$$

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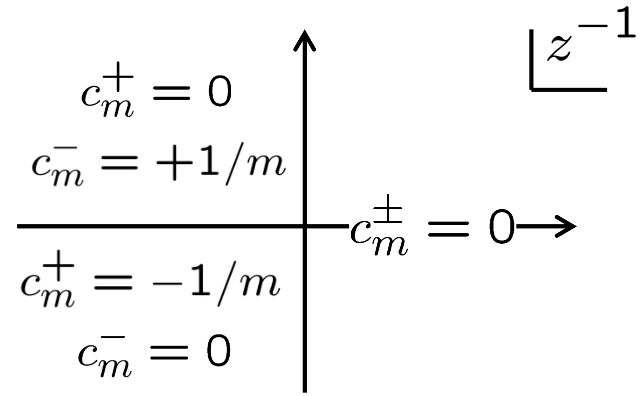
Improved Stirling's formula:

[cf. Nemes '14]

$$\log \Gamma(z) \sim z \log z - z - \frac{1}{2} \log \frac{z}{2\pi} + I_{\rm pert}(z) + \sum_{\pm} \sum_{m=1}^{\infty} c_m^{\pm} e^{\pm 2\pi i m z}$$

$$I_{\rm pert}(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}},$$

$$\sim \sum_{n} \frac{(2n)!}{z^{2n-1}}$$



Stokes phenomena!

(Jump of the form of asymptotic expansion)

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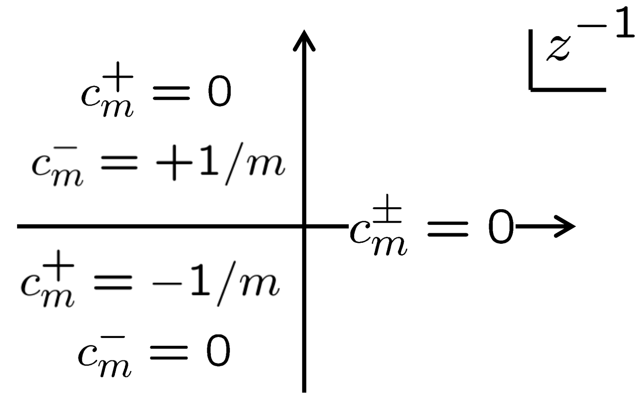


Stokes phenomena!

(Jump of the form of asymptotic expansion)

Borel resum. in perturbative sector:

$$S_{\arg z^{-1}} I_p(z) = \int_0^{e^{i\arg z^{-1}}\infty} dt\ e^{-zt}\mathcal{B}I_p(t) = \int_0^{e^{i\arg z^{-1}}\infty} dt\ \frac{e^{-zt}}{t}\left[\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right]$$

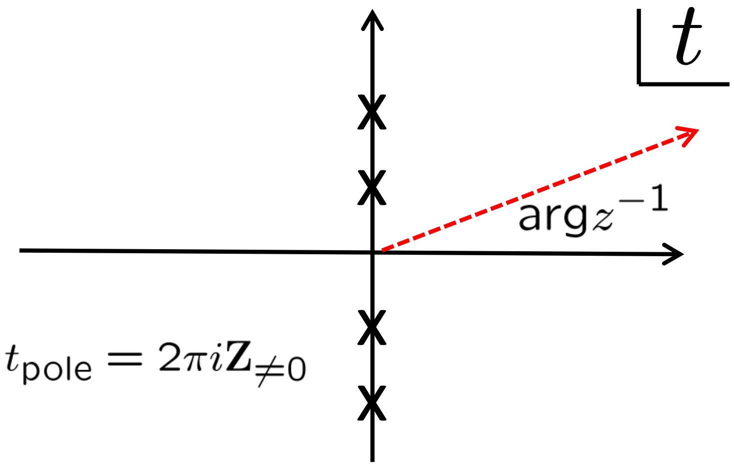
It is known for Re(z)>0,

[Binet's formula]

$$\log\Gamma(z) = z\log z - z - \frac{1}{2}\log\frac{z}{2\pi} + \int_0^{\infty} dt\ \frac{e^{-zt}}{t}\left[\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right]$$

What for Re(z)≦0?

Perturbative sector:



$$S_{\arg z^{-1}} I_p(z) = \int_0^{e^{i \arg z^{-1}} \infty} dt \, \frac{e^{-zt}}{t} \left[\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right]$$

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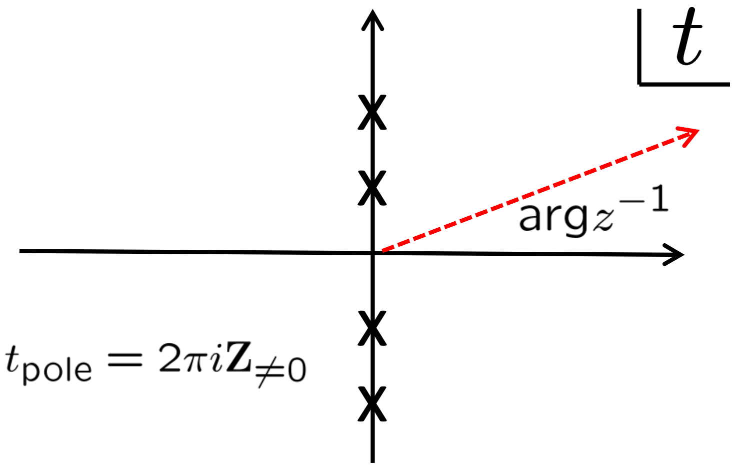

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Borel ambiguity at $\arg z^{-1} = \pi/2$:

$$(S_{\pi/2+0_+} - S_{\pi/2-0_+}) I_p(z)$$

$$= -\sum_{m=1}^{\infty} \mathrm{Res}_{t=2m\pi i}\left(e^{-zt}\mathcal{B}I_p(t)\right)$$

$$= -\sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

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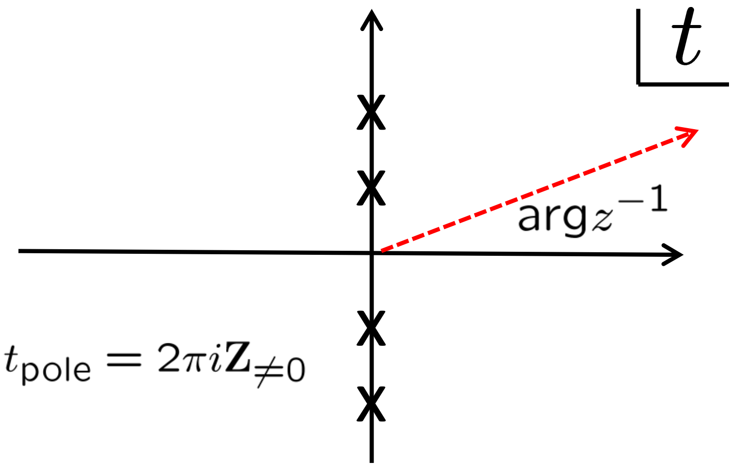

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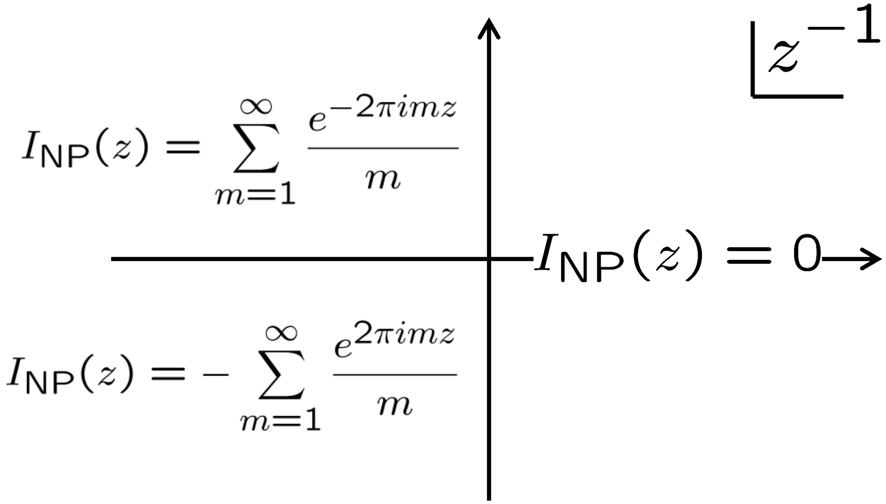
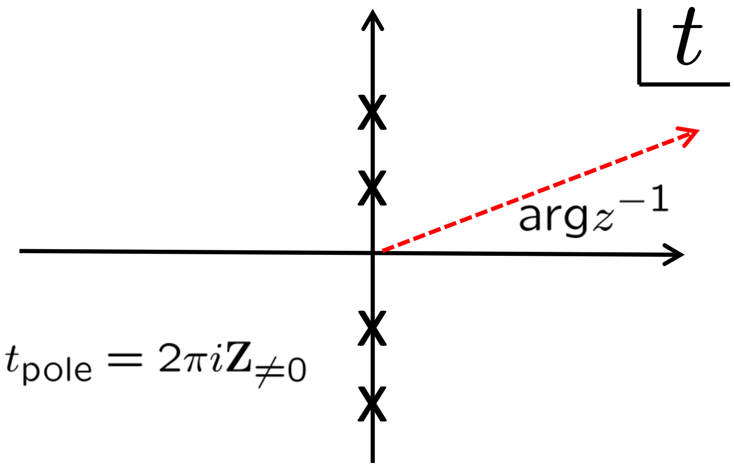


Stokes phenomena generates ambiguities

Perturbative sector:



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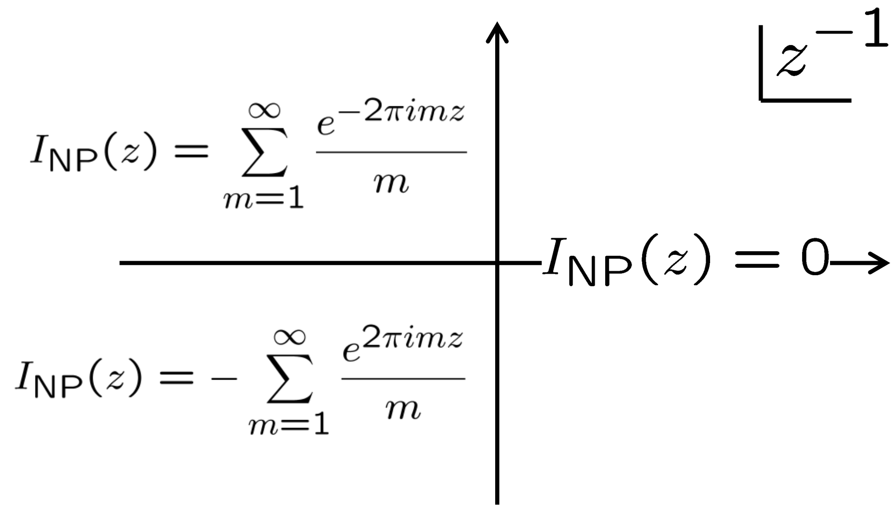


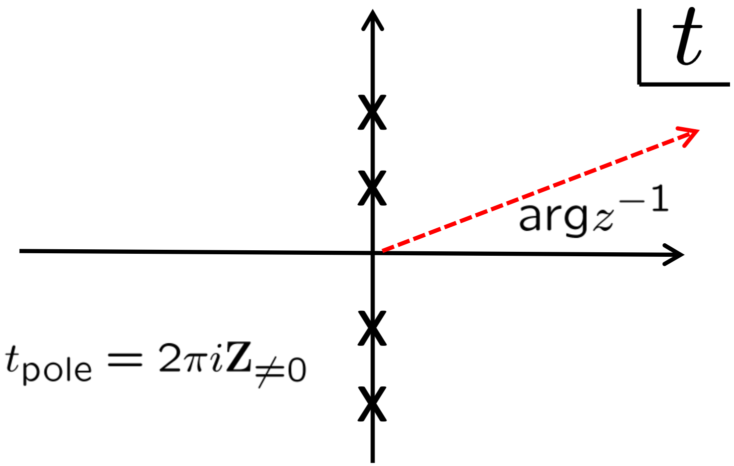
Stokes phenomena generates ambiguities

Ambiguity at $\arg z^{-1} = \pi/2$:

$$I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2}+0_+} - I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2}-0_+}$$

$$= +\sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

Perturbative sector:



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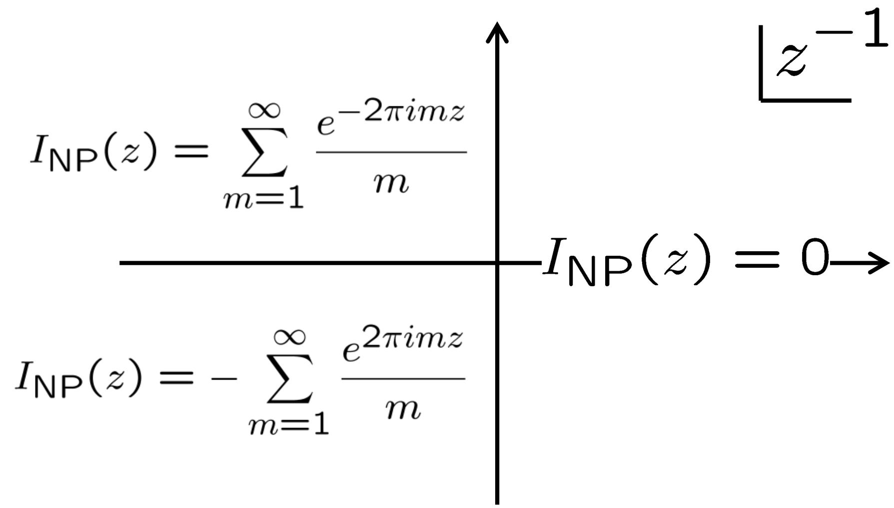

Stokes phenomena generates ambiguities

Ambiguity at $\arg z^{-1} = \pi/2$:

$$I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2} + 0_+} - I_{\text{NP}}(z)|_{\arg z^{-1} = \frac{\pi}{2} - 0_+}$$

$$= +\sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i m z}$$

Canceled! (similar for $\arg z^{-1} = -\pi/2$)

An example more like QFT

[Cherman-Dorigoni-Unsal '14,
Cherman-Koroteev-Unsal '14]

0d Sine-Gordon model:

$$Z(g) = \frac{1}{\sqrt{g}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \; e^{-\frac{1}{2g} \sin^2 x} = \frac{\pi}{\sqrt{g}} e^{-\frac{1}{4g}} I_0\left(\frac{1}{4g}\right)$$

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Saddle point:

$$0 = \frac{d}{dx}\sin^2 x\Big|_{x=x_*} = \sin(2x_*) \quad \Rightarrow \quad x_* = 0, \ \pm\frac{\pi}{2}$$

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"Action":

$$\left(S(x) = \frac{1}{2g}\sin^2 x\right)$$

$$S(x=0) = 0$$ *trivial*

$$S\left(x = \pm\frac{\pi}{2}\right) = \frac{1}{2g}$$ *Non-perturbative*

Expansion around the saddle pts:

$$Z(g) \sim \underbrace{\sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell}_{x_* = 0} + \underbrace{e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell}_{x_* = \pm\frac{\pi}{2}} \; ??$$

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Trivial saddle:

$$Z(g)|_{x_*=0} = \sqrt{2\pi} \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1/2)^2 2^\ell}{\Gamma(\ell+1)\Gamma(1/2)^2} g^\ell \equiv \Phi_0(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_0(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell^{(0)}}{\ell!} t^\ell = \sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; 2t\right)$$

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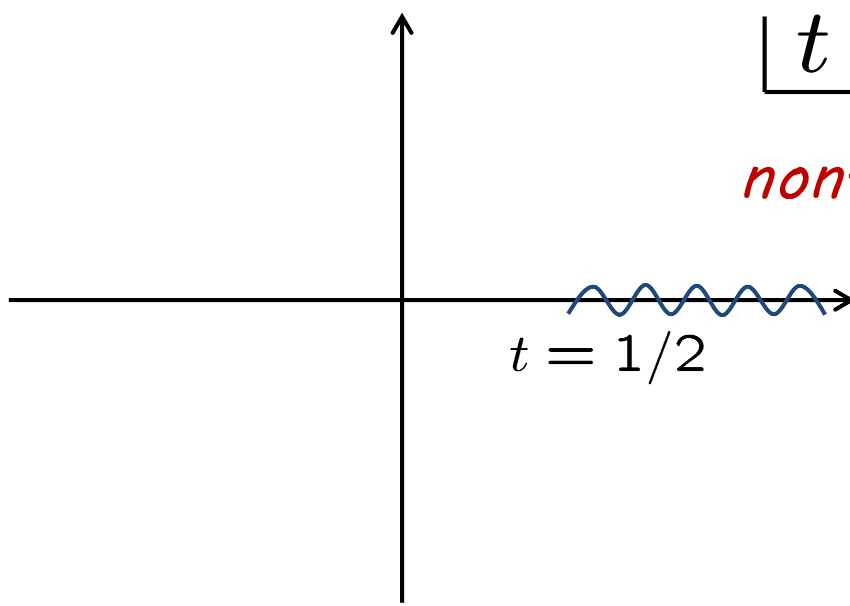
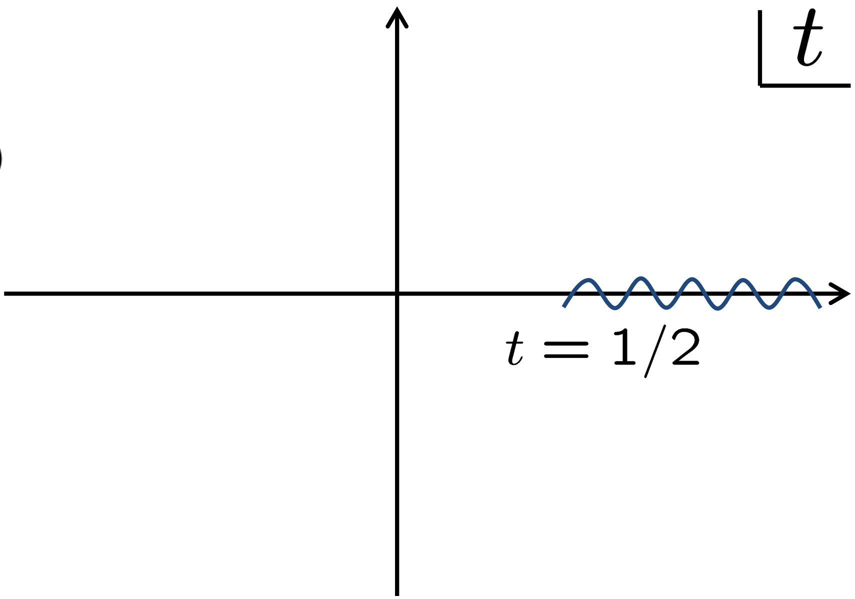


non-Borel summable!

$$\begin{cases} S_\theta \Phi_0(g) = \dfrac{1}{g}\displaystyle\int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\Phi_0(t) \\ \mathcal{B}\Phi_0(t) = \sqrt{2\pi}\ {}_2F_1\left(\dfrac{1}{2},\dfrac{1}{2},1;2t\right) \\ g = |g|e^{i\theta} \end{cases}$$



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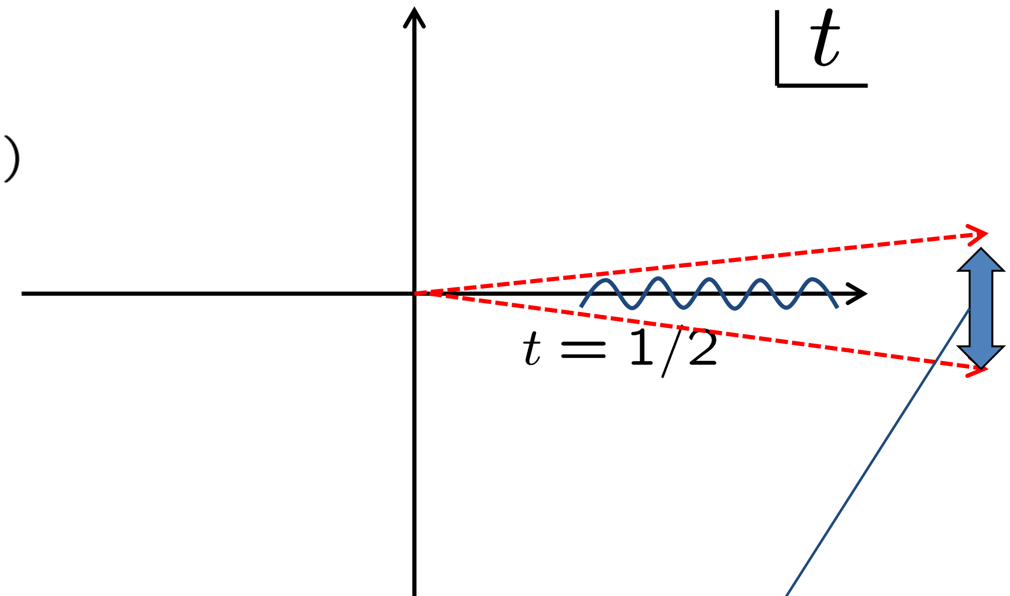

Ambiguity:

$$\left(S_{0+} - S_{0-}\right)\Phi_0(g) = e^{-\frac{1}{2g}} \times \frac{2i\sqrt{2\pi}}{g}\int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right) \neq 0$$

Related to contribution from $x_* = \pm\frac{\pi}{2}$?

Expansion around nontrivial saddle

$$\begin{cases} e^{-S(x)} = e^{-\frac{1}{2|g|}e^{-i\theta}x^2+\cdots} & x_* = 0 \\ e^{-S(x)} = e^{-\frac{1}{2g}} \times e^{\frac{1}{2|g|}e^{-i\theta}(x-\pm\frac{\pi}{2})^2+\cdots} & x_* = \pm\frac{\pi}{2} \end{cases}$$

$$\left(g = |g|e^{i\theta}\right)$$

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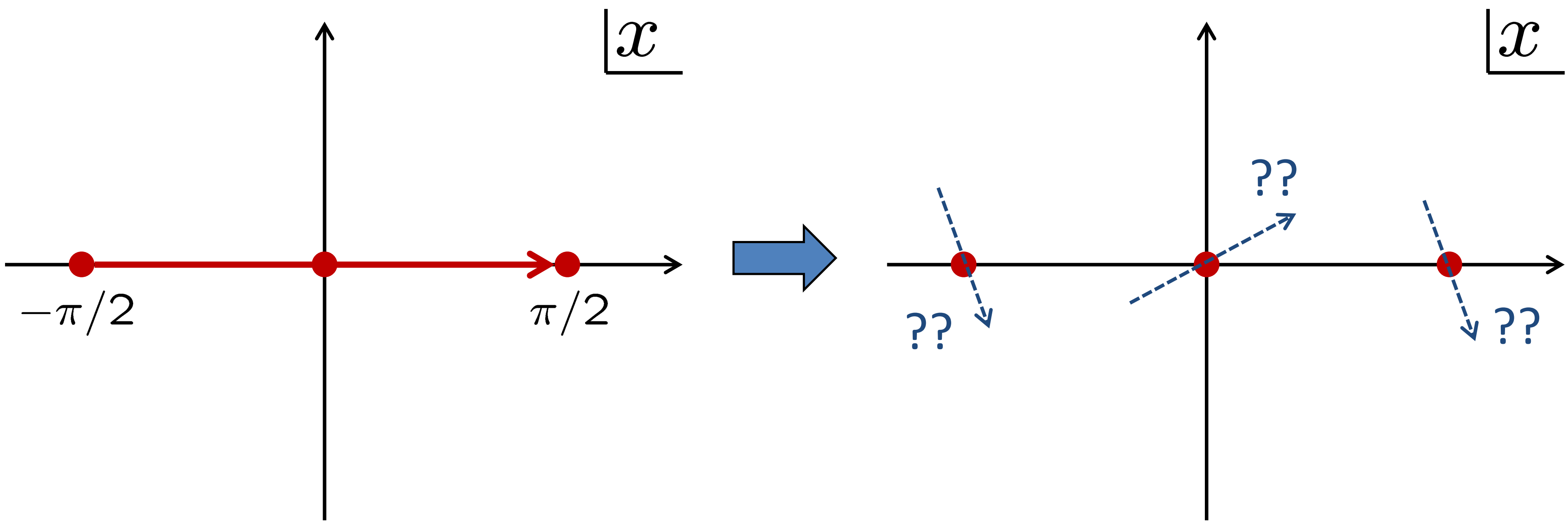
To pick up saddles, change the integral contour to steepest descent s.t.

1. passes the saddles w/ appropriate angle
2. Keep $\mathrm{Im}[S(x)]$ to avoid oscillation
3. Keep the final result (use Cauchy integration theorem)



Appropriate contour = Lefschetz thimble

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., $^{\exists}$unique Lefschetz thimble J_I :

$$\frac{dz(t)}{dt} = \overline{\frac{\partial S(z)}{\partial z}}, \quad \text{with } z(t \to -\infty) = z_I$$

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Properties:

$a)$ $\mathrm{Im}S(z)|_{J_I} = \mathrm{Im}S(z_I)$ $\left(\frac{d}{dt}\mathrm{Im}S \propto \frac{d}{dt}(S - \bar{S}) = \frac{dz}{dt}\frac{\partial S}{\partial z} - \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 0 \right)$

$b)$ $\mathrm{Re}S(z)|_{J_I} \geq \mathrm{Re}S(z_I)$ $\left(\frac{d}{dt}\mathrm{Re}S \propto \frac{dz}{dt}\frac{\partial S}{\partial z} + \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 2\frac{\partial S}{\partial z}\overline{\frac{\partial S}{\partial z}} \geq 0 \right)$

$c)$ Decomposition of cycle: (if we are not on Stokes line)

$$\int_C = \sum_{I \in \text{saddle}} n_I \int_{J_I} \qquad (n_I \in \mathbf{Z})$$

may jump as changing parameters

Appropriate contour = Lefschetz thimble

Dual thimble = steepest ascent

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., ∃unique dual thimble K_I :

$$\frac{dz(t)}{dt} = -\overline{\frac{\partial S(z)}{\partial z}}, \text{ with } z(t\to-\infty) = z_I$$

Properties:

$a)$ $\mathrm{Im}S(z)|_{K_I} = \mathrm{Im}S(z_I)$

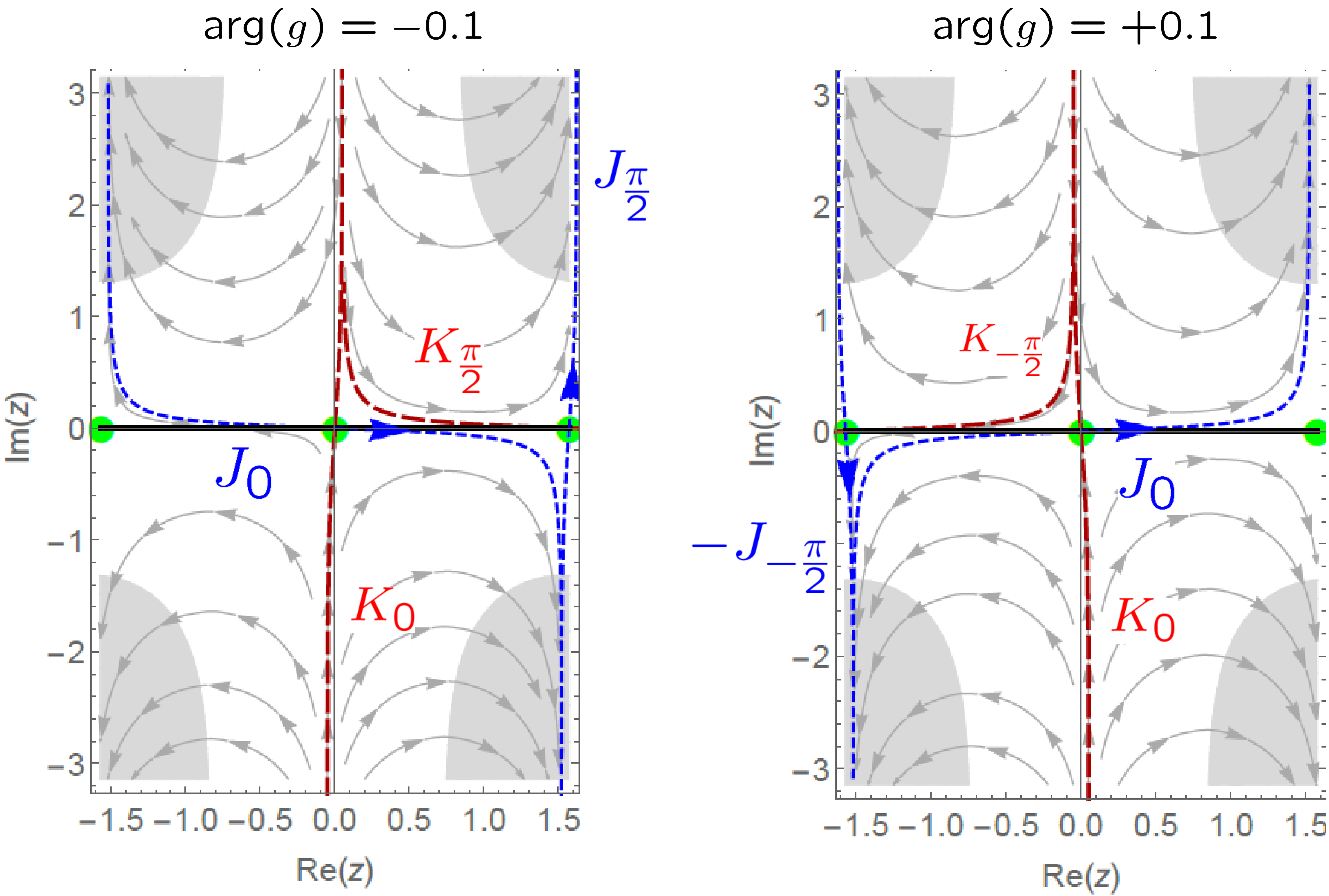
$b)$ $\mathrm{Re}S(z)|_{K_I} \leq \mathrm{Re}S(z_I)$

$c)$ Decomposition of cycle: (if we are not on Stokes line)

$$\int_C = \sum_{I\in\text{saddle}} n_I \int_{J_I}, \quad n_I = \text{intersection} \sharp \text{ of } (C, K_I)$$

Thimble structures in the toy model

[similar to fig.1 in Cherman-Dorigoni-Unsal ’14]



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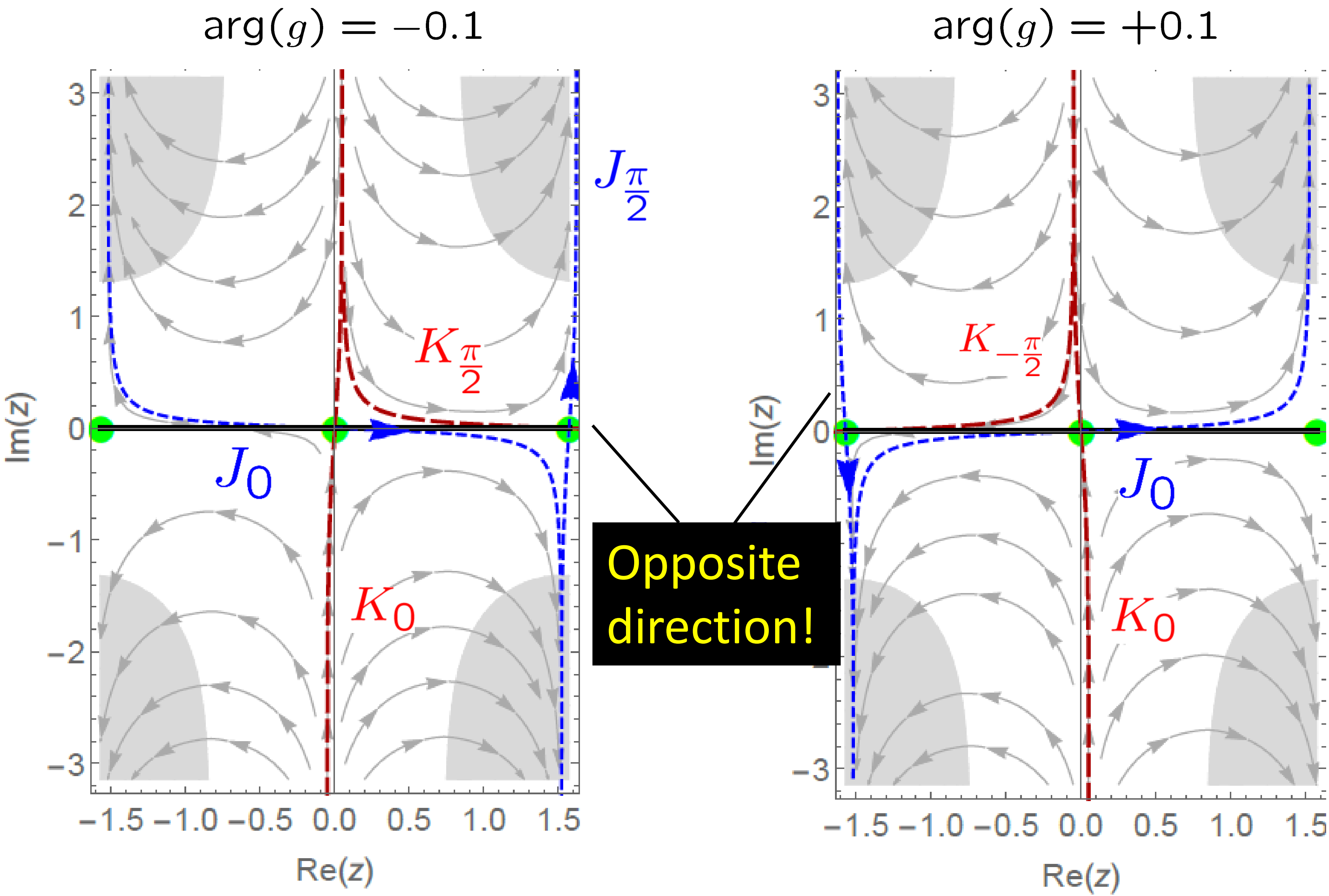
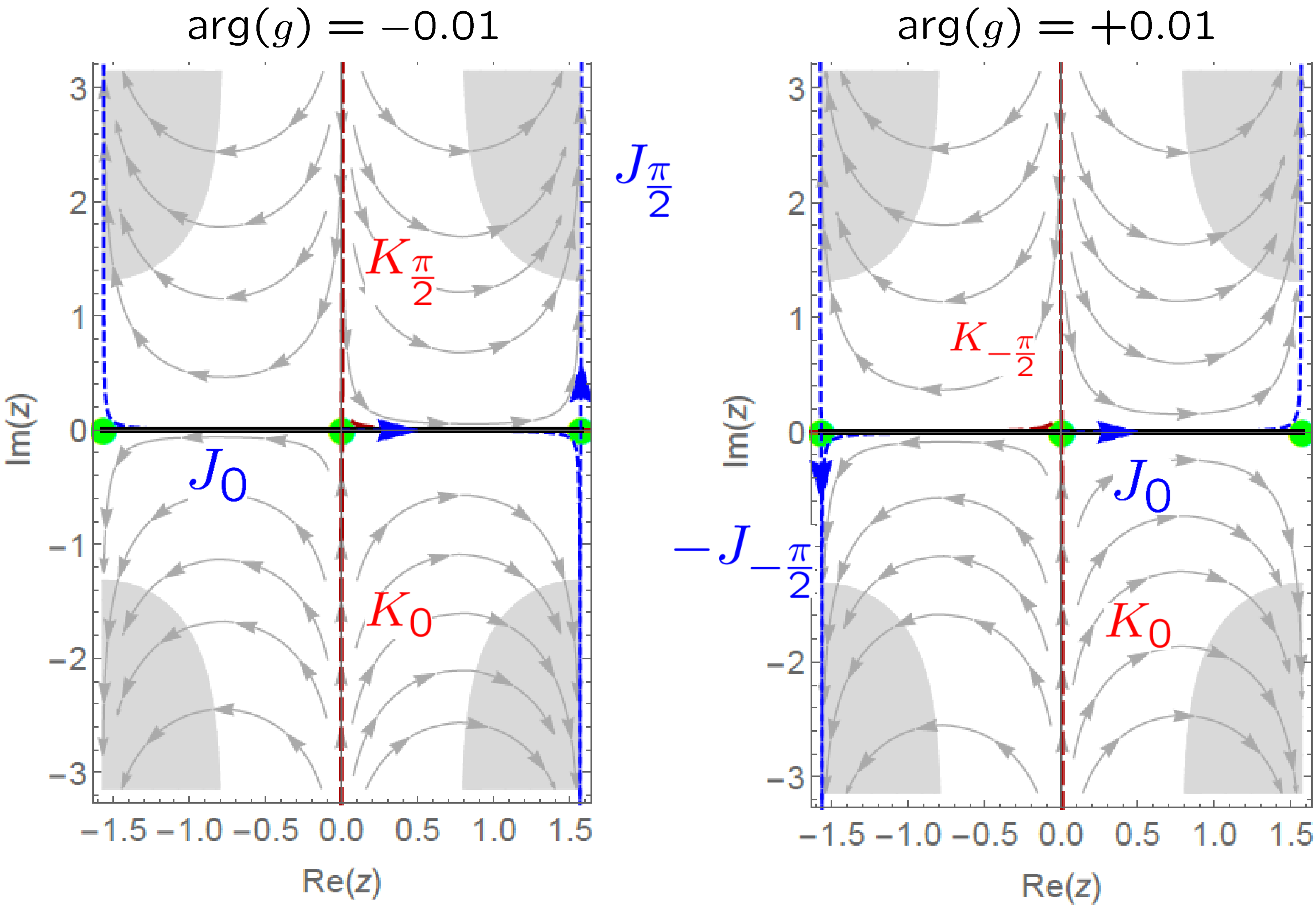
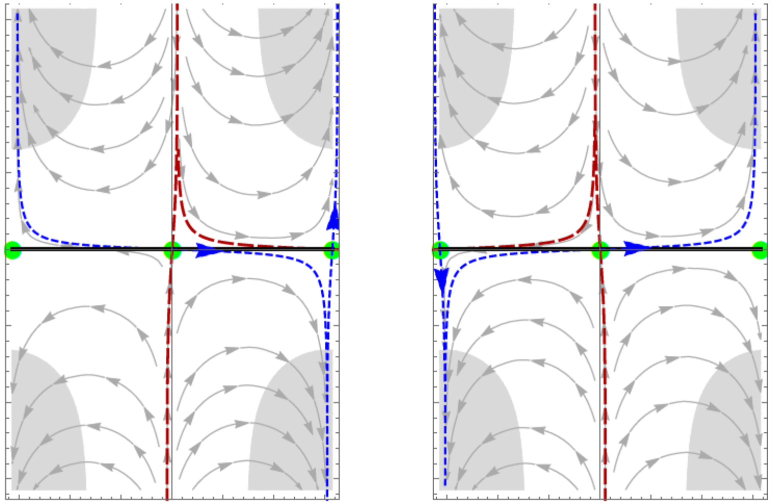


Thimble structures in the toy model (Cont'd)



Contribution from nontrivial saddle

- Either $x=+\pi/2$ or $-\pi/2$ contributes
- Contours smoothly change in the ranges $0<\theta<\pi$ and $-\pi<\theta<0$
- Contours through nontrivial saddles are opposite between $\theta<0$ & $\theta>0$



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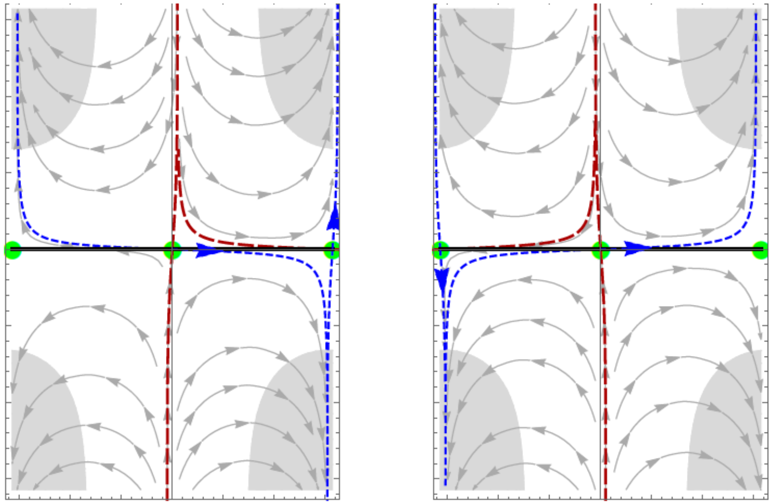

$$Z(g)|_{x_*=\pm\frac{\pi}{2}} = \begin{cases} +e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta < 0) \\ -e^{-\frac{1}{2g}} \sum_{\ell=0}^{\infty} c_\ell^{(1)} g^\ell & (\theta > 0) \end{cases}$$

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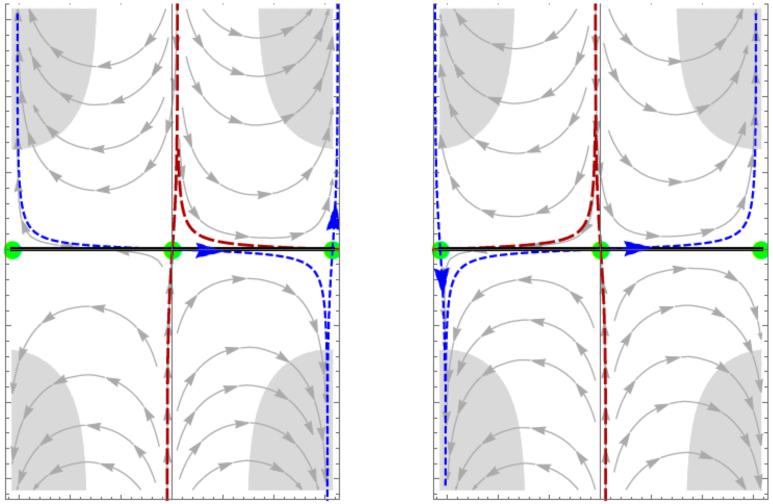

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∃ Jump at θ=0!! ("Stokes phenomenon")

Expansion around nontrivial saddle is also ambiguous at θ=0

Expansion around nontrivial saddle

$$\pm e^{-\frac{1}{2g}}\sqrt{2\pi}\sum_{\ell=0}^{\infty}\frac{(-2)^{\ell}\Gamma(\ell+1/2)^2}{\Gamma(\ell+1)\Gamma(1/2)^2}g^{\ell}\equiv\pm e^{-\frac{1}{2g}}\Phi_1(g)$$

$$\Rightarrow \quad \mathcal{B}\Phi_1(t)=\sum_{\ell=0}^{\infty}\frac{c_{\ell}^{(1)}}{\ell!}t^{\ell}=\sqrt{2\pi}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

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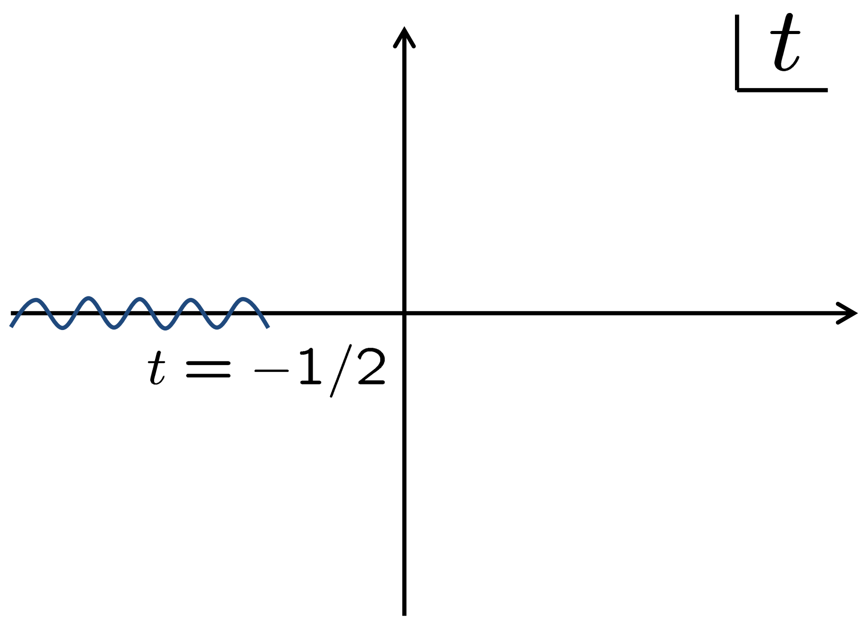
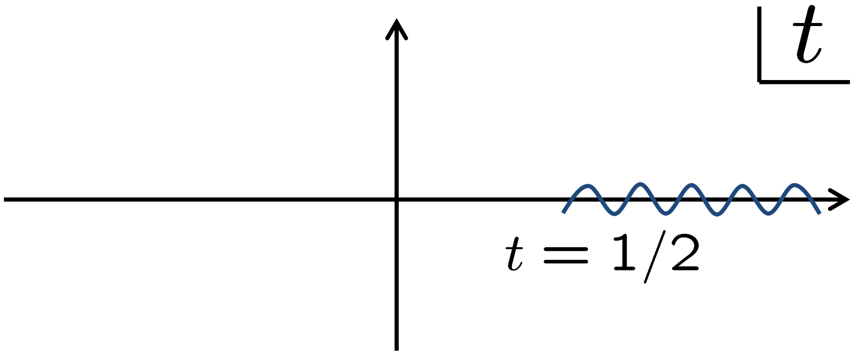


Borel trans. itself is OK but $^\exists$ambiguity at θ=0 because of Stokes phenomena

Comparison of ambiguities (at θ=0)

Trivial saddle



By the branch cut, ambiguity:

$$\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$
$$= e^{-\frac{1}{2g}}\frac{2i\sqrt{2\pi}}{g}\int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

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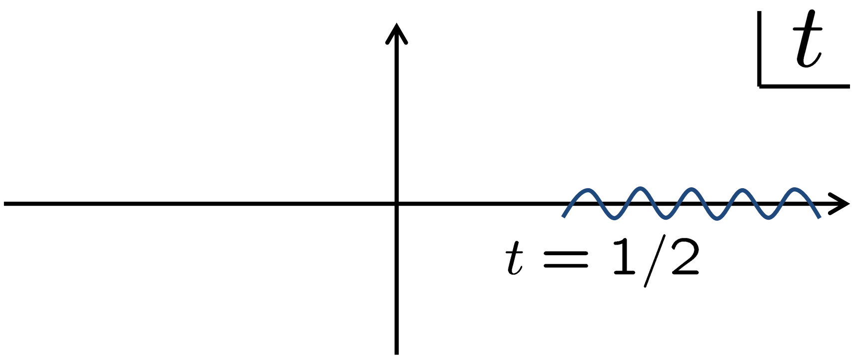

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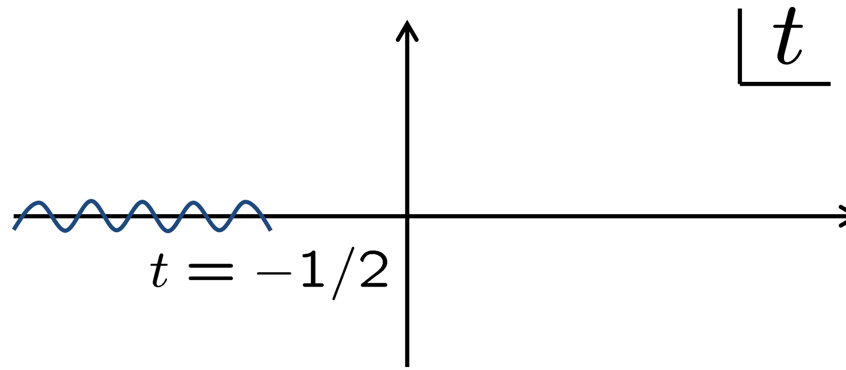

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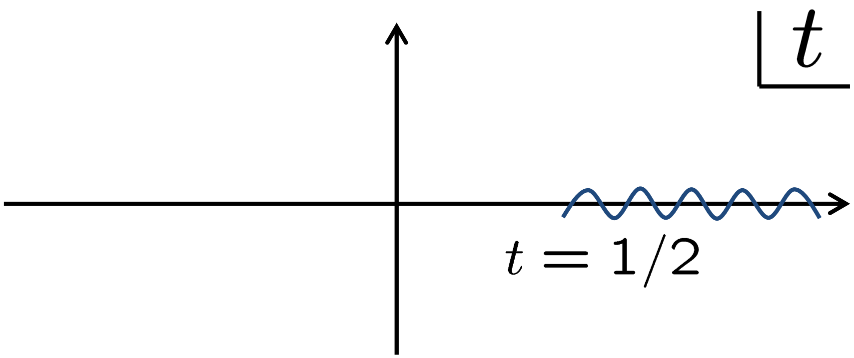

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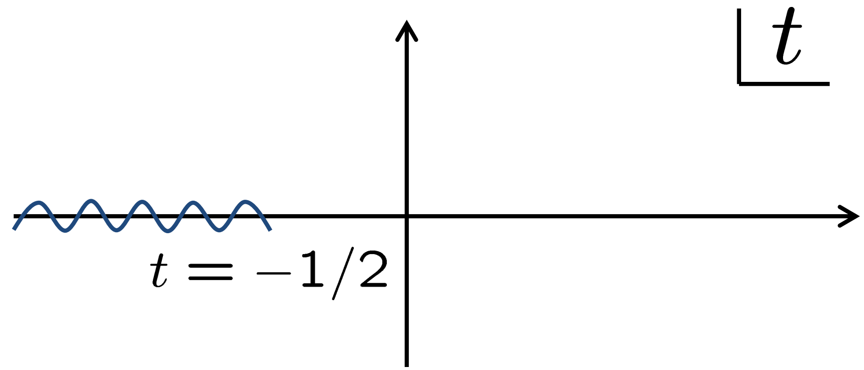

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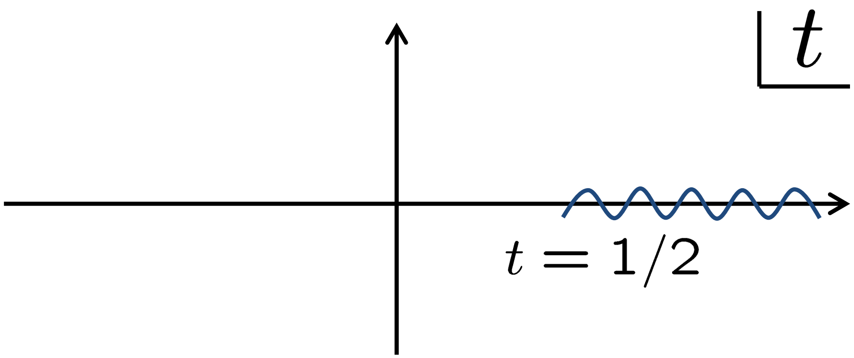

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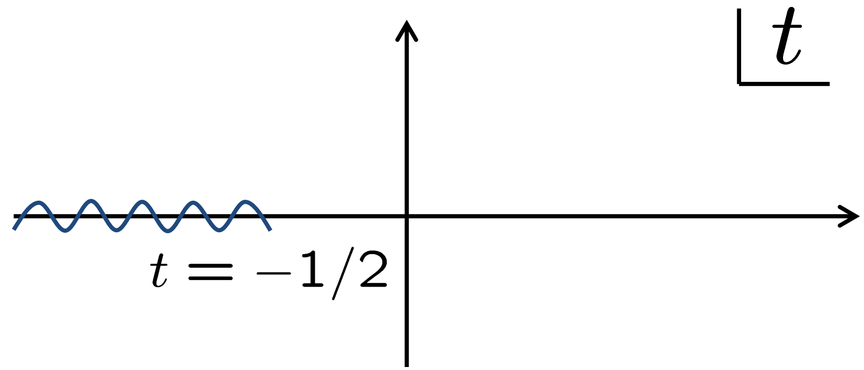

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$$= -\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$

Resurgence

(Ambiguity from trivial saddle point)

||

−(Ambiguity from nontrivial saddle point)

Resummation from a saddle point may be ambiguous but the ambiguity is cancelled by other saddles

Resurgence

(Ambiguity from trivial saddle point)

$$||$$

$-$(Ambiguity from nontrivial saddle point)

Resummation from a saddle point may be ambiguous but the ambiguity is cancelled by other saddles

In the toy model, resurgence gives the exact result:

$$Z(g \in \mathbf{R}_{\geq 0}) = \lim_{\theta \to 0_\pm} \left[S_\theta \Phi_0(g) \mp i e^{-\frac{1}{2g}} S_\theta \Phi_1(g) \right] = \mathrm{Re} S_0 \Phi_0(g)$$

It's natural to ask if resurgence can be applied to QFT

Remark 1/4: perturbative ↔ non-perturbative

Ambiguity cancellation:

$$(S_{0+} - S_{0-})\Phi_0(g) = 2ie^{-\frac{1}{2g}} S_0 \Phi_1(g)$$

⇨ Relation between perturbative coefficients around trivial & nontrivial saddles

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⇒ Relation between perturbative coefficients around trivial & nontrivial saddles

Note: Many talks on resurgence by physicists emphasize this point.
Then some physicists have an impression that definition of resurgence is relations between perturbative and non-perturbative sectors.
If there are ambiguities, there should be cancellations of them but if not, such relations do not have to exist.

Ex.) Ground state energy in system w/ SUSY breaking by non-perturbative effects,
Seiberg-Witten prepotential, SUSY obs. in 4d N=2 & 5d N=1 theories on sphere [MH '16]

[Some deformations have nontrivial structures: Dunne-Unsal , Kozcaz-Sulejmanpasic-Tanizaki-Unsal, Dorigoni-Glass]

Remark 2/4: The toy model is useful but very special

- We can compute all order perturbative coefficients

—— In realistic QFT, computing higher order itself deserves to write a paper

- ∃ only one nontrivial saddle points

—— ∃ ∞ many saddles in QFT

- Perturbative series in all the sectors are related

—— Resurgence doesn't relate different topological sectors

- We can explicitly draw thimbles

—— impossible in more than two dim. integral

- Perturbative sector knows everything: $Z(g) = \mathrm{Re} S_0 \Phi_0(g)$

—— not true in more complicated cases

Remark 3/4: A “Mathematical” viewpoint

Resurgence $\sim$ “Extension” of analyticity

Remark 3/4: A "Mathematical" viewpoint

Resurgence $\sim$ "Extension" of analyticity

Analytic function:

$$f(z) = \begin{cases} \sum_n f_n z^n, & |z| < \text{radius of convergence} \\ \text{(analytic continuation)} & \text{everywhere} \end{cases}$$

$\longrightarrow \{1,\ z,\ z^2,\ \cdots\}$ are "good basis" to express f(z)

Remark 3/4: A "Mathematical" viewpoint

Resurgence ~ "Extension" of analyticity

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$\longrightarrow \{1,\ z,\ z^2,\ \cdots\}$ are "good basis" to express f(z)

For more general function, we need more "basis":

$$\{z^\sharp,\ z^\sharp \log z,\ z^\sharp e^{-\frac{\sharp}{z}},\ \cdots\}$$

Ex.) The toy example needed $\{g^n,\ g^n e^{-\frac{1}{2g}}\}$

Remark 4/4: Finite order approximation

$$\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

To compute Borel trans.,

we need all order perturbative coefficients in principle.

Remark 4/4: Finite order approximation

$$\mathcal{BO}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$$

To compute Borel trans.,

we need all order perturbative coefficients in principle.

But when we know only up to finite order,

we can use Pade approximation for Borel trans.:

("Borel-Pade approximation")

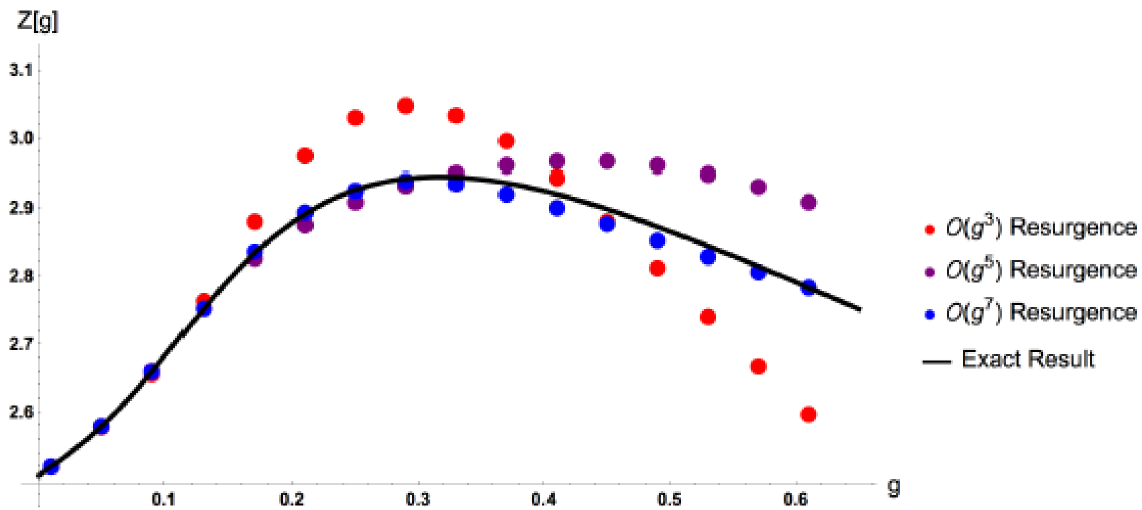
$$P_{m,n}(t) = \frac{\sum_{k=0}^{m} c_k t^k}{1 + \sum_{\ell=1}^{n} d_\ell t^\ell}$$

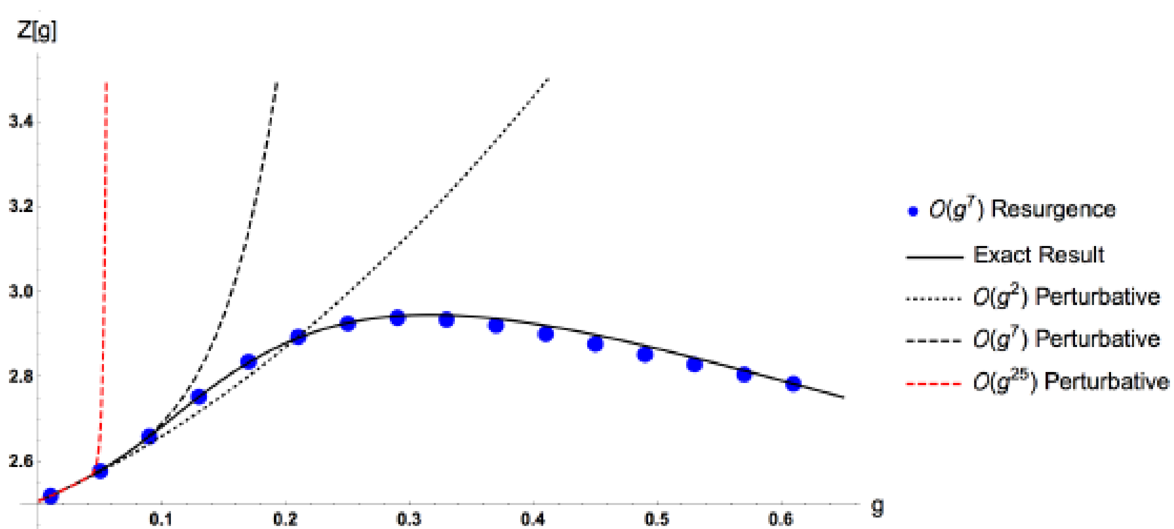
where coefficients are determined s.t.
small-t expansion gives the one of Borel trans.

Remark 4/4: Finite order approximation (Cont'd)

Result in the toy model:

[Fig.4 in Cherman-Koroteev-Unsal '14]

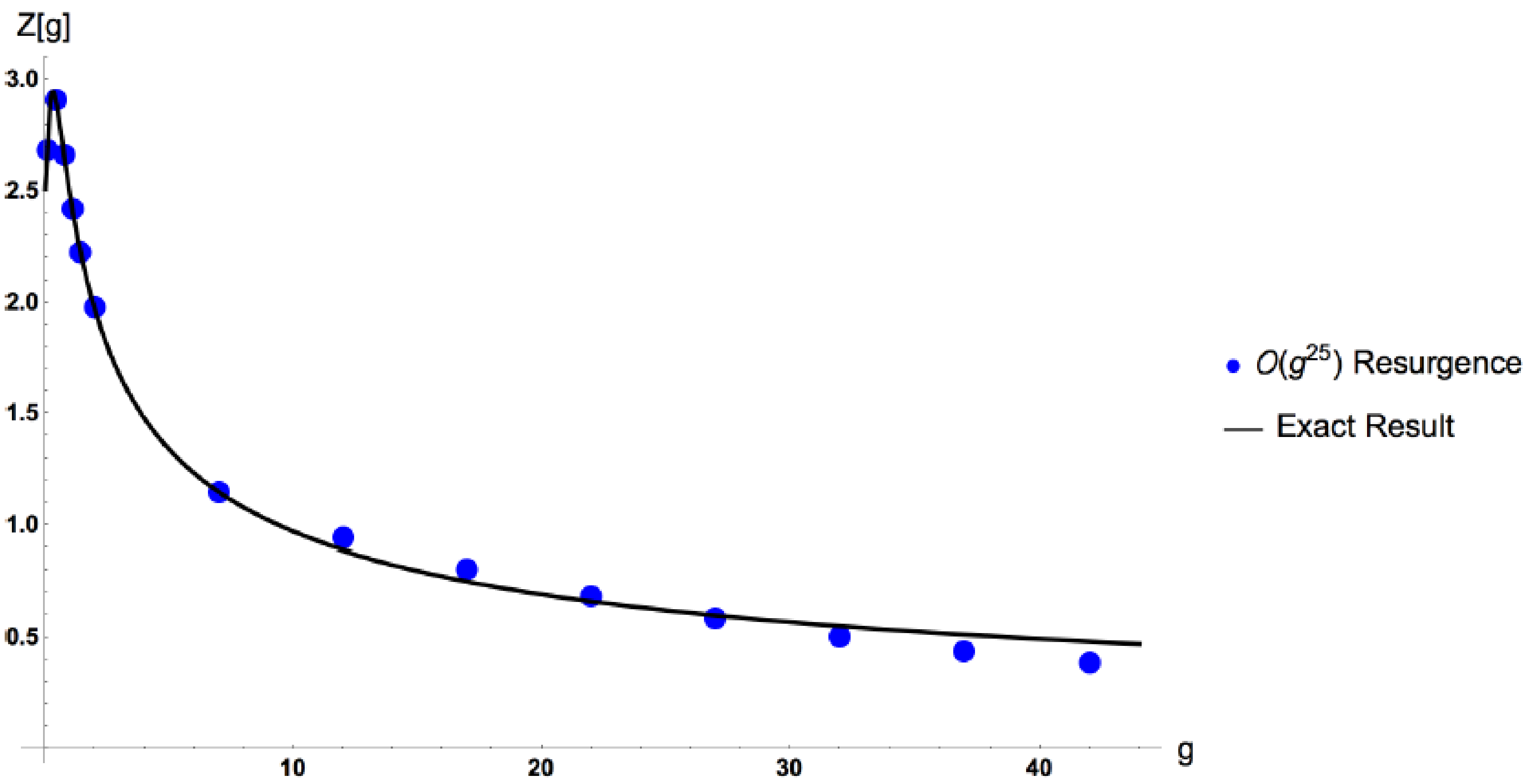




Remark 4/4: Finite order approximation (Cont'd)

[Fig.5 in Cherman-Koroteev-Unsal '14]

Result in the toy model:



Contents of day 1: Basics

0. Prologue

1. Expectations on weak coupling perturbative series in QFT

2. What is resurgence?

3. Summary of day 1

4. Preview of day 2 (Application to QFT)

Summary of day 1

- Perturbative series in QFT is typically non-convergent

- Borel singularities $\leftrightarrow$ Nontrivial saddle points

- At first sight, Borel resummation seems usually dead
  & ambiguous due to singularities along R+

- But it may be resurgent.
  The ambiguities from a saddle pt. may be cancelled
  by other saddles

- We should rewrite (path) int. in terms of Lefschetz thimble

More than weak coupling expansion in QFT

We could apply resurgence to other types of expansions.

For example,

- $1/N$ expansion (~string perturbation if AdS/CFT is correct)
- strong coupling expansion (~α'-expansion if AdS/CFT is correct)
- Weak coupling expansion in gravity (string)
- high/low temperature expansion
- ϵ-expansion
- Derivative expansion in effective theory

etc...

Preview of day 2

(Application to QFT)

Q. Can we apply resurgence to QFT?

This is essentially asking two questions:

Q1. Can we obtain resummation w/o ambiguities
by resurgence?

Q2. If yes,
is the resummation the same as exact result?

Q. Can we apply resurgence to QFT?

This is essentially asking two questions:

Q1. Can we obtain resummation w/o ambiguities by resurgence?

Q2. If yes,

is the resummation the same as exact result?

Q1. Can we obtain resummation w/o ambiguities by resurgence?

(Ideal) steps to answer Q1:

1. Find all critical pts.

   (including configurations outside original path)

2. Take complex coupling &

   rewrite path integral in terms of Lefschetz thimble

   [done for pure CS, Liouville, some QM: Witten, Harlow-Maltz-Witten]

3. Compute perturbation around contributing saddles

4. Check cancellation of ambiguities

Sounds difficult? Sometimes we can simplify it.

See you next week! Thank you for attention!!