Resummation of Perturbative Series & Resurgence in Quantum Field Theory

Day2: Application to QFT

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KEK-PH lecture series

Summary of day 1

- Perturbative series in QFT is typically non-convergent
- Borel singularities $\leftrightarrow$ Nontrivial saddle points
- At first sight, Borel resummation seems usually dead & ambiguous due to singularities along $\boldsymbol{R}_+$
- But it may be resurgent.
  The ambiguities from a saddle pt. may be cancelled by other saddles
- We should rewrite (path) int. in terms of Lefschetz thimble

Borel resummation

Borel transformation:

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell g^{a+\ell} \quad \Rightarrow \quad \boxed{\mathcal{B}\mathcal{O}(t) = \sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}}$$

Borel resummation (along θ):

$$\boxed{S_\theta \mathcal{O}(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t)}$$

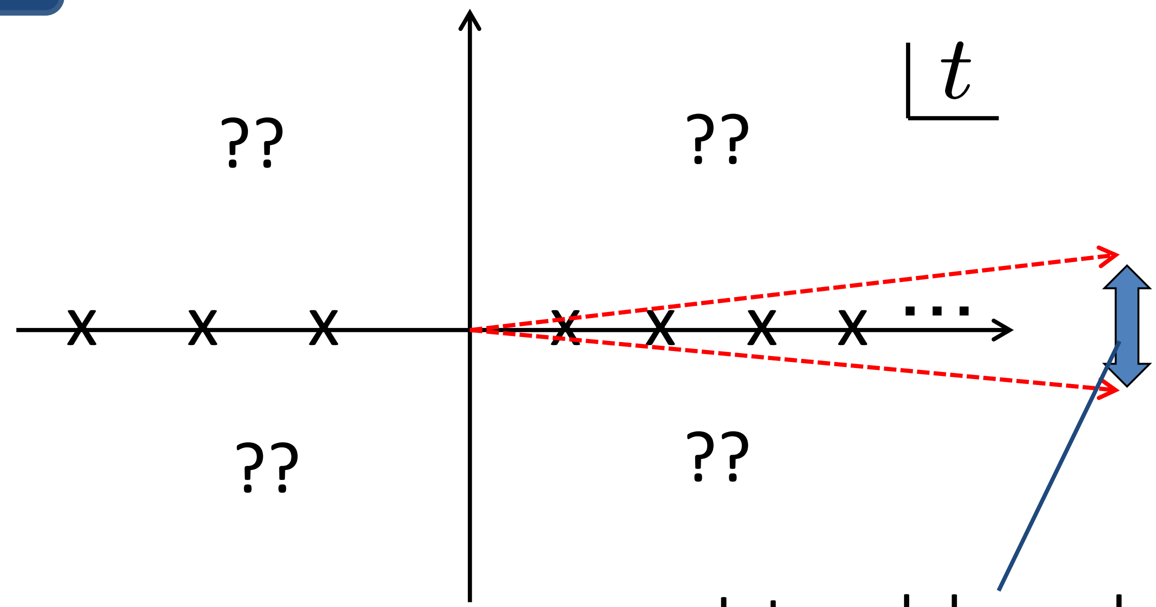
(usually, $\theta=\arg(g)=0$)

Expectations in typical QFT

['t Hooft '79]

Non-Borel summable due to singularities along R_+

Borel plane: (singularities of Borel trans.)

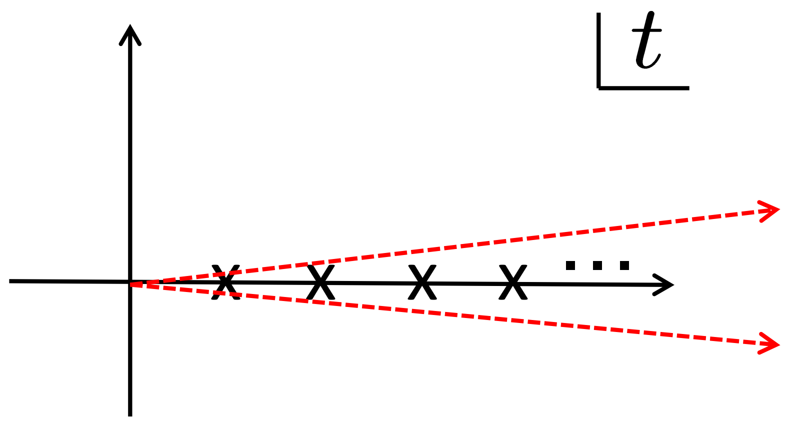


Integral depends on a way to avoid singularities

$$S_{\theta=0}\mathcal{O}(g) = \int_0^\infty dt\ e^{-\frac{t}{g}}\ \mathcal{B}\mathcal{O}(t) \Rightarrow (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Non-perturbative effect?

Resurgence



$$(\text{Ambiguities}) \sim (\text{Residue}) \sim e^{-\frac{\sharp}{g}}$$

Idea of resurgence:

This is precisely canceled by ambiguities of perturbative series around other saddle points (∼ non-pert. sector):

(perturbative ambiguity) = −(non-perturbative ambiguity)

⇒ (unambiguous answer)

Lefschetz thimble

[Extension to path integral: Witten '10]

1. Extends real x to complex z
2. Critical pt. : $\left.\frac{dS(z)}{dz}\right|_{z=z_I} = 0$
3. Associated w/ critical pt., $^\exists$unique Lefschetz thimble J_I :

$$\frac{dz(t)}{dt} = \overline{\frac{\partial S(z)}{\partial z}}, \quad \text{with } z(t \to -\infty) = z_I$$

Properties:

a) $\mathrm{Im}S(z)|_{J_I} = \mathrm{Im}S(z_I)$ $\quad \left(\frac{d}{dt}\mathrm{Im}S \propto \frac{d}{dt}(S - \bar{S}) = \frac{dz}{dt}\frac{\partial S}{\partial z} - \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 0\right)$

b) $\mathrm{Re}S(z)|_{J_I} \geq \mathrm{Re}S(z_I)$ $\quad \left(\frac{d}{dt}\mathrm{Re}S \propto \frac{dz}{dt}\frac{\partial S}{\partial z} + \frac{d\bar{z}}{dt}\overline{\frac{\partial S}{\partial z}} = 2\frac{\partial S}{\partial z}\overline{\frac{\partial S}{\partial z}} \geq 0\right)$

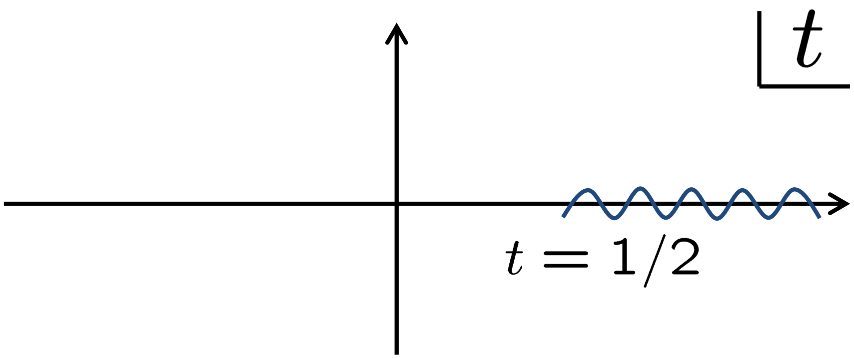
c) Decomposition of cycle: (if we are not on Stokes line)

$$\int_C = \sum_{I \in \text{saddle}} n_I \int_{J_I} \qquad (n_I \in \mathbf{Z})$$

may jump as changing parameters

What we saw in the toy model

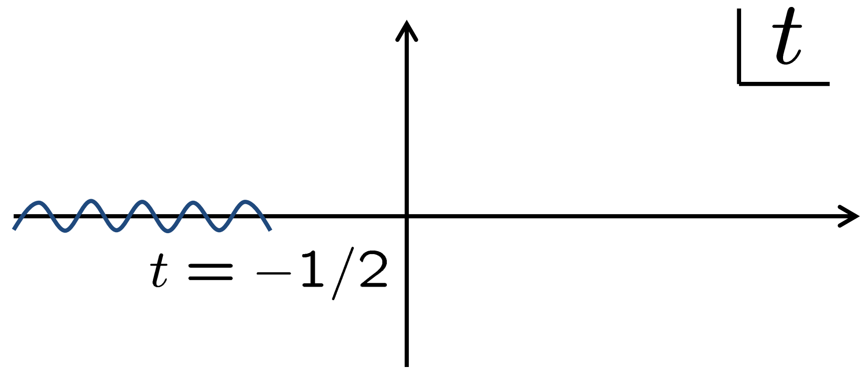
Trivial saddle



By the branch cut, ambiguity:

$$\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$
$$= e^{-\frac{1}{2g}}\frac{2i\sqrt{2\pi}}{g}\int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

Nontrivial saddle



By the Stokes phenomena,

$$Z(g)|_{x_*=\pm\frac{\pi}{2}} = \begin{cases} +ie^{-\frac{1}{2g}}S_\theta\Phi_1(g) & (\theta < 0) \\ -ie^{-\frac{1}{2g}}S_\theta\Phi_1(g) & (\theta > 0) \end{cases}$$

Ambiguity:

$$-2ie^{-\frac{1}{2g}}S_0\Phi_1(g) = -\frac{2i\sqrt{2\pi}}{g}e^{-\frac{1}{2g}}\int_0^\infty dt\ e^{-\frac{t}{g}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-2t\right)$$

$$= -\left(S_{0+} - S_{0-}\right)\Phi_0(g)$$

A "Mathematical" viewpoint

Resurgence $\sim$ "Extension" of analyticity

Analytic function:

$$f(z) = \begin{cases} \sum_n f_n z^n, & |z| < \text{radius of convergence} \\ \text{(analytic continuation)} & \text{everywhere} \end{cases}$$

$\longrightarrow \{1,\ z,\ z^2,\ \cdots\}$ are "good basis" to express f(z)

For more general function, we need more "basis":

$$\{z^\sharp,\ z^\sharp \log z,\ z^\sharp e^{-\frac{\sharp}{z}},\ \cdots\}$$

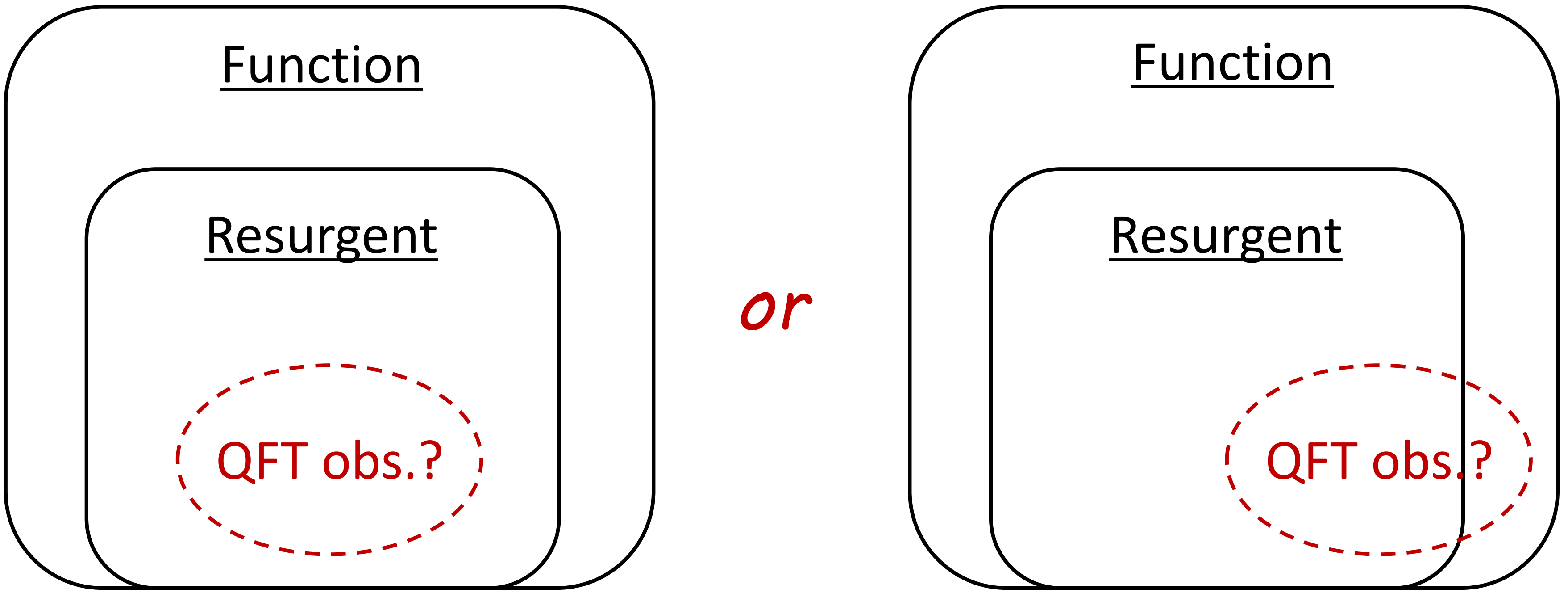
Ex.) The toy example needed $\{g^n,\ g^n e^{-\frac{1}{2g}}\}$

Day 2: Application to QFT

Q. Can we apply resurgence to QFT?

A. may or may not depend on setup.

At this moment, we don’t know whether or not all observables in all Lagrangian QFTs are resurgent



Q. For what observables in what Lagrangian QFTs, does resurgence work?

Aim of this lecture

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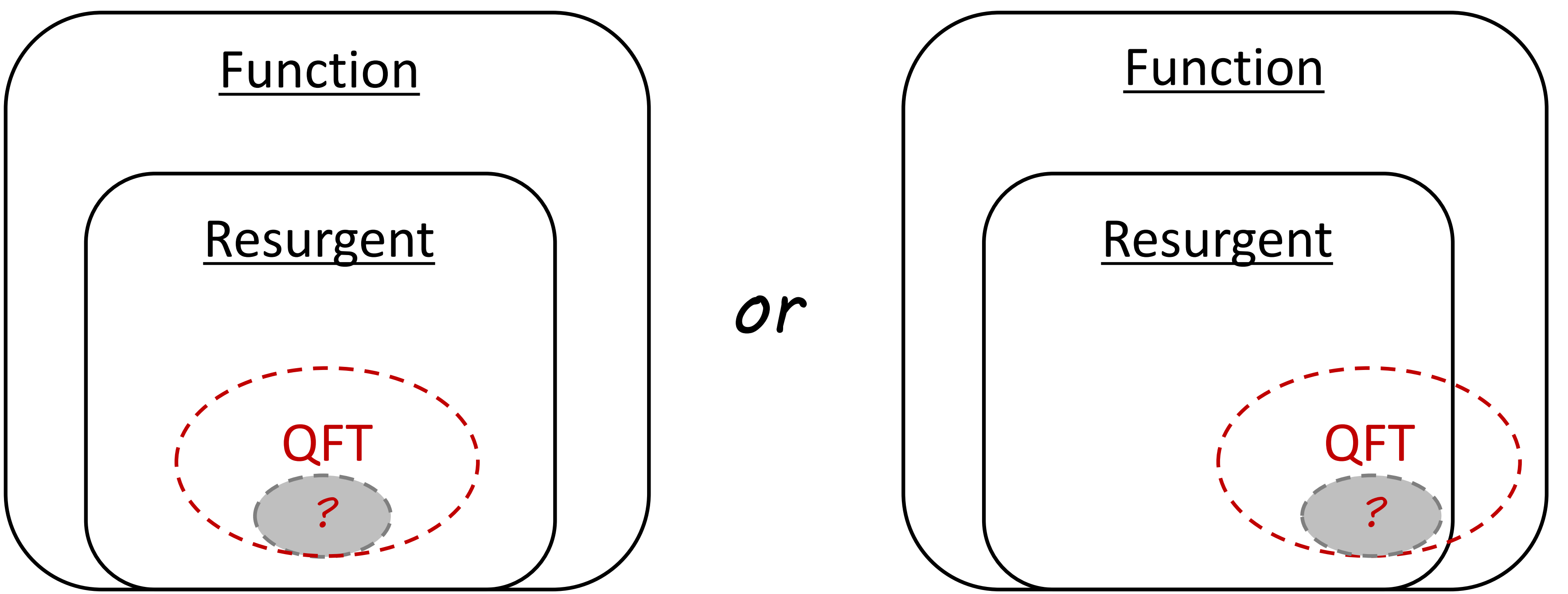
To give a partial answer to the question



Strategy here

Focus on special classes of resurgent function

&

Identify classes of QFTs associated w/ them

Trivial classes:

- Analytic function
- $\exists$ non-perturbative corrections but each sector is analytic
- $\exists$ non-perturbative corrections but each sector is Borel summable

Non-trivial classes:

- Without IR renormalons
- With IR renormalons
- $\exists$ "non-perturbative effect of non-perturbative effect" $\sim e^{-e^{\frac{1}{g}}}$

A schematic answer

Function of coupling

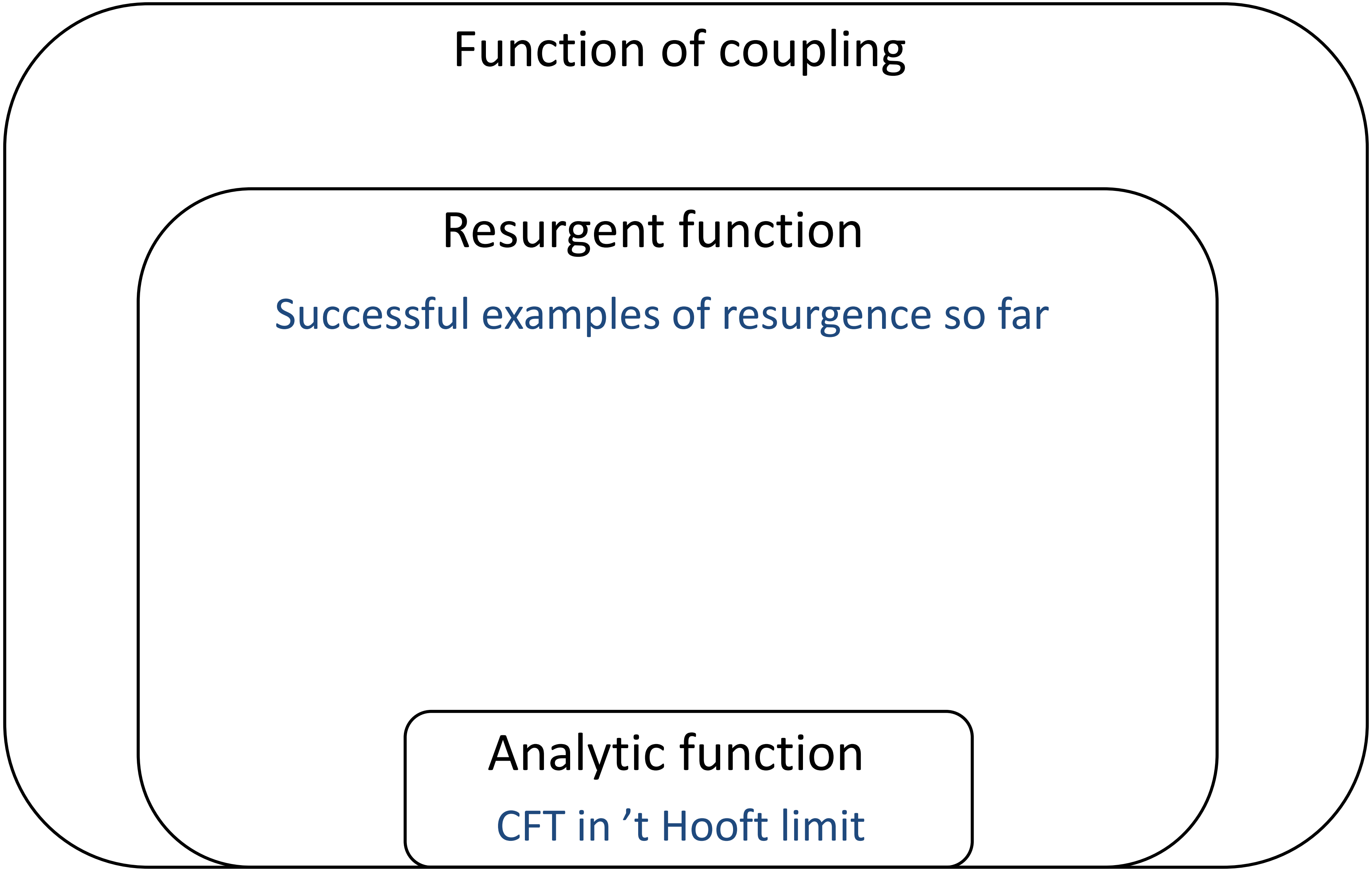
A schematic answer

Function of coupling

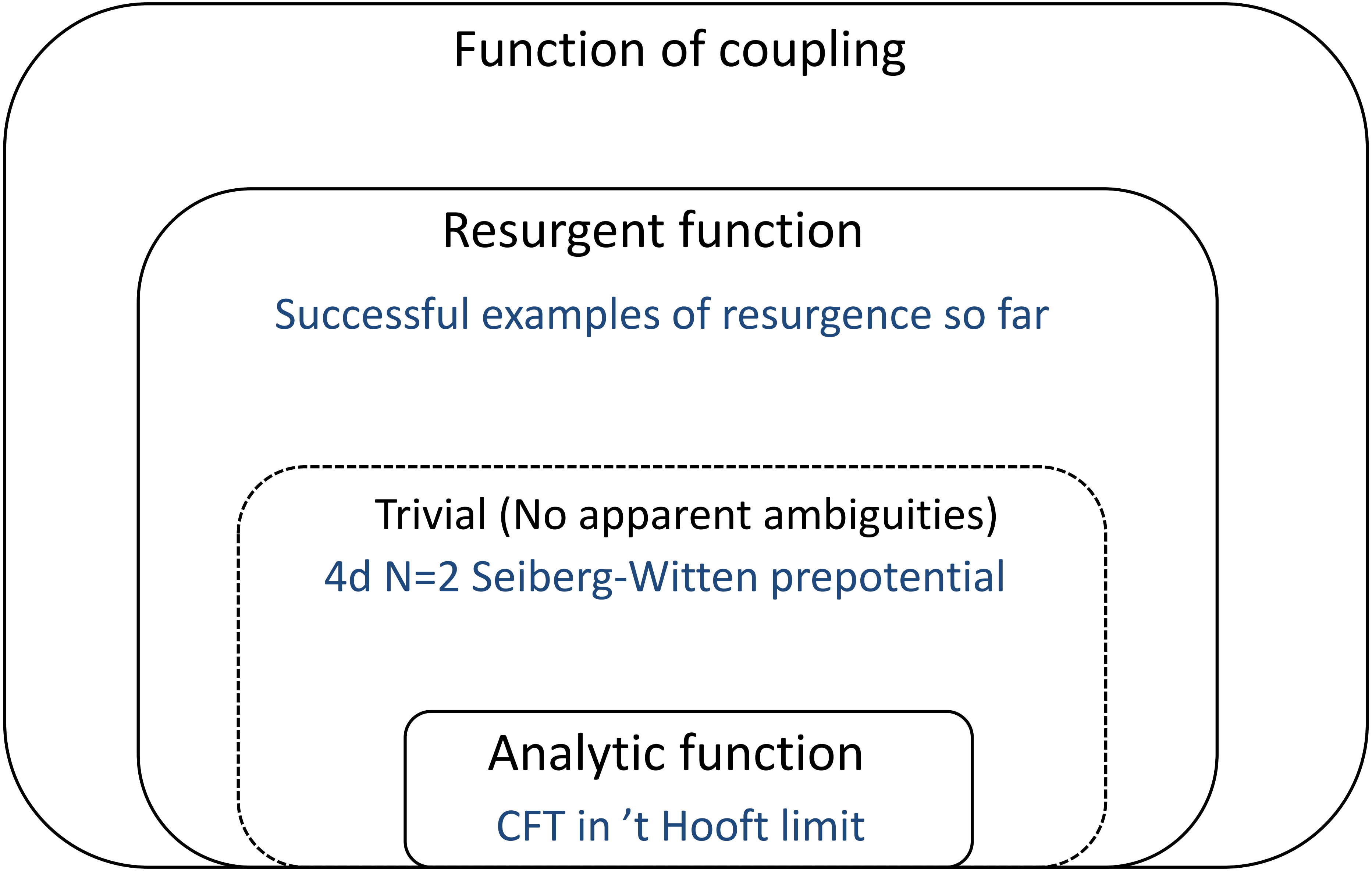
Analytic function

CFT in 't Hooft limit

A schematic answer



A schematic answer



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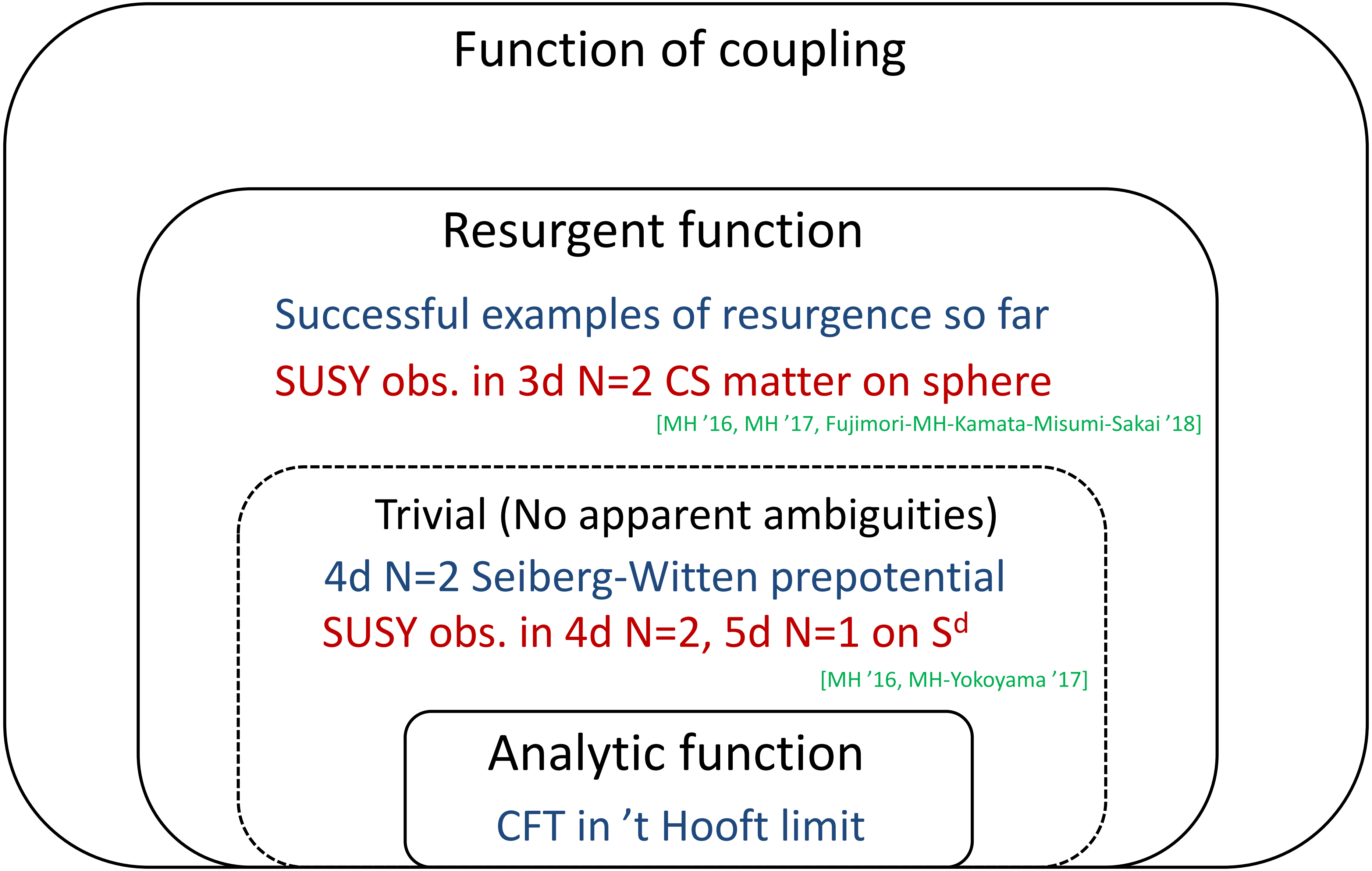

Contents

Exact result = Analytic function

Analytic function:

$$f(z) = \begin{cases} \sum_n f_n z^n, & |z| < \text{radius of convergence} \\ \text{(analytic continuation)} & \text{everywhere} \end{cases}$$

Exact result = Analytic function

Analytic function:

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It is known ['t Hooft '82]

(# of n-loop diagrams in large-N limit) $\sim (\text{Const.})^n \neq n!$

Therefore, unless $^\exists$diagrams w/ large values (renormalons), perturbative series should be convergent.

$\Rightarrow$ Large-N CFT seems in this class

A bit more nontrivial class

$^{\exists}$non-perturbative corrections but each sector is analytic

Ex1. SUSY QM w/ ~~SUSY~~ by non-perturbative effects

$$E_0^{\rm pert}(g) = 0, \qquad E_0^{\rm non-pert}(g) \neq 0 \text{ (unambiguous)}$$

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$$E_0^{\mathrm{pert}}(g) = 0, \qquad E_0^{\mathrm{non-pert}}(g) \neq 0 \text{ (unambiguous)}$$

Ex2. Seiberg-Witten prepotential in 4d $\mathcal{N} = 2$ theory on $\boldsymbol{R}^4$

[Seiberg-Witten '94]

$$\mathcal{F} = i\frac{1}{2\pi}\mathcal{A}^2 \ln\frac{\mathcal{A}^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{\mathcal{A}}\right)^{4k}\mathcal{A}^2$$

$^{\exists}$Instanton effects but each sector is 1-loop exact

More nontrivial class: 4d $\mathcal{N} = 2$ theories on S^4

[M.H. '16]

Set up:

- Theories w/ β≦0 and Lagrangians $(Z_{S^4} < \infty)$
- Perturbative expansion by g_{YM} around fixed # of instanton/anti-inst.

More nontrivial class: 4d $\mathcal{N} = 2$ theories on S^4

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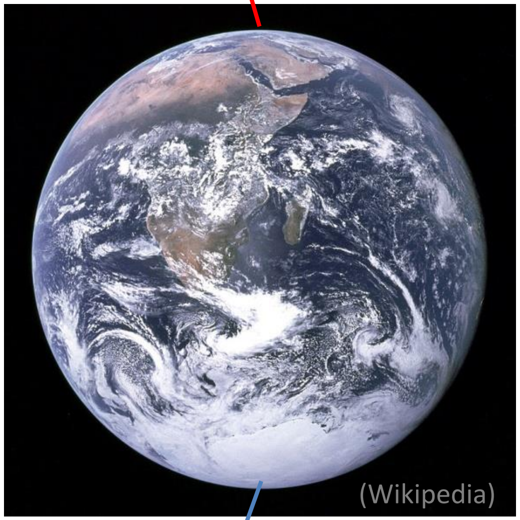

Result:

(similar for 5d N=1 case)

[cf. some low rank cases: Russo, Aniceto-Russo-Schiappa,
Gerchkovitz-Gomis-Ishtiaque-Karashik-Komargodski-Pufu]

- Find explicit finite dimensional integral rep. of Borel trans. for various observables
- $^\exists$ Singularities only along R- → Borel summable along R+
- (Exact) = $\sum_{\text{instantons}}$ (Borel resum)

Typical case: SU(2) w/ fundamentals

Borel trans. around trivial b.g. : $\mathcal{B}Z_{S^4}^{(0,0)}(t) \propto \sqrt{t} \prod_{n=1}^{\infty} \frac{\left(1+\frac{4t}{n^2}\right)^{2n}}{\left(1+\frac{t}{n^2}\right)^{2N_f n}}$

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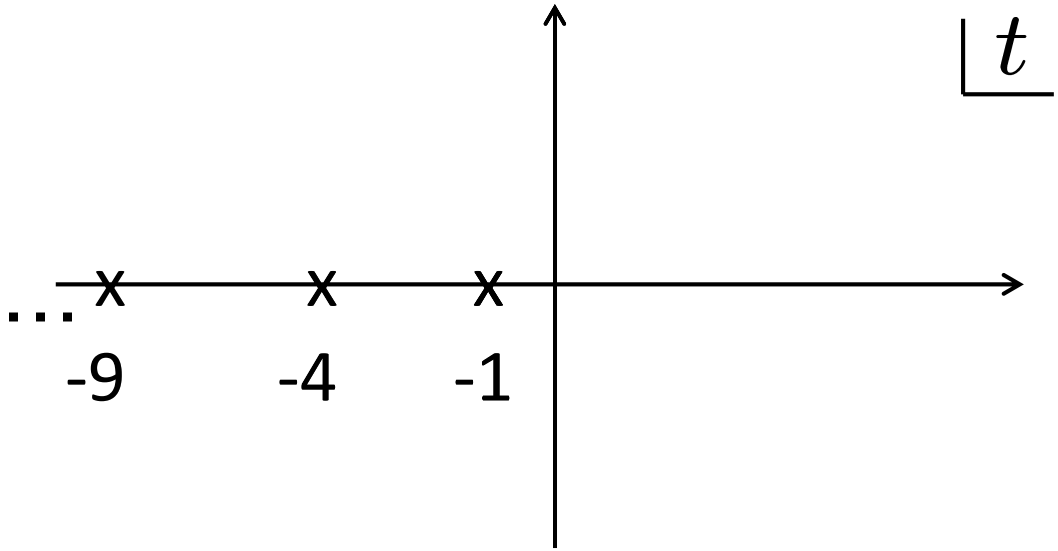

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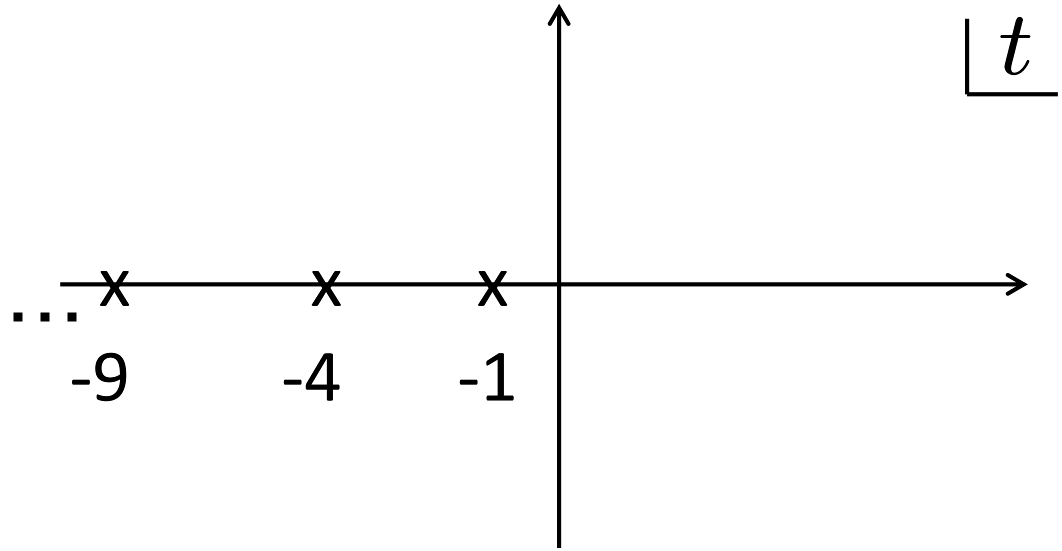


- $^\exists\infty$ singularities along R-
- All singularities are NOT instantons & IR/UV renormalons
- No qualitative difference between CFT and non-CFT

Partition function of SU(N) theory on S^4 ($\beta \leqq 0$)

Exact result:

[Pestun '07]

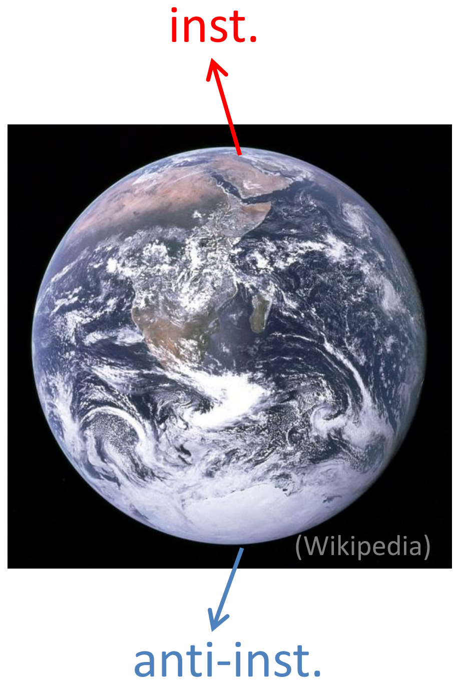
$$Z_{S^4}(g,\theta) = \int_{-\infty}^{\infty} d^N a\; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2}\,\tilde{Z}(a) Z_{\text{inst}}(g,\theta;a)$$

($\tilde{Z}(a)$: 1-loop determinant w/ traceless constraint)

$g \propto 1-$ loop effective g_{YM}^2 at scale $R_{S^4}^{-1}$

$$Z_{\text{inst}}(g,\theta;a) = \sum_{k,\bar{k}=0}^{\infty} e^{-\frac{k+\bar{k}}{g}+i(k-\bar{k})\theta} Z_{\text{inst}}^{(k,\bar{k})}(a)$$

$$Z_{S^4}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a\; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2}\,\tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$



We are interested in small-g expansion of this

Borel trans. hidden in localization formula

$$Z_{S^4}^{(k,\bar{k})}(g) = \int_{-\infty}^{\infty} d^N a \; e^{-\frac{1}{g}\sum_{j=1}^{N} a_j^2} \tilde{Z}(a) Z_{\text{inst}}^{(k,\bar{k})}(a)$$

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Taking polar coordinate $a_i = \sqrt{t}\hat{x}_i$ w/ $(\hat{x}^i)^2 = 1,$

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Taking polar coordinate $a_i = \sqrt{t}\hat{x}_i$ w/ $(\hat{x}^i)^2 = 1$,

$$\boxed{Z_{S^4}^{(k,\bar{k})}(g) = \int_0^{\infty} dt \; e^{-\frac{t}{g}} f^{(k,\bar{k})}(t)}$$

similar to Borel resummation formula?

$$\left(f^{(k,\bar{k})}(t) = \int_{S^{N-1}} d^{N-1}\hat{x} \; h^{(k,\bar{k})}(t,\hat{x}), \;\; h^{(k,\bar{k})}(t,\hat{x}) = \tilde{Z}(a) Z_{\rm inst}^{(k,\bar{k})}\Big|_{a^i=\sqrt{t}\hat{x}^i} \right)$$

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We can indeed prove (proof skipped)

$$f^{(k,\bar{k})}(t) = \mathcal{B}Z_{S^4}^{(k,\bar{k})}(t)$$

(Exact result)

$$||$$

$$\sum_{k,\bar{k}} \text{(Borel resummation along } R_+\text{)}$$

(up to resummation of instanton expansion)

Other observables

[M.H. ’16]

- SUSY Wilson loop on S^4

- Bremsstrahrung function in SCFT on R^4 [cf. Fiol-Gerchkovitz-Komargodski ’15]

$$(\text{Energy of quark}) = B \int dt\ \dot{a}^2$$

- Extremal correlator in SCFT on R^4 [cf. Gerchkovitz-Gomis-Ishtiaque -Karasik-Komargodski-Pufu ’16]

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \bar{\mathcal{O}} \rangle$$

- Partition function on squashed S^4 ～ SUSY Renyi entropy

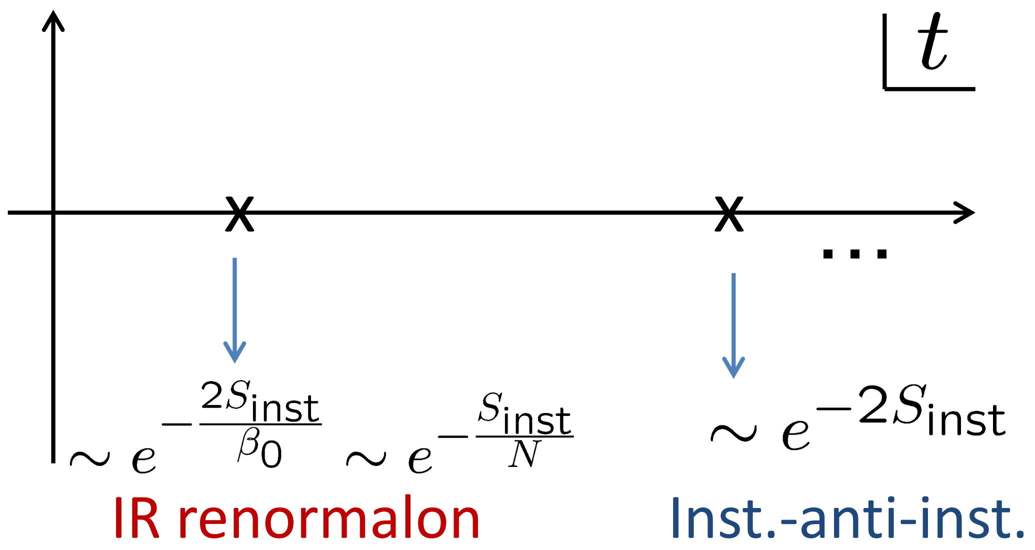
[cf. Hama-Hosomichi, Nosaka-Terashima Nishioka-Yaakov ’13, Crossley-Dyer-Sonner, Huang-Zhou]

- SUSY ’t Hooft loop ($^\exists$ monopole bubbling effects)

[MH-Yokoyama ’17]

Nontrivial consistency w/ a conjecture on QCD

Borel plane in typical gauge theory (?) :



Conjecture: (IR renormalon) = (Combination of monopoles)

[Argyres-Unsal '12]

But we don't have such solution for $\mathcal{N}=2$ [Popitz-Unsal]

$\longrightarrow$ No IR renormalon singularities for $\mathcal{N}=2$?

(But recently $^\exists$negative report to the conjecture)

[Morikawa-Takaura, Ashie-Morikawa-Suzuki-Takaura '20]

Contents

3d N=2 SUSY CS matter theory

- Borel transformation [MH '16]
- Resurgence structure [Fujimori-MH-Kamata-Misumi-Sakai '18]
- Lefschetz thimble decomposition
- Interpretation of Borel singularities [MH '17]

Results on 3d N=2 SUSY Chern-Simons theories

Set up:

(w/ 4 SUSY)

- N=2 SUSY Chern-Simons matter theories on S^3 $(Z_{S^3} < \infty)$
- Perturbative expansion by inverse CS levels

Results on 3d N=2 SUSY Chern-Simons theories

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- N=2 SUSY Chern-Simons matter theories on S³ $(Z_{S^3} < \infty)$
- Perturbative expansion by inverse CS levels

Results:

$$S_\theta I(g) = \int_0^{e^{i\theta}\infty} dt\ e^{-\frac{t}{g}}\ \mathcal{B}I(t)$$

- Find finite dimensional integral rep. for Borel trans. [MH '16]
- (exact result) = (Borel resum. along half imaginary axis)
- Nontrivial resurgence structure [Fujimori-MH-Kamata-Misumi-Sakai '18]
- Decomposition by Lefschetz thimble (=steepest descent)
- Borel singularities = Complexified SUSY solutions [MH '17]

Partition function of U(N) CS theory on S^3

Exact result: [$g \propto 1/k$, k>0: CS level]

[Kapustin-Willett-Yaakov, Jafferis, Hama-Hosomichi-Lee]

$$Z_{S^3}(g) = \int_{-\infty}^{\infty} d^N\sigma \; e^{\frac{i}{g}\sum_{j=1}^{N}\sigma_j^2}\,\tilde{Z}(\sigma)$$

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$$Z_{S^3}(g) = i\int_0^{-i\infty} dt \; e^{-\frac{t}{g}} f(it), \;\; if(\tau) = \mathcal{B}Z_{S^3}(-i\tau)$$

$$\left(f(\tau) = \int_{S^{N-1}} d^{N-1}\hat{x} \; h(\tau,\hat{x}), \;\; h(\tau,\hat{x}) = \tilde{Z}(\sigma)\Big|_{\sigma^i=\sqrt{\tau}\hat{x}^i} \right)$$

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Namely,

$$Z_{S^3}(g) = \int_0^{-i\infty} dt \; e^{-\frac{t}{g}} \mathcal{B}Z_{S^3}(t)$$

(exact result) = (Borel resum. along θ=-π/2)

More general cases

[M.H. ’16]

Other theories:

Similar results hold as long as $Z_{S^3} < \infty$

Other quantities:

- SUSY Wilson loop on S^3
- Bremsstrahrung function in SCFT on R^3 [cf. Lewkowycz-Maldacena ’13]
- 2-pt. function of U(1) flavor current in SCFT
- 2-pt. function of stress tensor in SCFT
- Partition function on squashed S^3～SUSY Renyi entropy
- Partition function on squashed lens space

3d N=2 SUSY CS matter theory

- Borel transformation [MH '16]
- Resurgence structure [Fujimori-MH-Kamata-Misumi-Sakai '18]
- Lefschetz thimble decomposition
- Interpretation of Borel singularities [MH '17]

The simplest nontrivial example

Let us consider

$\mathcal{N}=3\ U(1)$ CS theory w/ charge-1 hyper & real mass m

Partition function:

$$Z_{S^3}=\int_{-\infty}^{\infty}d\sigma\frac{e^{\frac{ik}{4\pi}\sigma^2}}{2\cosh\frac{\sigma-m}{2}}$$

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Exact result = Borel resummation:

$$Z_{S^3}=\int_0^{-i\infty} dt\ e^{-\frac{t}{g}}\mathcal{B}Z_{S^3}(t), \qquad \mathcal{B}Z_{S^3}(t)=\frac{i}{4\sqrt{it}}\sum_{\pm}\frac{1}{\cosh\frac{\sqrt{it}\pm m}{2}}$$

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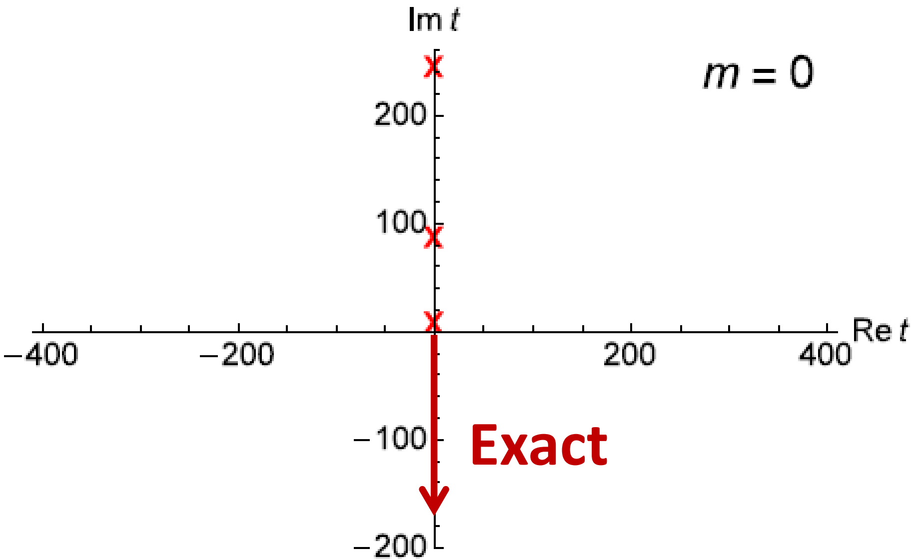
Borel singularities:

$$t_{\text{pole}}=-i\left[m+(2n+1)\pi i\right]^2\qquad (n\in\mathbf{Z})$$

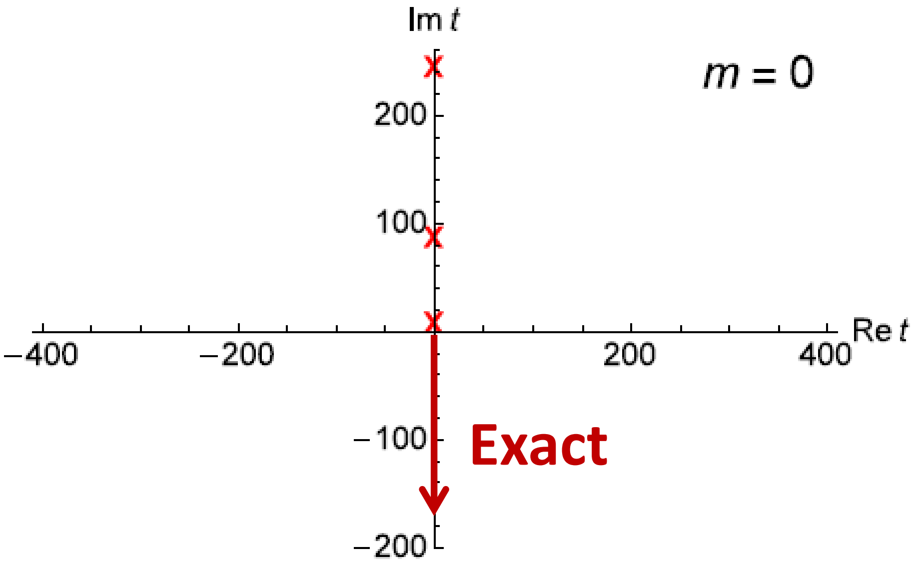
Analytic property of Borel trans.

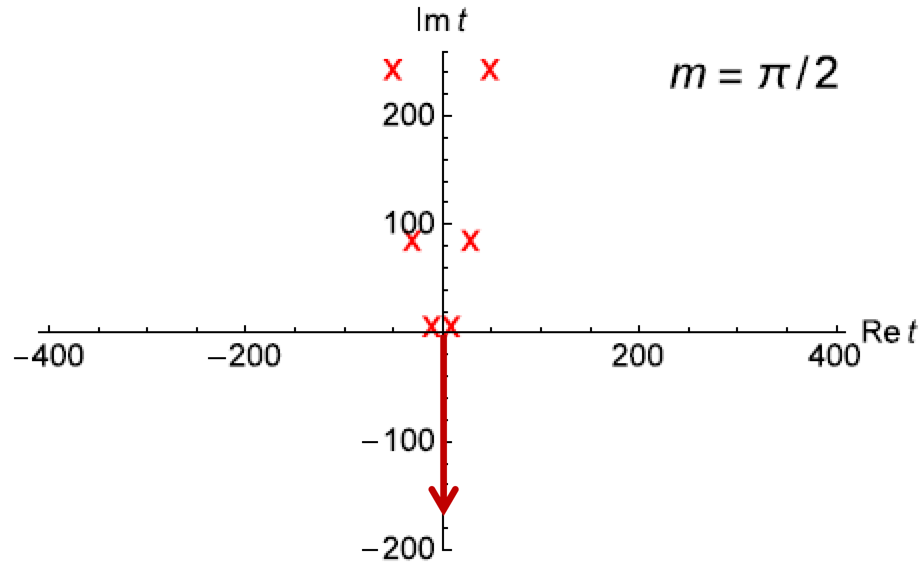
$$t_{\text{pole}} = -i\left[m + (2n+1)\pi i\right]^2$$



Analytic property of Borel trans.

$$t_{\text{pole}} = -i\left[m + (2n+1)\pi i\right]^2$$





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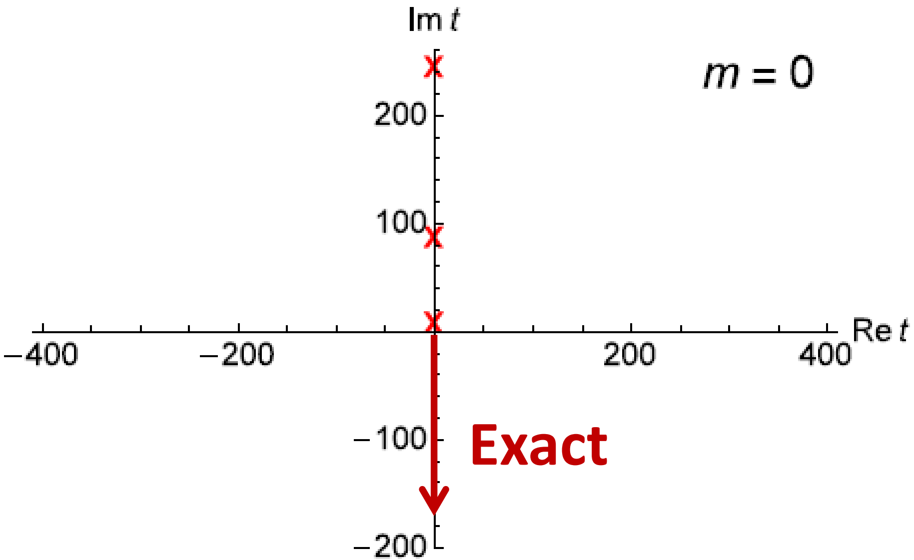

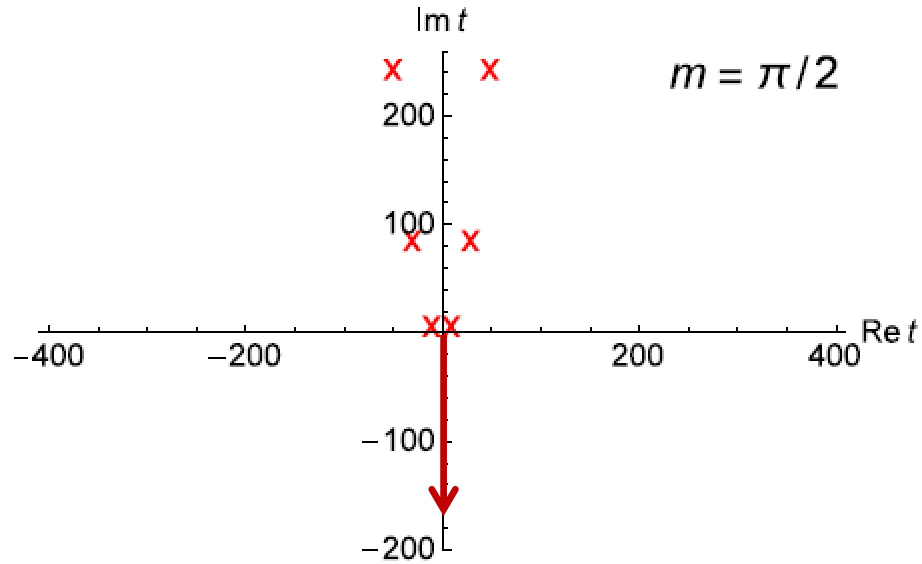

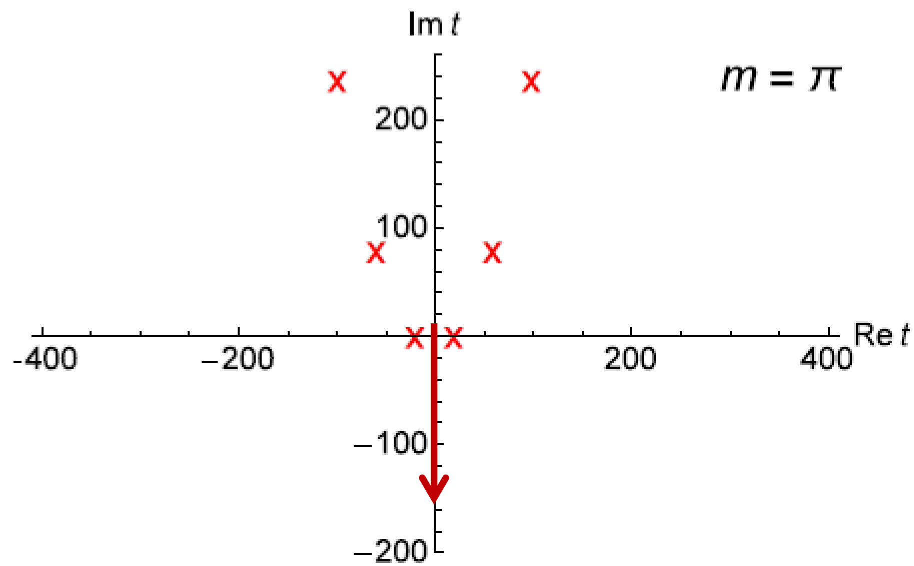

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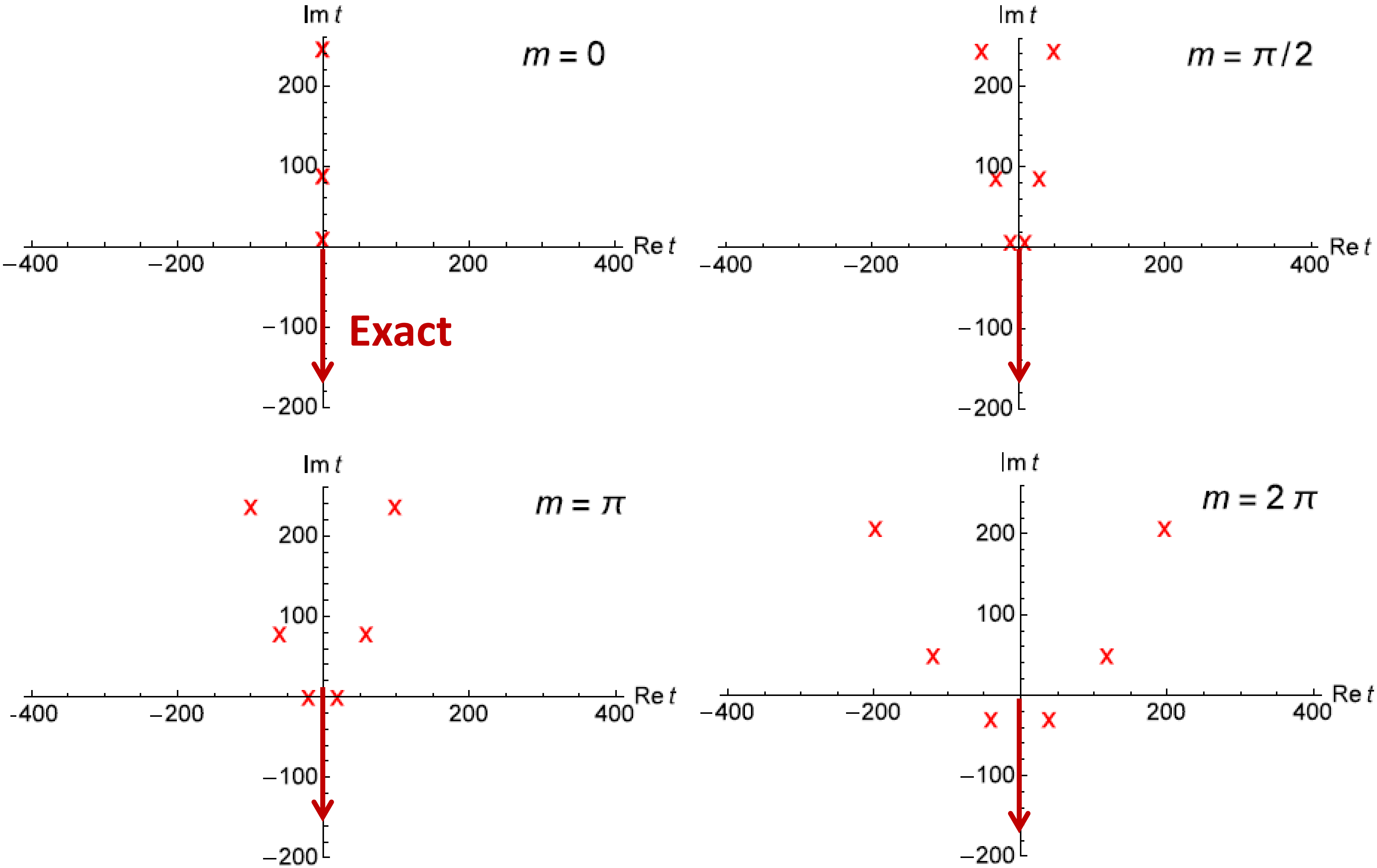

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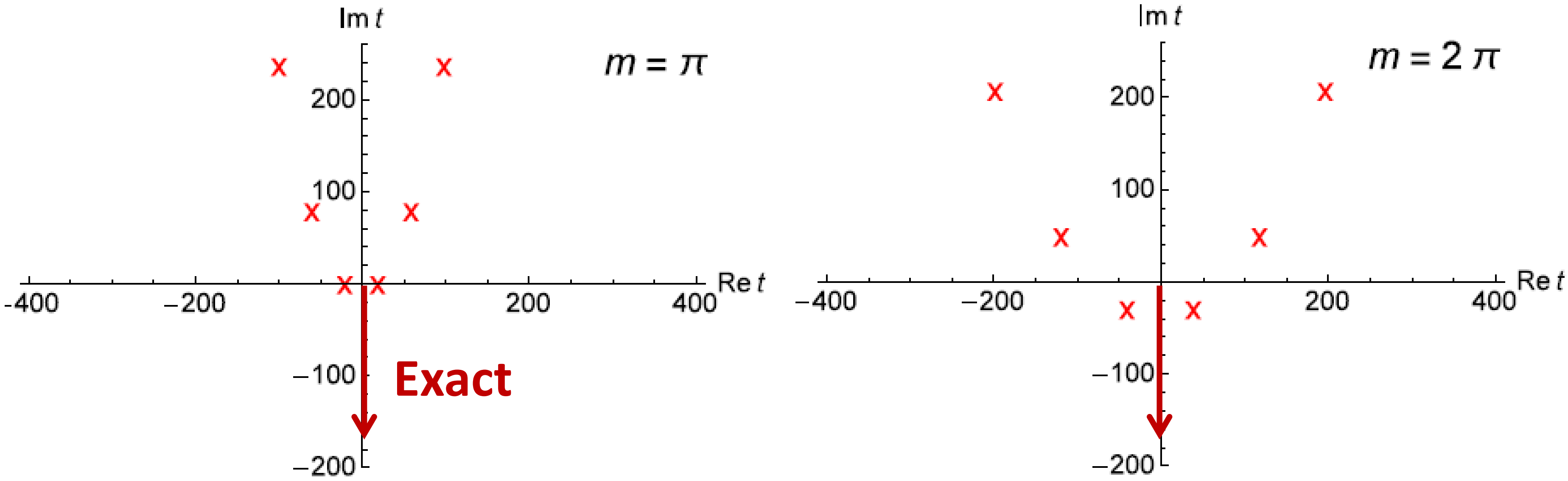

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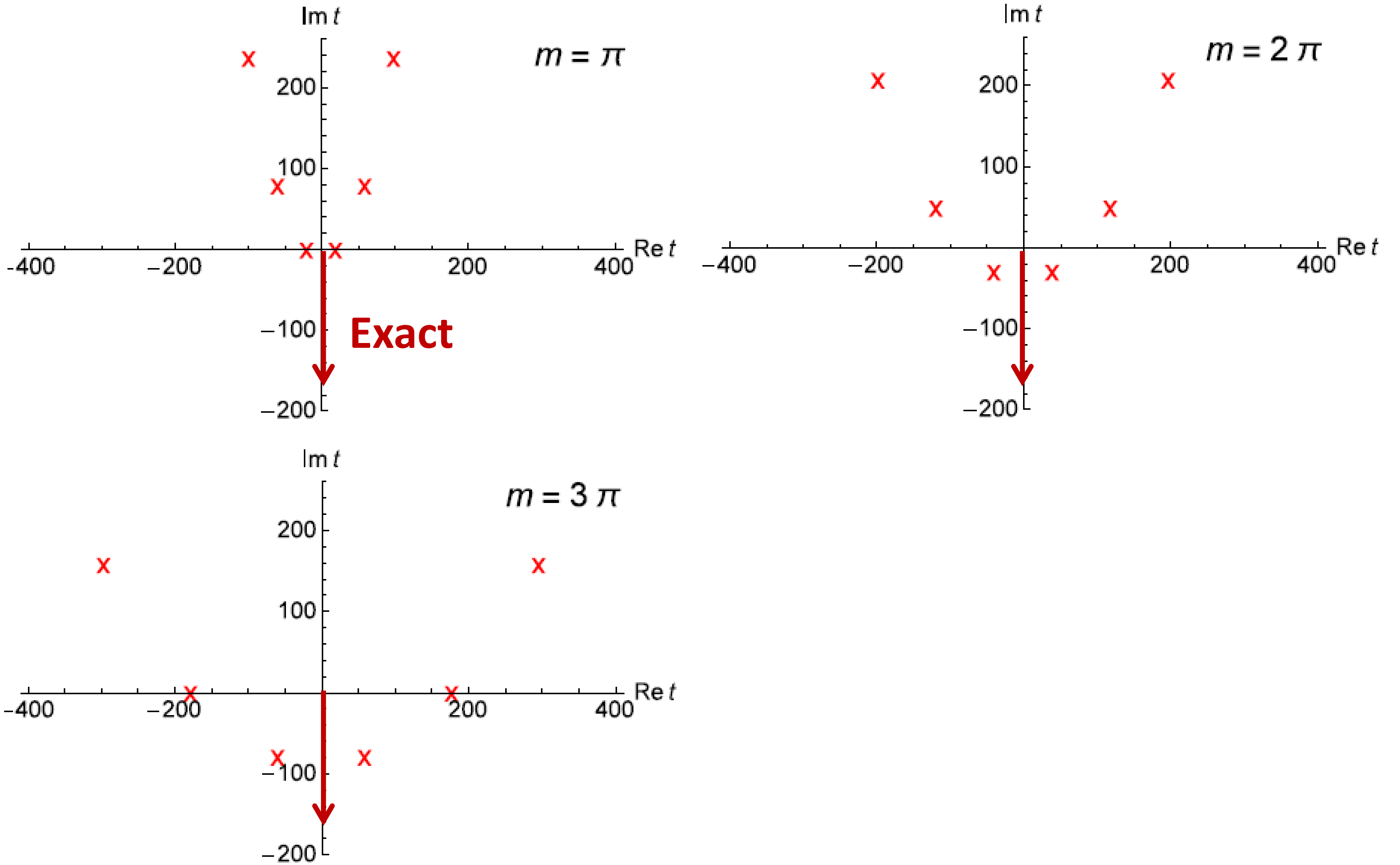

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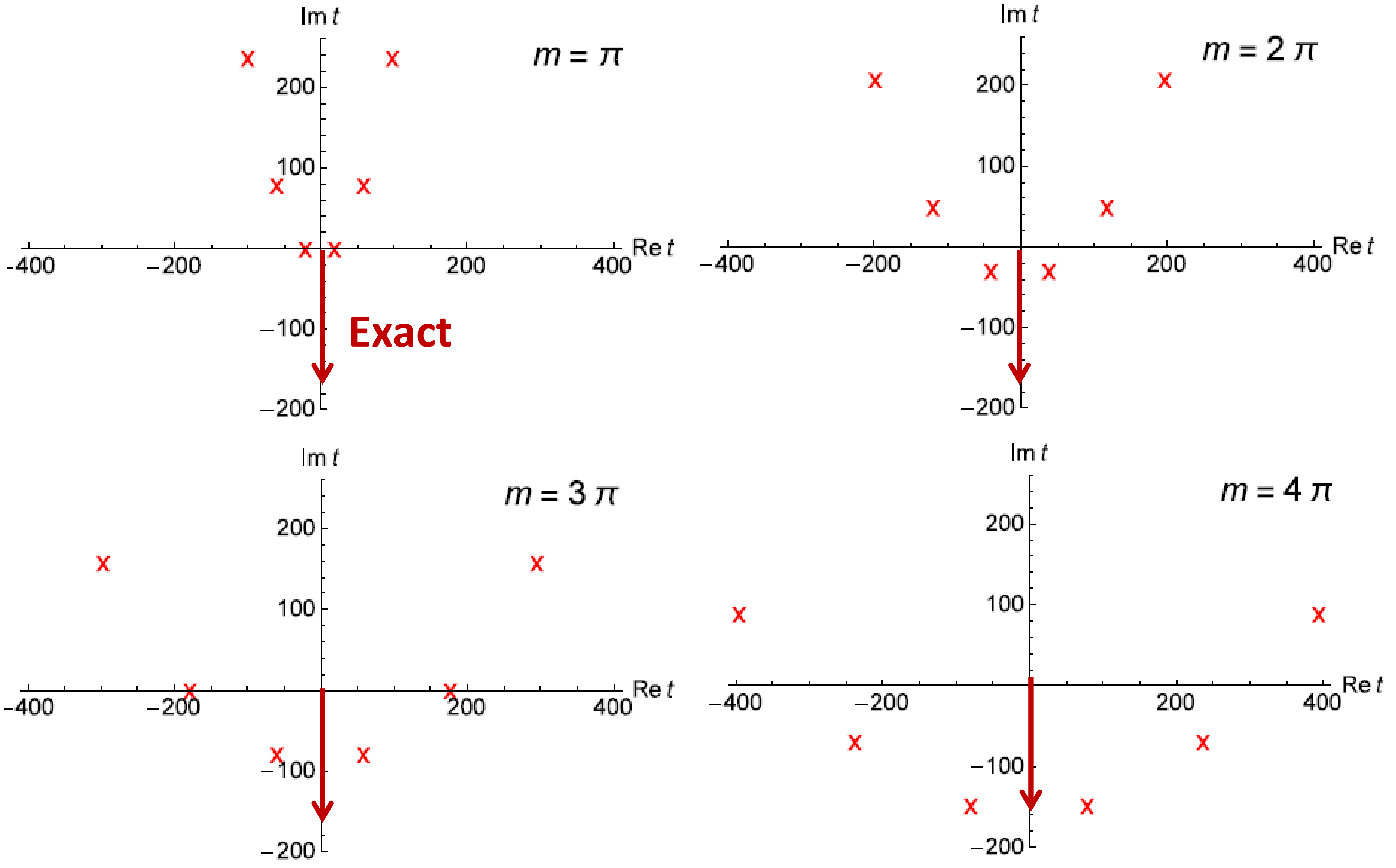

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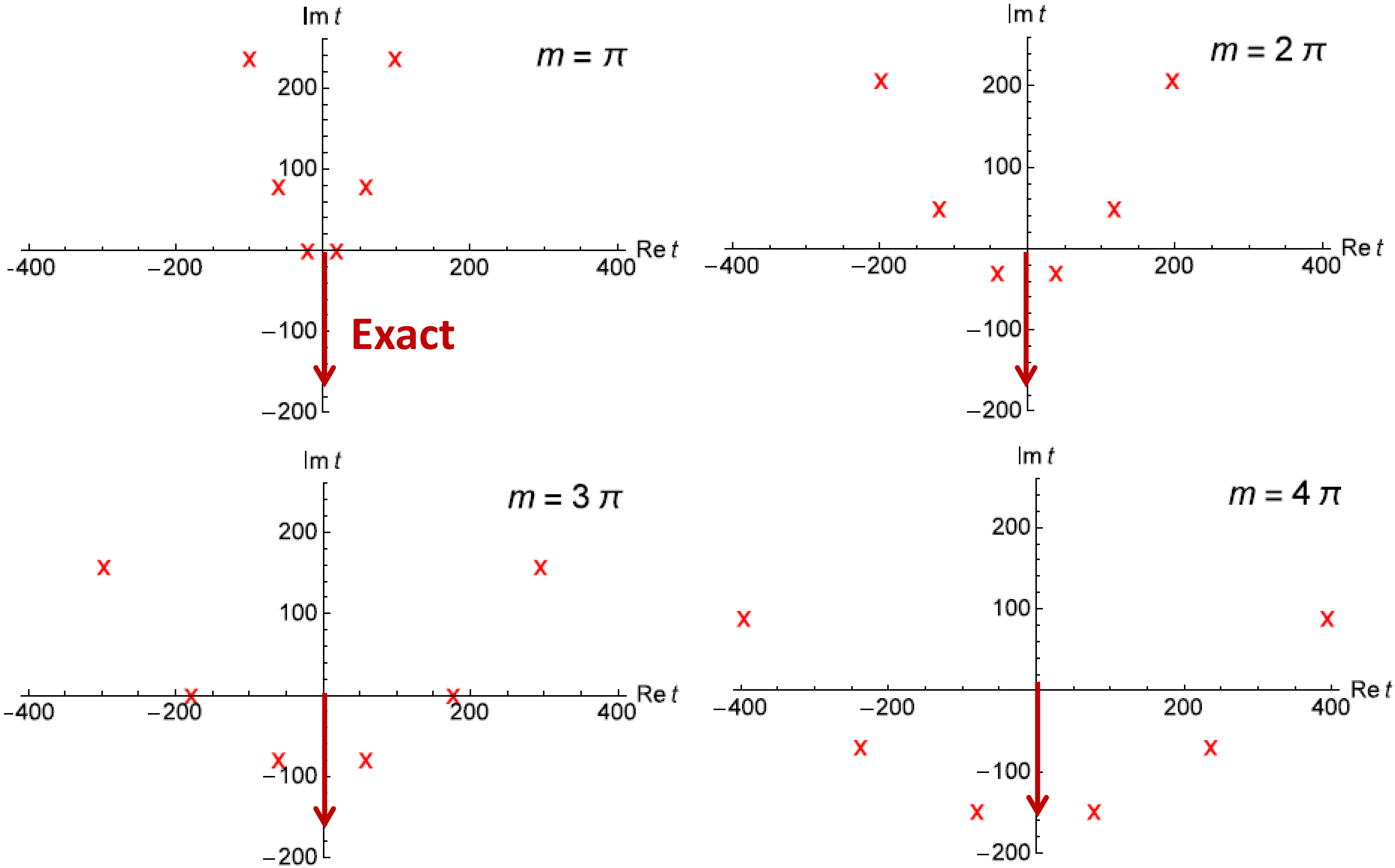
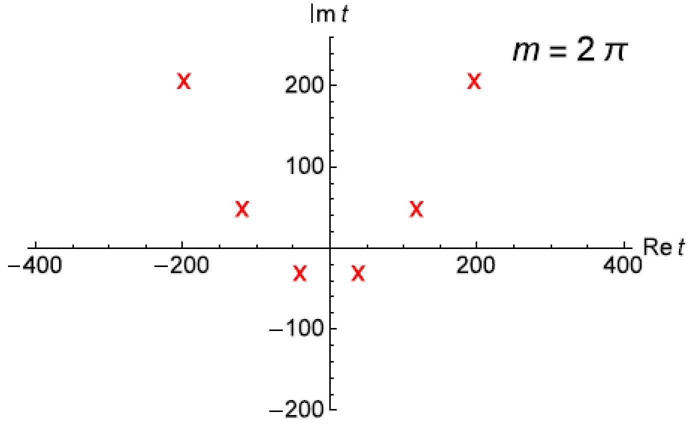


This is repeated infinitely many times…

Trans-series expression

In terms of Borel resum. along R+,

$$Z = \int_0^\infty dt\ e^{-\frac{t}{g}} \mathcal{B}Z(t) + \sum_{\text{poles} \in \text{4th quadrant}} \text{Res}_{t=t_{\text{pole}}} \left[e^{-\frac{t}{g}} \mathcal{B}Z(t) \right]$$



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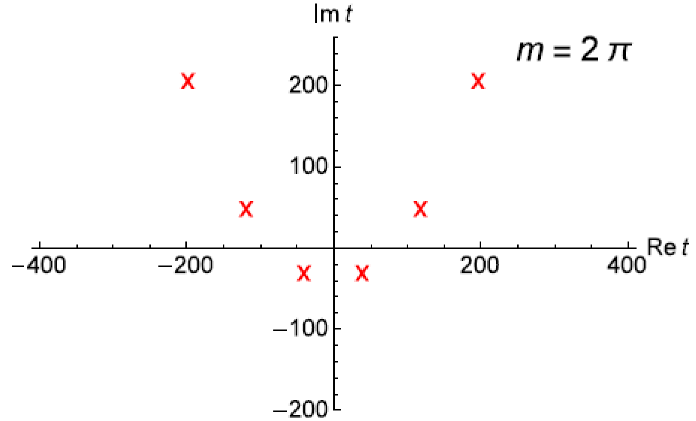

Decompose this into “perturbative part” & “non-pert. part”:

$$Z = Z_{\text{pt}} + \sum_{n=1}^{\infty} Z_{\text{np}}^{(n)} ,$$

Perturbative part:

$$Z_{\text{pt}} = \int_0^\infty dt\ e^{-\frac{t}{g}} \mathcal{B}Z(t),$$

$$\left(\frac{\sqrt{ig}}{2} \sum_{q=0}^{\infty} \sum_{a=0}^{\infty} \frac{E_{2(q+a)} \Gamma(q+1/2)}{2^{2(q+a)} \Gamma(2q+1) \Gamma(2a+1)} m^{2a} (ig)^q \right)$$

Non-pert. part:

$$Z_{\text{np}}^{(n)} = \theta\left(m - (2n-1)\pi\right) 2\pi (-1)^{n-1}\ e^{\frac{i}{g}[m+(2n-1)\pi i]^2}$$

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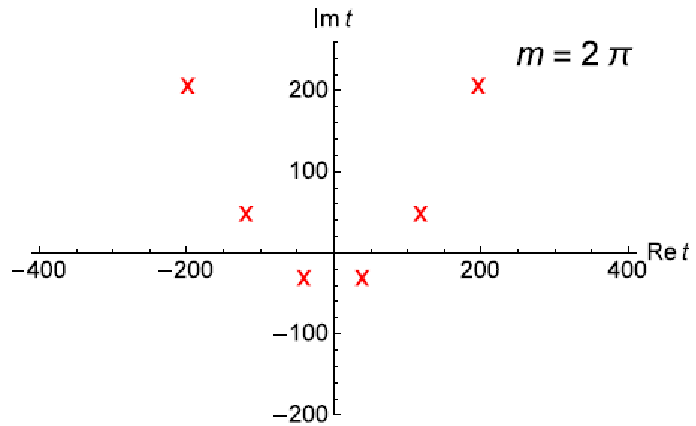

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Stokes phenomena!

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Trans-series expression (Cont'd)

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For m=(2n-1)π, perturbative part is ambiguous:

$$Z_{\rm pt}(g, m=(2n-1)\pi+0_+) - Z_{\rm pt}(g, m=(2n-1)\pi+0_-)$$

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Canceled! → Unambiguous answer

3d N=2 SUSY CS matter theory

- Borel transformation [MH '16]
- Resurgence structure [Fujimori-MH-Kamata-Misumi-Sakai '18]
- Lefschetz thimble decomposition
- Interpretation of Borel singularities [MH '17]

Application to the U(1) CS matter theory

$$Z = \int_{-\infty}^{\infty} d\sigma \ e^{-S[\sigma]}, \qquad S[\sigma] = -\frac{i}{g}\sigma^2 - \log \frac{1}{2\cosh\frac{\sigma - m}{2}}$$

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$$\left.\frac{\partial S[z]}{\partial z}\right|_{z=z^c} = -\frac{2i}{g} z^c + \frac{1}{2}\tanh\frac{z^c - m}{2} = 0\,.$$

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Flow equation:

$$\begin{cases} \left.\dfrac{dz}{ds}\right|_{\mathcal{J}_I} = \overline{\dfrac{\partial S[z]}{\partial z}} = +\dfrac{2i}{g}\bar{z} + \dfrac{1}{2}\tanh\dfrac{\bar{z} - m}{2} \, , \\ \lim\limits_{s \to -\infty} z(s) = z_I^c , \end{cases}$$

We solve these numerically but let us first understand weak coupling behavior analytically

Analytic argument for weak coupling

Critical point:

$$-\frac{2i}{g} z^c + \frac{1}{2} \tanh \frac{z^c - m}{2} = 0 \,. \quad \xrightarrow{g \to 0} \quad z^c \cosh \frac{z^c - m}{2} = 0$$

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$$\lim_{g \to 0} z^c(g,m) = 0, \ \underline{m + (2\ell + 1)\pi i}$$

pole of integrand

$$S[0] \simeq 0, \ S[m+(2\ell+1)\pi i] \simeq -\frac{i}{g}[m+(2\ell+1)\pi i]^2$$

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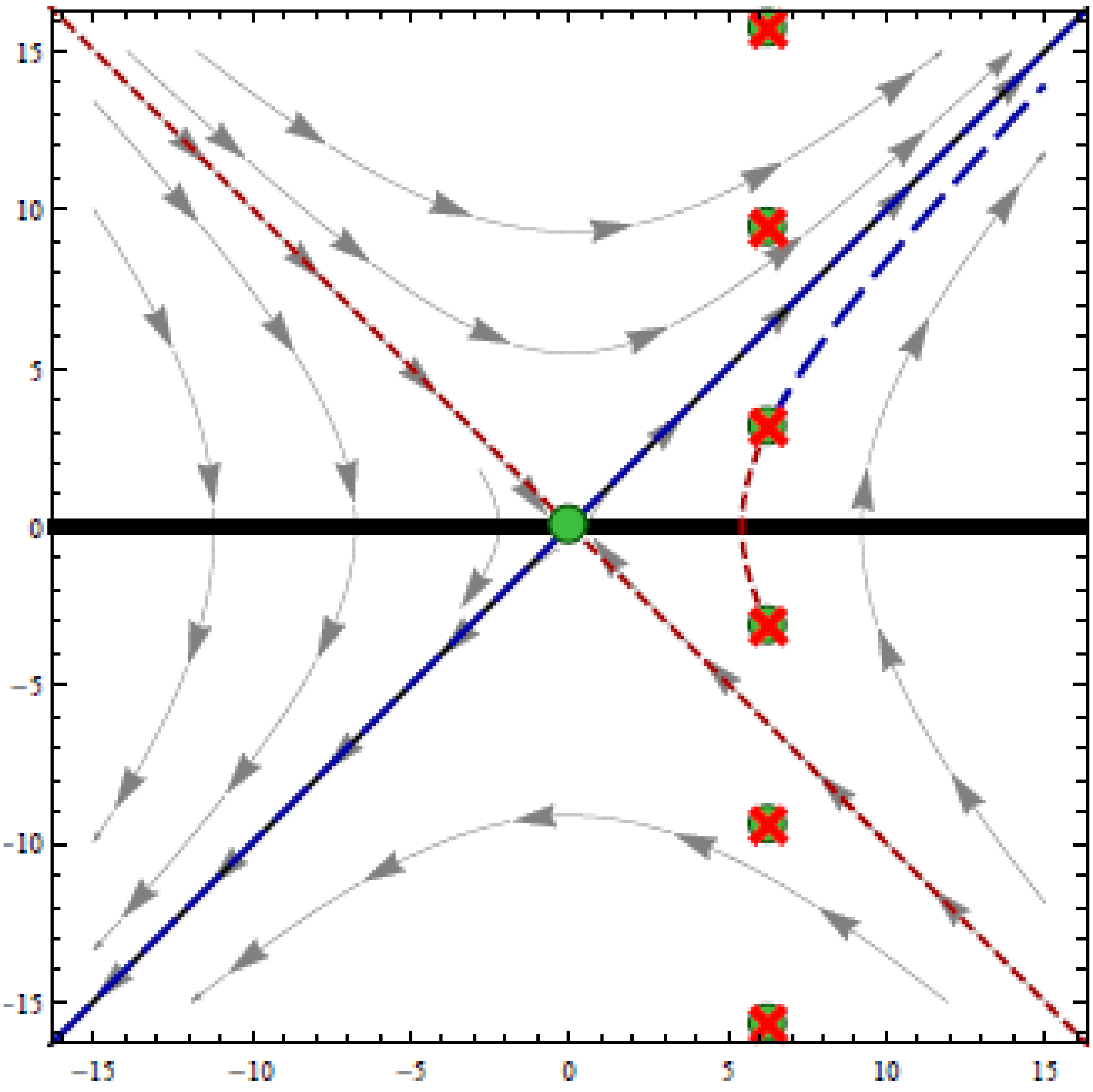
Lefschetz thimble associated w/ zc=0 :

$$\frac{dz}{ds} = +\frac{2i}{g}\bar{z} + \frac{1}{2}\tanh\frac{\bar{z} - m}{2} \quad \xrightarrow{g \to 0} \quad \frac{dz}{ds} = +\frac{2i}{g}\bar{z}$$

$$\lim_{g\to 0} z_{\text{pt}}(g,m;s) = \epsilon \exp\left(\frac{2}{g}s + \frac{\pi i}{4}\right)$$ (∼contour for Fresnel integral)

Numerical result for g=0.1 & m=2π



● : critical pt.

x : pole

line: steepest desc.

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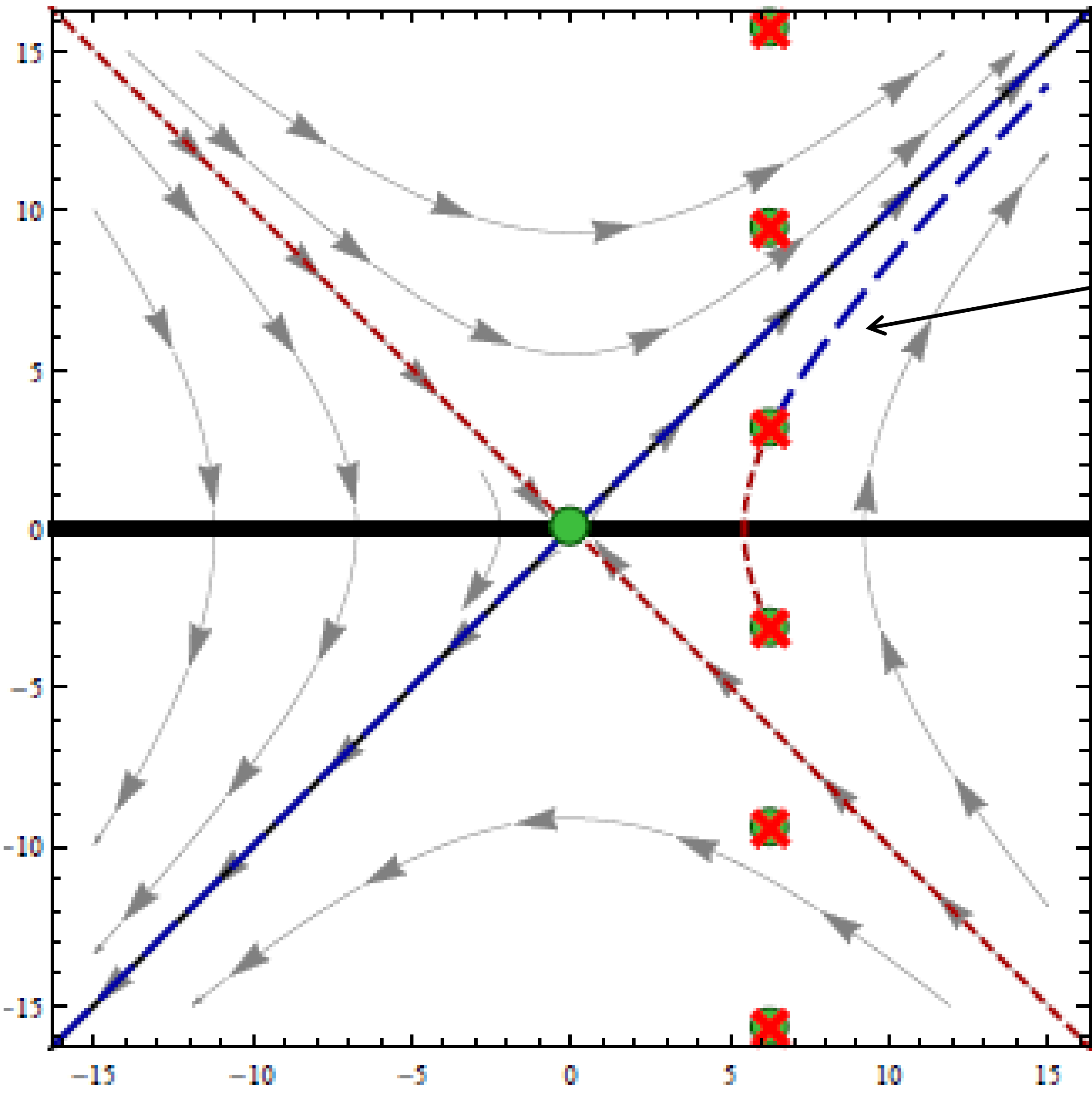

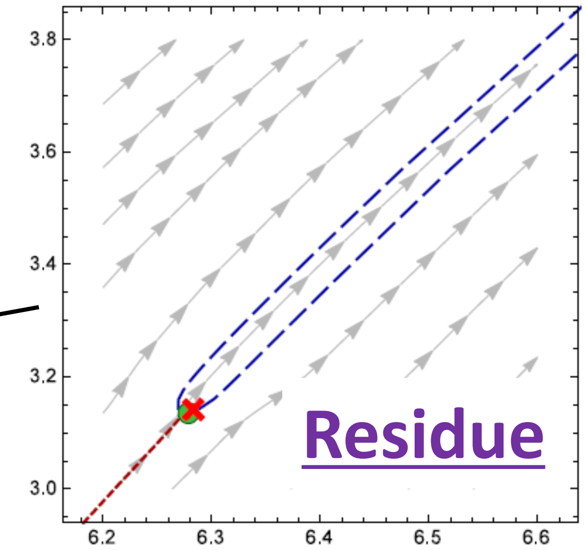

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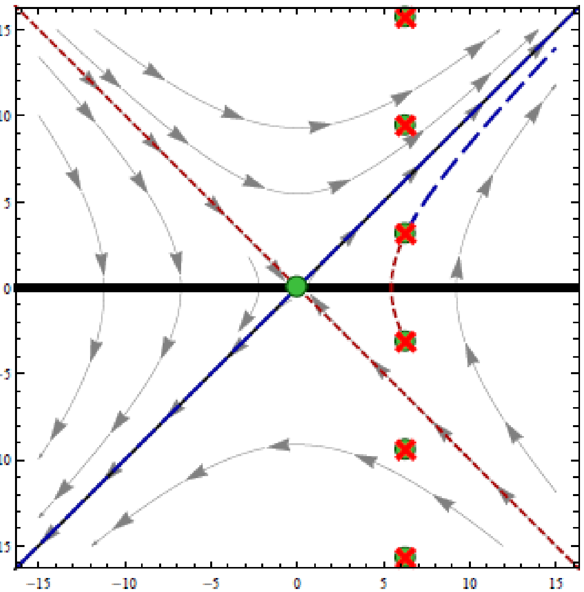

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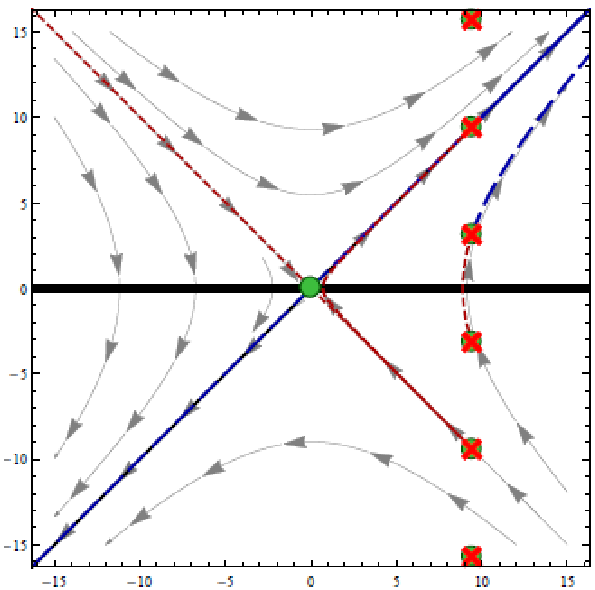

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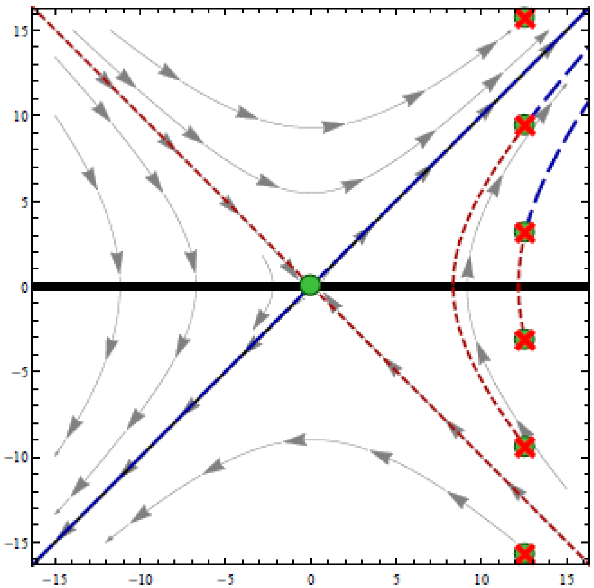

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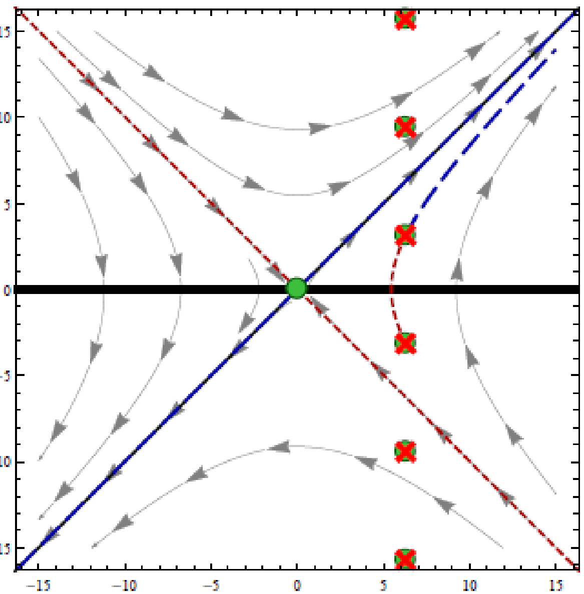

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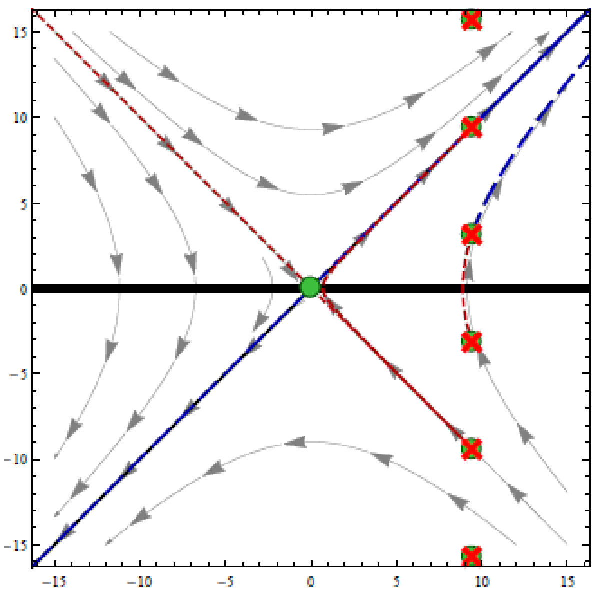

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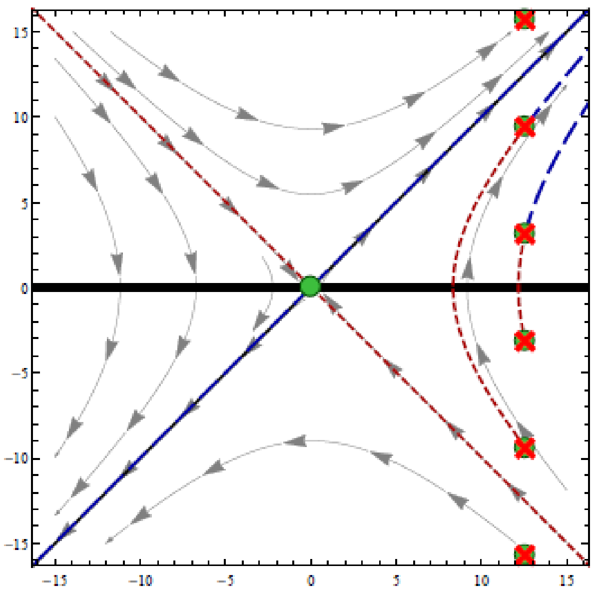

Stokes phenomena!

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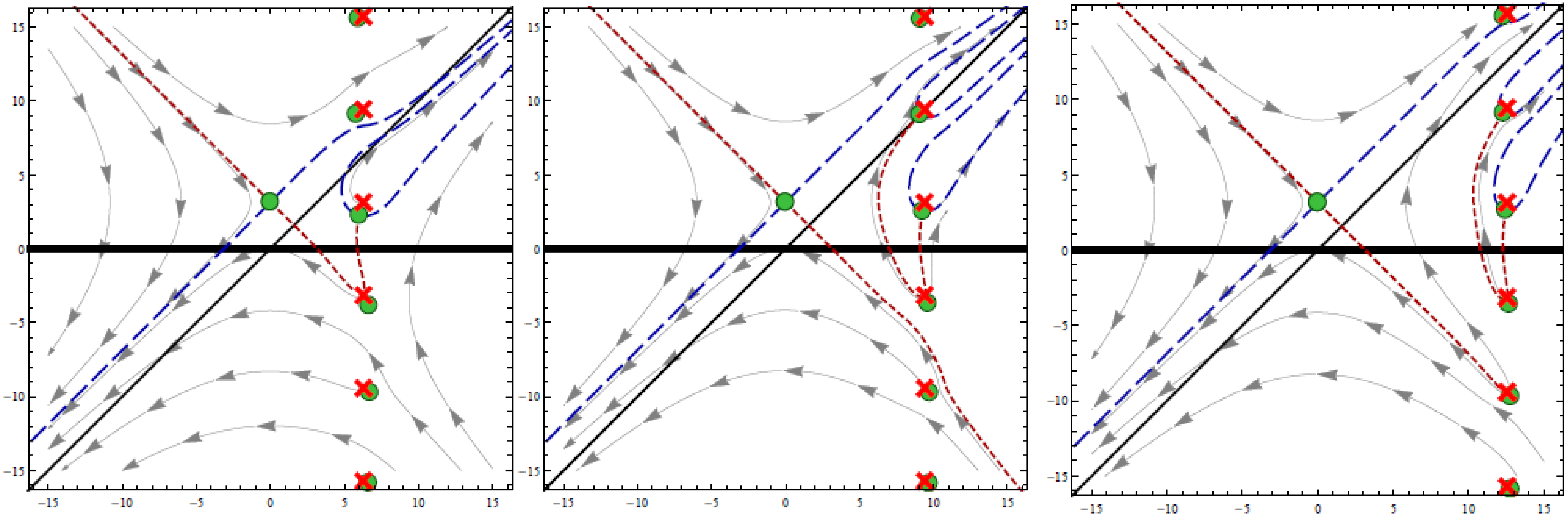

Thimble decomposition & Resurgent trans-series

Let us label the critical points by

$$\lim_{g\to 0} z^{\rm c}_{\rm pt}(g,m) = 0, \quad \lim_{g\to 0} z^{\rm c}_{\ell}(g,m) = m + (2\ell+1)\pi i.$$

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Small-g expansion

$$Z(g,m) = Z_{\mathrm{pt}}(g,m) + \sum_{\ell=0}^{\infty} \theta\left(m - (2\ell+1)\pi\right) \mathrm{Res}_{z=z^*_\ell}\left[e^{-S[z]}\right],$$

Resurgent trans-series

3d N=2 SUSY CS matter theory

- Borel transformation [MH '16]
- Resurgence structure [Fujimori-MH-Kamata-Misumi-Sakai '18]
- Lefschetz thimble decomposition
- Interpretation of Borel singularities [MH '17]
  = Complexified SUSY solutions

Interpretation of Borel singularities (3d)

[M.H. '17]

All the singularities can be explained by

Complexified SUSY Solutions

which are not on original contour of path integral
but formally satisfy SUSY conditions: $Q(\text{fields}) = 0$

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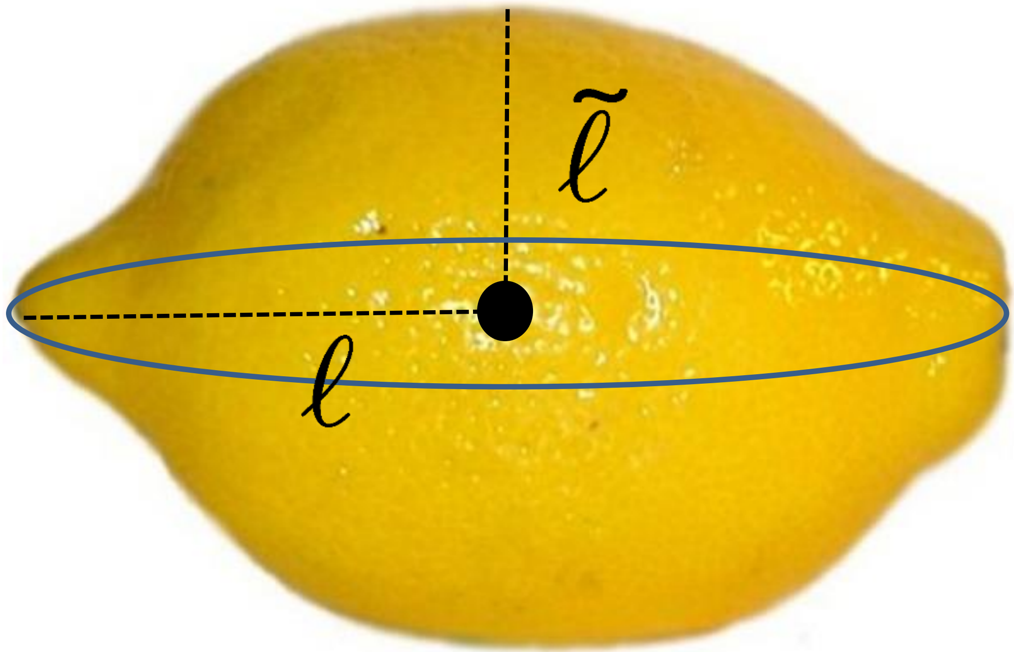
Proposal:

If there are n_B bosonic & n_F fermionic solutions with action $S=S_c/g$, then

$$(\text{Borel trans.}) \supset \prod_{\text{solutions}} \frac{1}{(t - S_c)^{n_B - n_F}}$$



S_b^3

$$b = \sqrt{\tilde{\ell}/\ell}$$

For a technical convenience,

we consider 3d N=2 theories on ellipsoid

(Round sphere corresponds to b=1)

Bosonic Complexified SUSY Solutions

Under the Coulomb branch solution (constant σ),

we look for solutions w/ $\psi = \bar{\psi} = F = \bar{F} = 0$

Nontrivial condition for scalar: $0 = Q\psi = -\gamma^{\mu}\epsilon D_{\mu}\phi - \epsilon\sigma\phi - \frac{i\Delta}{f(\vartheta)}\epsilon\phi$

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Useful eigenvalue problem:

[already solved in Hama-Hosomichi-Lee]

$$\begin{cases} \gamma^\mu \epsilon D_\mu \Phi + \epsilon\sigma\Phi + \frac{i\Delta}{f(\vartheta)}\epsilon\Phi = M\epsilon\Phi \\ M = M_{m,n} = \sigma + i\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta\right), \qquad m, n \in \mathbf{Z}_{\geq 0} \end{cases}$$

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SUSY condition is M=0 but this cannot be realized for $\sigma \in \mathbf{R}$

(=original path)

If we relax this, we have

$$\boxed{\sigma = -i\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta\right), \quad \phi = \Phi_{m,n}}$$

Fermionic Complexified SUSY Solutions

We look for solutions w/ $\phi = \bar{\phi} = F = \bar{F} = 0$

Nontrivial condition for fermion: $$\epsilon(-\gamma^{\mu} D_{\mu} + \sigma)\psi + \frac{i(2\Delta - 1)}{2f(\vartheta)}\epsilon\psi = 0.$$

Fermionic Complexified SUSY Solutions

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Useful eigenvalue problem:

[already solved in Hama-Hosomichi-Lee]

$$\begin{cases} \epsilon(-\gamma^\mu D_\mu \Psi + \sigma\Psi) + \frac{i(2\Delta - 1)}{2f(\vartheta)}\epsilon\Psi = M\epsilon\Psi \\ M = M_{m,n} = \sigma - i\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta - 2)}{2}\right), \qquad m, n \in \mathbf{Z}_{\geq 0} \end{cases}$$

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$$\sigma = i\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta - 2)}{2}\right), \; \psi = \Psi_{m,n}$$

Comparison w/ Borel trans.

For U(1) theory w/ charge q_a chiral multiplets,

$$\mathcal{B}Z_{S_b^3}(t) = \frac{1}{2\sqrt{-it}\prod_{a=1}^{N_f} s_b\left(q_a\sqrt{it} - \frac{iQ(1-\Delta_a)}{2}\right)}$$

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Locations of poles & zeroes:

$$t_{\rm pole}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta_a\right)^2,$$

$$t_{\rm zero}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta_a - 2)}{2}\right)^2$$

Comparison w/ Borel trans.

For U(1) theory w/ charge q_a chiral multiplets,

$$\mathcal{B}Z_{S_b^3}(t) = \frac{1}{2\sqrt{-it}\prod_{a=1}^{N_f} s_b\left(q_a\sqrt{it} - \frac{iQ(1-\Delta_a)}{2}\right)}$$

Locations of poles & zeroes:

$$t_{\rm pole}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta_a\right)^2,$$

$$t_{\rm zero}^{m,n} = -\frac{i}{q_a^2}\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta_a - 2)}{2}\right)^2$$

Actions of the solutions:

$$S_{\rm bos} = \frac{i\pi k}{q_a^2}\left(mb + nb^{-1} + \frac{b+b^{-1}}{2}\Delta_a\right)^2 = \frac{t_{\rm pole}^{m,n}}{g}$$

$$S_{\rm fer} = \frac{i\pi k}{q_a^2}\left(mb + nb^{-1} - \frac{(b+b^{-1})(\Delta_a - 2)}{2}\right)^2 = \frac{t_{\rm zero}^{m,n}}{g}$$

Remarks

- Degeneration of poles & zeroes in round sphere limit:

$$s_b(z) = \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \frac{mb + nb^{-1} + \frac{b+b^{-1}}{2} - iz}{mb + nb^{-1} + \frac{b+b^{-1}}{2} + iz} \quad \xrightarrow[b \to 1]{} \quad s_1(z) = \prod_{n=1}^{\infty} \left(\frac{n - iz}{n + iz} \right)^n$$

actions of solutions become degenerate

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- Contribution from hyper multiplet:

$$\frac{1}{s_1(z - i/2)\, s_1(-z - i/2)} = \frac{1}{2\cosh(\pi z)}$$

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- In the planar limit: N→∞, gN=fixed,

(actions)→∞ ⟹ Borel singularities →∞

consistent w/ expected convergence in the planar limit

Cheshire Cat Resurgence

SUSY QM w/ ~~SUSY~~ by non-perturbative effects:

$$E_0^{\text{pert}}(g) = 0, \qquad E_0^{\text{non-pert}}(g) \neq 0$$ (unambiguous)

$\exists$ non-perturbative corrections but each sector is analytic

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$^\exists$ non-perturbative corrections but each sector is analytic

Explicit ~~SUSY~~ deformations $\longrightarrow$ nontrivial resurgence structure

"Cheshire Cat Resurgence"

[Kozcaz-Sulejmanpasic-Tanizaki-Unsal '16, Dunne-Unsal '16]

$^\exists$ extensions to 2d & 3d SUSY QFTs for FI-parameter expansion

[Dorigoni-Glass '17,19]

Contents

Cases w/ IR renormalons

Are renormalons associated w/ semiclassical configurations?

If yes:

($^\exists$conjecture on semiclassical realization of IR renormalons
but $^\exists$negative report against the conjecture)

[Argyres-Unsal '12 , Morikawa-Takaura, Ashie-Morikawa-Suzuki-Takaura '20]

Cancellation of renormalon ambiguities should be understood in terms of Borel ambiguity and Stokes phenomena of thimbles:

$$\int_C D\phi\, \mathcal{O}(\phi)\, e^{-S[\phi]} = \sum_{I \in \mathrm{saddles}} n_I \int_{J_I} D\phi\, \mathcal{O}(\phi)\, e^{-S[\phi]}$$

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includes renormalons

If no:

Path integral interpretation sounds beyond current understanding

$$\int_C D\phi\, \mathcal{O}(\phi)\, e^{-S[\phi]} = \sum_{I \in \text{saddles}} n_I \int_{J_I} D\phi\, \mathcal{O}(\phi)\, e^{-S[\phi]}$$

doesn't include renormalons, where?

New renormalons?

There was a folklore that renormalons appear only in renormalizable QFTs

However, recently

$^{\exists}$renormalons in QM & super renormalizable QFTs

[Pazarbasi-Van Den Bleeken ’19, Marino-Reis ’19,’20]

What is renormalon? Should we go back to QM?

“Non-perturbative effect of non-perturbative effect”

$$\mathcal{O}(g) \simeq \sum_{\ell=0}^{\infty} c_\ell^{(0)} g^\ell + \sum_{I \in \text{saddles}} e^{-S_I(g)} \sum_{\ell=0}^{\infty} c_\ell^{(I)} g^\ell$$

We have focused on convergence here

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We have focused on convergence here

But is this convergent?

This may lead us to existence of $e^{-e^{\frac{1}{g}}}$

$^\exists$ proposal that QCD has this type of corrections

[Aitken-Cherman-Poppitz-Yaffe ’17]

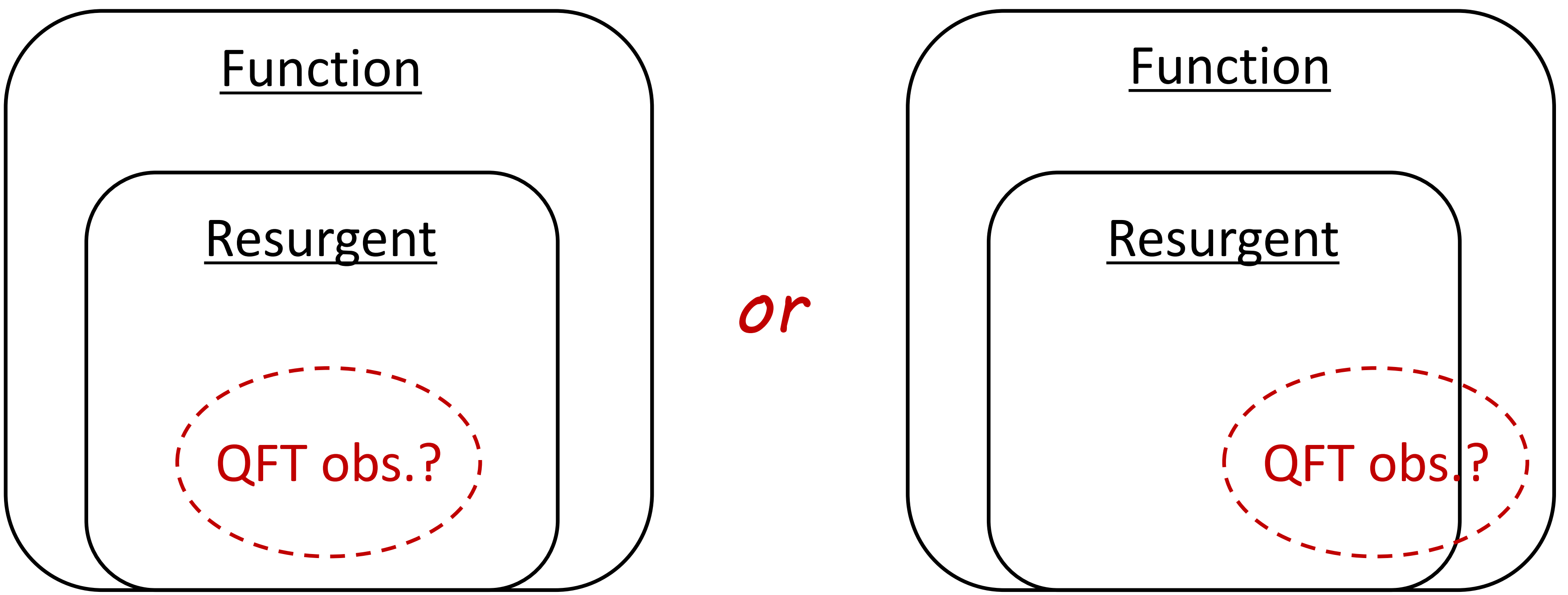
[cf. similar suspection in 1/N-expansion of SYK model :
Cotler-Gur-Ari-Hanada-Polchinski-Saad-Shenker-Stanford-Streicher-Tezuka]

Summary & Outlook

Q. Can we apply resurgence to QFT?

A. may or may not depend on setup.

At this moment, we don't know whether or not all observables in all Lagrangian QFTs are resurgent



What are “sufficient basis” to express QFT observables?

General function

$$\{g^\sharp,\ g^\sharp \log g,\ g^\sharp e^{-\frac{1}{g^\sharp}},\ g^\sharp e^{-\sharp e^{\frac{\sharp}{g}}}\}$$

QCD? [Aitken-Cherman-Poppitz-Yaffe ’17]

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$$\{g^\sharp,\ g^\sharp \log g,\ g^\sharp e^{-\frac{\sharp}{g}}\}$$

Successful examples of resurgence so far

(QM, 2d QFTs, 3d $\mathcal{N}=2$ CS matter on S^3 etc...)

Analytic function $\{g^n\}$

CFT in ’t Hooft limit

Outlook

- More successful examples wanted!
  (especially $d \geq 3$, non-topological, non-conformal, non-SUSY)
- Search of counter example?
- Revisit renormalons!
- Effect of renormalization [cf. 2d CP^N model: Fujimori-Kamata-Misumi-Nitta-Sakai '18]
- "Non-perturbative effect of non-perturbative effect"?
- Revisit your saddle point analysis in the past including complex saddles

Works in progress:

(Resurgence) x (phase transition, SUSY breaking, wall crossing, black hole etc…)

Thanks!