Magnetized T^4 orbifold model

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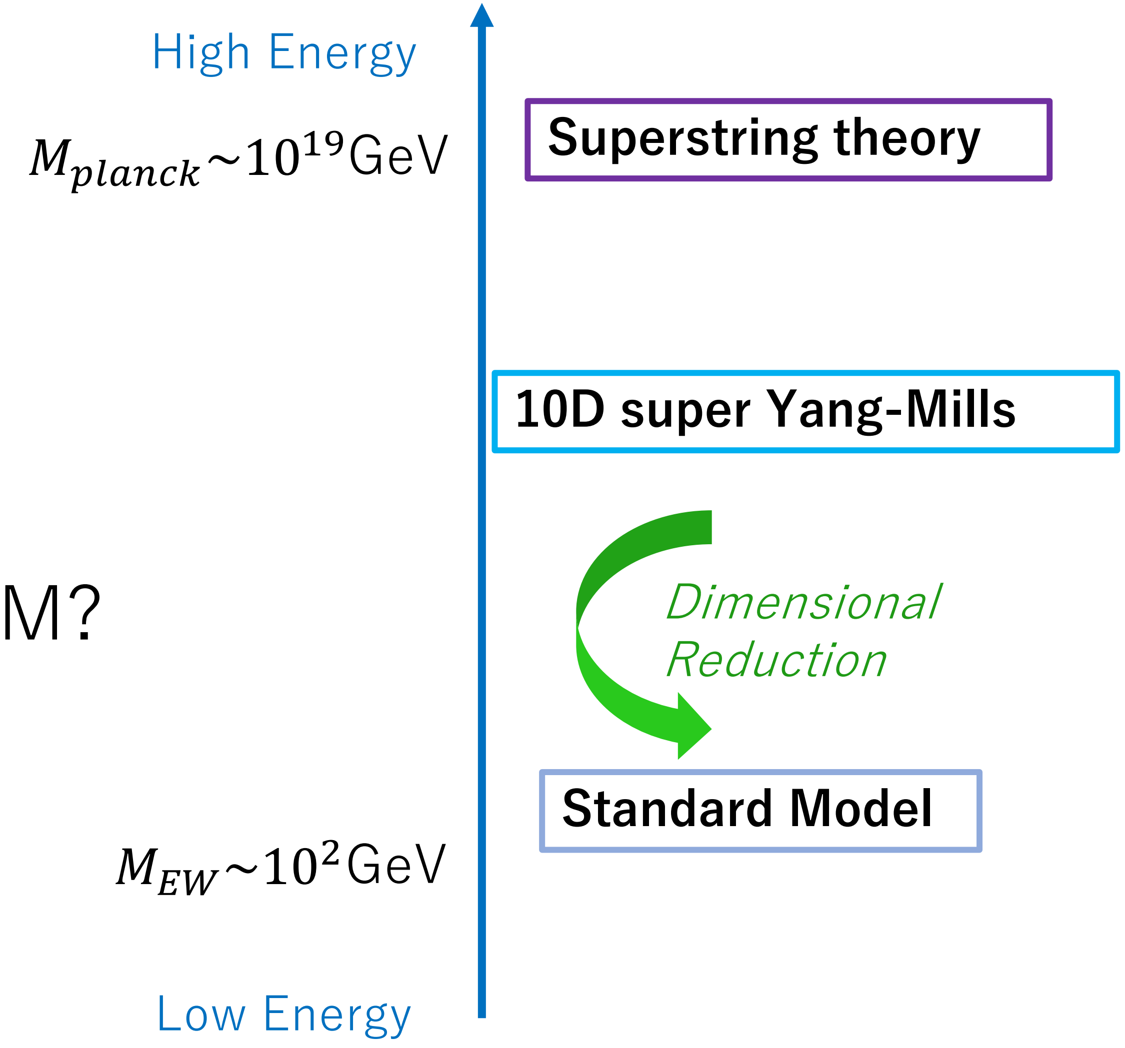
Introduction

- Puzzles in the Standard model(SM):
  - Three generations Main focus of this talk
  - Parameters (e.g. Yukawa couplings)

- What compactifications realize the SM?



10D, $\mathcal{N} = 1$ super Yang-Mills

We start with $$S_{10D} = \int d^4x \int d^6z \, Tr\left(-\frac{1}{4g^2} F^{MN} F_{MN} - \frac{i}{2g^2} \bar{\lambda} \Gamma^M D_M \lambda\right)$$

$(M, N = 0,1,2,3,4,..9)$ — $0,1,2,3$: $\mathbb{R}^{1,3}$; $4,..9$: 6D compact space

Decompose the 10D gaugino λ by energy eigenstates $\{\psi_n^j\}$ of the Dirac operator in the compact space

$$i\not{D}_6 \psi_n^j = m_n \psi_n^j$$

$$\lambda(x^\mu, z^m) = \sum_j^{(degeneracy)} \eta_0^j(x^\mu) \otimes \psi_0^j(z^m) \quad + \quad (K.K.modes, m_n > 0)$$

$\eta_0^j(x^\mu)$: **SM fermion** $\psi_0^j(z^m)$: **Zero-modes ($m_n = 0$)**

e.g. $e_L = \eta_0^{j=1}(x^\mu), \mu_L = \eta_0^{j=2}(x^\mu), \tau_L = \eta_0^{j=3}(x^\mu)$

#(Generations of fermions) = #(Degeneracy of the zero-modes)

Magnetized torus and orbifold compactification

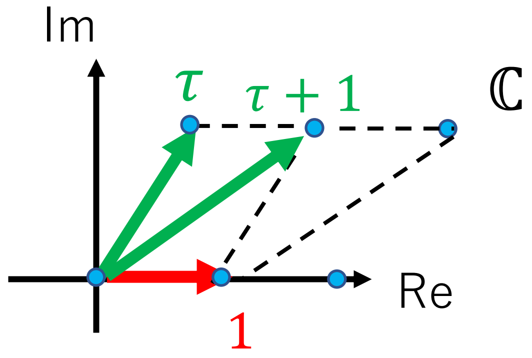
D. Cremades, L. E. Ibanez, and F. Marchesano (2004)

Why is it interesting ?

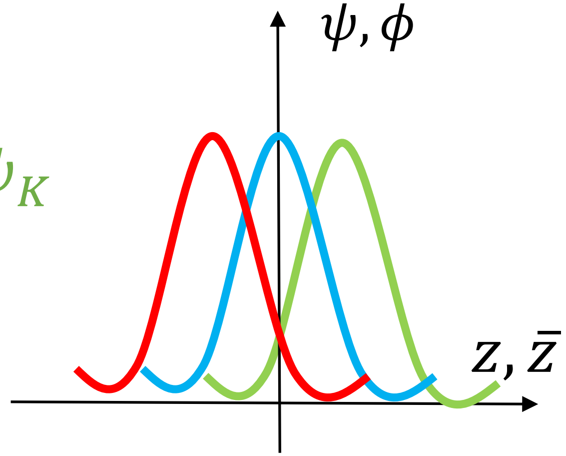
- Analytical computation
  - ➢ Zero-mode wavefunctions, ψ
  - ➢ Yukawa couplings
- Chiral 4D field theory
  - ➢ Background magnetic flux
- Modular symmetry
  - ➢ Phenomenologically interesting
  - ➢ Zero-mode number analysis in orbifold models

Study the conditions needed
to realize the 3 generations

e.g. T^2



$$Y_{IJK} \sim \int_{T^2} d^2 z\, \psi_I^{\dagger} \phi_{J,i} \Gamma^i \psi_K$$



Torus compactification with magnetic fluxes

-D. Cremades, L. E. Ibanez, and F. Marchesano (2004),
-T. Abe, Y. Fujimoto, T. Kobayashi,
T.Miura, K. Nishiwaki, M. Sakamoto (2014)
-T. Kobayashi, S. Nagamoto (2017)

Less General

- T^2 (or T^2/Z_N)  $\times$ $\mathcal{M}_4$: 4D compact space

3 degeneracies: 3 $\times$ 1

$$\psi = \begin{pmatrix} \psi^{j=1} \\ \psi^{j=2} \\ \psi^{j=3} \end{pmatrix} \otimes \phi$$

- T^4(or T^4/Z_N) $\times \mathcal{M}_2$

3 degeneracies: 3 $\times$ 1

How do we produce 3 degeneracy of zero-modes?

More General

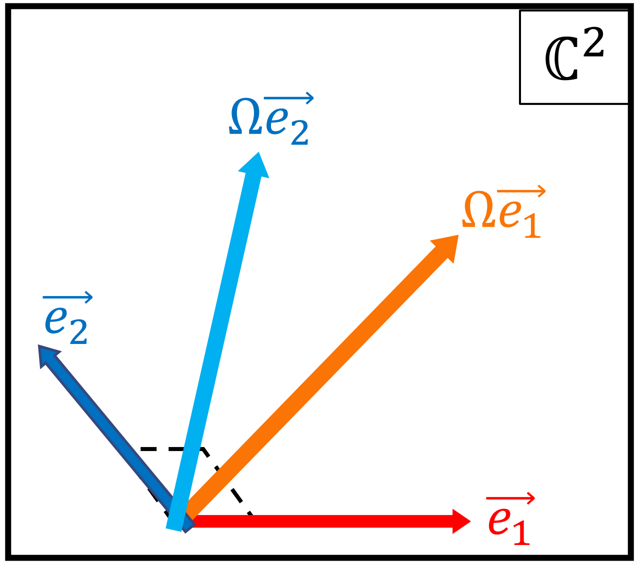
$T^4, T^4/Z_N$

- $T^4 \simeq \mathbb{C}^2/\Lambda$

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim \vec{z} + \vec{e_i} \sim \vec{z} + \Omega\vec{e_i}\,, \quad (i = 1,2).$$

➢Ω complex structure moduli
(symmetric complex 2×2 matrix),

$$\Omega = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$



- T^4/Z_N

$$\vec{z} \sim U_{twist}\vec{z}$$

➢U_{twist}: 2×2 unitary matrix satisfying $(U_{twist})^N = I_2$.

$N = 2,3,4,5,6,8,10,12$ are possible

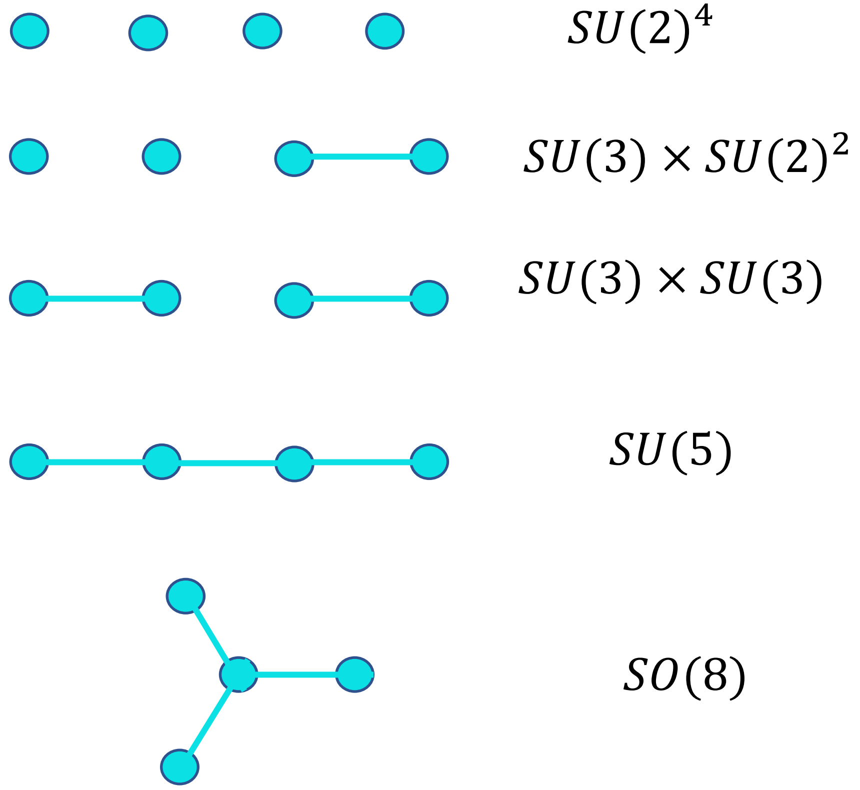
We have only T^2/Z_N,(N=2,3,4,6)

T^4/Z_N

- We have variety of orbifolds.

Orbifolds	Lattice
T^4/Z_2	Unfixed
T^4/Z_3	$SU(3) \times SU(3)$
T^4/Z_4	$SU(2)^4$
T^4/Z_5	$SU(5)$
T^4/Z_6	$SU(3) \times SU(3)$
T^4/Z_8	$SO(8)$
T^4/Z_{10}	$SU(5)$
T^4/Z_{12}	$SU(3) \times SU(2)^2$

Lattice shapes are fixed.



- First step is to analyze the number of zero-modes.

Zero-modes on magnetized T^4

- Uniform magnetic field on T^4:

I. Antoniadis, A. Kumar, and B. Panda' (2009)

$$F = \pi[M^T \cdot (\mathrm{Im}\Omega)^{-1}]_{ij}\left(i dz^i \wedge d\bar{z}^j\right), \qquad (i,j = 1,2)$$

  - (1,1)-form with $(M\Omega)^T = M\Omega$: 4D N=1 SUSY (F-term condition)
  - M denotes 2 × 2 integer matrix

Dirac quantization

- Covariant derivatives:

$$D_{z^i} = \partial_i - \frac{\pi}{2}([M\vec{\bar{z}}](\mathrm{Im}\Omega)^{-1})_i,$$
$$\bar{D}_{\bar{z}^i} = \bar{\partial}_i + \frac{\pi}{2}([M\vec{z}](\mathrm{Im}\Omega)^{-1})_i.$$

Zero-modes on magnetized T^4

- Fermion zero-modes satisfy the Dirac equation, $i\not{D}_4\Psi = 0$,

D. Cremades, L. E. Ibanez, and F. Marchesano (2004)
I. Antoniadis, A. Kumar, and B. Panda' (2009)

$$\Psi(\vec{z},\bar{\vec{z}}) = \begin{pmatrix} \psi_+^1 \\ \psi_-^2 \\ \psi_-^1 \\ \psi_+^2 \end{pmatrix}.$$

Positive Chirality (ψ_+^1, ψ_+^2)

Negative Chirality (ψ_-^2, ψ_-^1)

Riemann theta function

- Zero-mode wavefunctions:
  - If $\mathrm{Im}(M\Omega) > 0$:

$$\psi_M^{\vec{J}} = \mathcal{N}_{\vec{J}} \cdot e^{\pi i [M\vec{z}]^T \cdot (\mathrm{Im}\Omega)^{-1} \cdot \mathrm{Im}\vec{z}} \cdot \vartheta \begin{bmatrix} \vec{J}^T M^{-1} \\ 0 \end{bmatrix} (M\vec{z}, M\Omega),$$

$$\Psi = \begin{pmatrix} < \psi_{T4}^0, \ldots, \psi_{T4}^{detM-1} > \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

***detM* degenerated zero-modes on T^4**

$$\psi_+^1 = \sum_{j=0}^{detM-1} a_j \psi_{T4}^j$$

a_j: *arbitrary*

Zero-modes on magnetized T^4/Z_N

- Zero-mode wavefunctions on T^4/Z_N satisfy

$$\varphi_{T^4/Z_N}(U_{twist}\vec{z}) = \exp(2\pi i k/N)\varphi_{T^4/Z_N}(\vec{z})$$

  - $(k = 0,1,\ldots,N-1)$, N different sectors with different eigenvalues
  - Zero-modes on T^4/Z_N are given by the linear combinations of $\psi_{T^4}^{j}$:

Basis which diagonalize the twist, U_{twist}

Modular Transformation is very useful !

Solution space of the T^4 model

$$< \psi_{T4}^{0}, \ldots, \psi_{T4}^{detM-1} >$$

dim=$detM$

e.g.T^4/Z_4

Orbifolding by $U_{twist} = iI_2$

$\sum_{j=0}^{detM-1} b_j^{(+1)}\psi_{T4}^{j}$ **+1**	$\sum_{j=0}^{detM-1} b_j^{(+i)}\psi_{T4}^{j}$ **+i**
-1 $\sum_{j=0}^{detM-1} b_j^{(-1)}\psi_{T4}^{j}$	**-i** $\sum_{j=0}^{detM-1} b_j^{(-1)}\psi_{T4}^{j}$

Modular transformation

- Change of basis vectors defining the torus:
  - ➢In T^4 case, we consider,

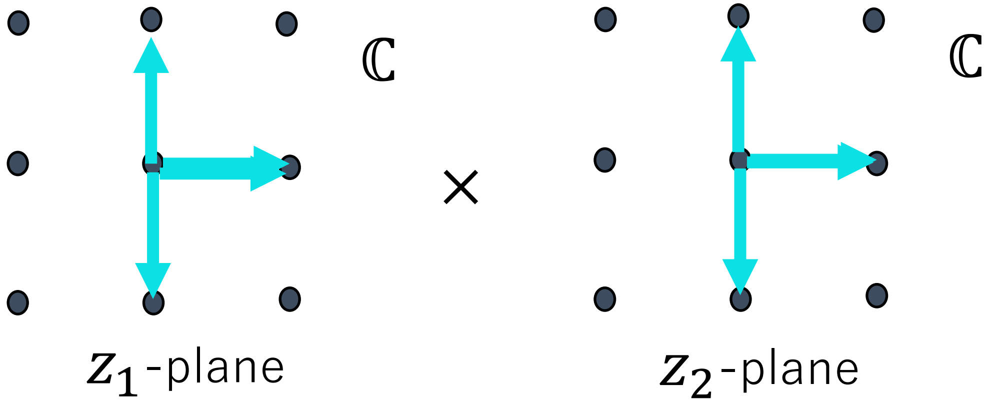
$$Sp(4,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} A_{2\times2} & B_{2\times2} \\ C_{2\times2} & D_{2\times2} \end{pmatrix} \middle| \gamma^T J \gamma = J \right\}, \text{ where } J = \begin{pmatrix} 0 & I_{2\times2} \\ -I_{2\times2} & 0 \end{pmatrix}.$$

  - ➢We have 4 generators, $S, T_i (i = 1,2,3)$
- Modular transformation can reproduce the Z_N twist.

e.g. T^4/Z_4

Z_4-twist is realized by the S transformation

$$\vec{z} \longrightarrow i\vec{z}, \quad (U_{twist} = iI_2)$$



Zero-modes on magnetized T^4/Z_N

- Behavior of $\psi_{T^4}^{j}$ under the modular transformation:

e.g. S trans.(Z_4-twist)

$$\psi_{T^4}^{\vec{J}}(i\vec{z}, iI_2) = \frac{1}{\sqrt{\det M}} \sum_{\vec{K}\in\Lambda_M} \exp(2\pi i \vec{J}^T M^{-1} \vec{K})\, \psi_{T^4}^{\vec{K}}(\vec{z}, iI_2)$$

$= \rho(S)$: Representation of S on $< \psi_{T4}^{0}, \ldots, \psi_{T4}^{detM-1} >$

- Zero-modes on T^4/Z_4 can be obtained by analyzing the eigensystem of $\rho(S)$.

$detM$	conditions	$+1$	-1	$+i$	$-i$
1		1	0	0	0
2		1	1	0	0
3	$(\frac{n_i}{3})_L = 1,\ (i = 1 \text{ or } 2)$	1	1	1	0
3	$(\frac{n_i}{3})_L = -1,\ (i = 1 \text{ or } 2)$	1	1	0	1
4	$\gcd(n_i, m) = 1,\ (i = 1 \text{ or } 2)$	2	1	1	0
4	$\gcd(n_i, m) = 2,\ (i = 1 \text{ and } 2)$	2	2	0	0
5	$(\frac{n_i}{5})_L = 1,\ (i = 1 \text{ or } 2)$	2	1	1	1
5	$(\frac{n_i}{5})_L = -1,\ (i = 1 \text{ or } 2)$	1	2	1	1
6		2	2	1	1
7		2	2	2	1
8	$n_i \overset{\text{mod}8}{\equiv} 1,\ (i = 1 \text{ or } 2)$	3	2	2	1
8	$n_i \overset{\text{mod}8}{\equiv} 3,\ (i = 1 \text{ or } 2)$	2	3	2	1
8	$\gcd(n_i, m) \overset{\text{mod}2}{\equiv} 0,\ \ (i = 1 \text{ and } 2)$	3	3	1	1
9	$\gcd(n_i, m) = 1,\ \ (i = 1 \text{ or } 2)$	3	2	2	2
9	$\gcd(n_i, m) = 3,\ \ (i = 1 \text{ and } 2)$	2	3	2	2
10		3	3	2	2
11	$(\frac{n_i}{11})_L = 1,\ (i = 1 \text{ or } 2)$	3	3	3	2
11	$(\frac{n_i}{11})_L = -1,\ (i = 1 \text{ or } 2)$	3	3	2	3
12	$n_i \overset{\text{mod}4}{\equiv} 1,\ (i = 1 \text{ or } 2)$	4	3	3	2
12	$n_i \overset{\text{mod}4}{\equiv} -1,\ (i = 1 \text{ or } 2)$	3	4	3	2
12	$n_i \overset{\text{mod}4}{\equiv} 2,\ (i = 1 \text{ or } 2)$	4	4	2	2
12	$n_i \overset{\text{mod}4}{\equiv} 0,\ (i = 1 \text{ and } 2)$	4	4	3	1
13	$(\frac{n_i}{13})_L = 1,\ (i = 1 \text{ or } 2)$	4	3	3	3
13	$(\frac{n_i}{13})_L = -1,\ (i = 1 \text{ or } 2)$	3	4	3	3
14		4	4	3	3
15		4	4	4	3
16	$\gcd(n_i, m) = 1,\ \ (i = 1 \text{ or } 2)$	5	4	4	3
16	$\gcd(n_i, m) = 2,\ \ (i = 1 \text{ or } 2)$	5	5	3	3

2×2 symmetric integer matrix:

$$M = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix}$$

In T^4/Z_4, 3 generation models are only found when $8 \le \det M \le 16$.

Negative-chirality modes

- Non-holomorphic transformations in the compact space relate different components of the spinor.

Positive Chirality

Negative Chirality

$$\Psi(\vec{z},\bar{\vec{z}}) = \begin{pmatrix} \psi_+^1 \\ \psi_-^2 \\ \psi_-^1 \\ \psi_+^2 \end{pmatrix}$$

ψ_+^1: $\mathrm{Im}(M\Omega) > 0$

ψ_-^2, ψ_-^1: $\mathrm{Im}(M\Omega)$ is indefinite

ψ_+^2: $\mathrm{Im}(M\Omega) < 0$

P transformation

e.g.

Exchange of real parts:

$$\mathrm{Re}z_1 = \mathrm{Re}z_2^{(p)},$$
$$\mathrm{Re}z_2 = \mathrm{Re}z_1^{(p)},$$
$$\mathrm{Im}z_i = \mathrm{Im}z_i^{(p)}, \quad (i = 1, 2),$$

Summary

- #(generations) = #(degeneracy of zero-modes)
- Zero-modes on magnetized T^4/Z_N
- Conditions to realize the 3 generation and the use of modular transformation.
- P, CP transformation in the compact space

Future work:

- Extension to T^6/Z_N
- More studies in P and CP
- Yukawa couplings

Background magnetic flux on T^4

$4C2 = 6$ real d.o.f. $x^i, y^j,\ (i,j = 1,2)$: T^4real coordinates

$\vec{z} = \vec{x} + \Omega\vec{y}$

$$F = \underbrace{p_{x^1x^2}dx^1 \wedge dx^2 + p_{y^1y^2}dy^1 \wedge dy^2}_{\to 0} + p_{x^iy^j}dx^i \wedge dy^j$$

I. Antoniadis, A. Kumar, and B. Panda' (2009)

Then we have

$$F = p_{x^iy^j}dx^i \wedge dy^j$$

4 real d.o,f.

$$p_{x^1y^1}, p_{x^1y^2}, p_{x^2y^1}, p_{x^2y^2}$$

Background magnetic flux on T^4

$$\vec{z} = \vec{x} + \Omega\vec{y}$$

$$F = p_{x^i y^j} dx^i \wedge dy^j$$
$$= F_{z^i z^j} dz^i \wedge dz^j + F_{\bar{z}^i \bar{z}^j} d\bar{z}^i \wedge d\bar{z}^j + F_{z^i \bar{z}^j}\left(i dz^i \wedge d\bar{z}^j\right)$$

F-term condition,

$$F_{z^i z^j} = F_{\bar{z}^i \bar{z}^j} = 0$$

$$F = F_{z^i \bar{z}^j}\left(i dz^i \wedge d\bar{z}^j\right) = \frac{1}{2}[p_{xy} \cdot (\mathrm{Im}\Omega)^{-1}]_{ij} \ \left(i dz^i \wedge d\bar{z}^j\right)$$

Dirac quantization:

$$p_{xy} = 2\pi M^T$$

M: 2×2 integer matrix

$$F = \pi[M^T \cdot (\mathrm{Im}\Omega)^{-1}]_{ij}\left(i dz^i \wedge d\bar{z}^j\right)$$

Algebraic relations

$$S^4 = I_4,$$
$$(ST_1T_2)^3 = I_4,$$
$$(ST_1T_3)^5 = I_4,\ (ST_2T_3)^5 = I_4,$$
$$(ST_3)^6 = I_4,\ (ST_1T_2^{-1})^6 = I_4,\ ((T_1T_2)^{-1}S)^6 = I_4,\ (ST_1T_2\gamma_P)^6 = I_4,$$
$$(ST_1T_2^{-1}T_3^{-1})^8 = I_4,$$
$$((T_1T_3)^{-1}S)^{10} = I_4,\ ((T_2T_3)^{-1}S)^{10} = I_4,$$
$$(ST_1)^{12} = I_4,\ (ST_2)^{12} = I_4,$$

Complex structure moduli of orbifolds

e.g.

- T^4/Z_3: $(ST_1\ \ T_2\ \ )^3 = I_4$
  - ST_1T_2-invariant moduli

$$\Omega_{(ST_1T_2)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}.$$

- T^4/Z_5: $(ST_1\ \ T_3\ \ )^5 = I_4$
  - ST_1T_3-invariant moduli

$$\Omega_{(ST_1T_3)} = \begin{pmatrix} -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} & -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} \\ -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} & i\sqrt{\frac{5+\sqrt{5}}{10}} \end{pmatrix}.$$