Swampland program from amplitudes: gravitational positivity bounds

Junsei Tokuda (IBS, CTPU/ Kobe U.)

Based on:

[JHEP11(2020)054 **JT**, K. Aoki, S. Hirano]

[PRL127,091602(2021), K. Aoki, T.Q. Loc. T. Noumi, **JT**]

[PRD104,066022(2021) T. Noumi, **JT**]

[arXiv: 2205.12835 T. Noumi, S. Sato, **JT**]

2022.12.01 KEK-PH2022 Workshop@KEK

Introduction

- I think it would be very exciting if we can verify

What is **the quantum gravity theory?**

- **We need predictions which can be tested** experimentally.

- **Very difficult.** $M_{\mathrm{pl}} \sim 10^{18}$ GeV: Very high.
  - Phenomenology (@ $E \ll M_{\mathrm{pl}}$) → QFT coupled to gravity in 4D. These models = effective field theories (EFTs).

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  - ➢ Phenomenology (@ $E \ll M_{\mathrm{pl}}$) → QFT coupled to gravity in 4D. These models = effective field theories (EFTs).

- But, **hidden predictions may exist.**→ **Swampland program:**

[C. Vafa ('05)]

"Not all consistent-looking EFTs are consistent with quantum gravity."

* Inconsistent EFTs are said to be in the Swampland.

Hidden predictions exist! (without gravity)

- S-matrix **Unitarity** has been useful for finding new physics.

W-boson scattering → Unitarity predicts (Higgs mass) $\lessapprox 1\ \mathrm{TeV}$

[Lee–Quigg–Thacker ('77)]

Euler-Heisenberg: $\mathcal{L} \sim -F^2 + \frac{c_2}{m_e^4} F^4 + \cdots$

→ Unitarity requires new physics below $E \sim m_e$.

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- S-matrix **Unitarity** + **Analyticity** etc.

→ More information on UV theory : "**Positivity bounds**"

[Adams – Arkani-Hamed – Dubovsky – Nicolis – Rattazzi ('06)]

[Pham –Truong ('85)]

e.g.) $\mathcal{L} \sim -F^2 + \frac{c_2}{m_e^4} F^4 + \cdots$ → $c_2 > 0$

"Hidden prediction" of UV completion.

Introduction

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  § **I. Formulation** of gravitational positivity bounds. (7 pages)

  § **II. Implication** (1 page)

  ✓ Bounds on **renormalizable** interactions!

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  ✓ Bounds on **renormalizable** interactions!

(My opinion): ***Not well established yet, but very interesting !***

§ I. Formulation

1. Positivity bounds without gravity (Review)

2. Gravitational positivity bounds

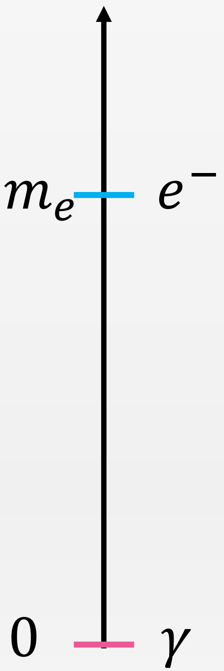
§ II. Implications

Positivity bound without gravity (1/3)

- Let's consider $\gamma\gamma \to \gamma\gamma$ amplitude $\mathcal{M}(s,t)$. $\quad {}^{*}\, s \sim (\mathrm{CM\ energy})^2$
- UV complete theory: Local, Unitary, Lorentz invariant, Causal.

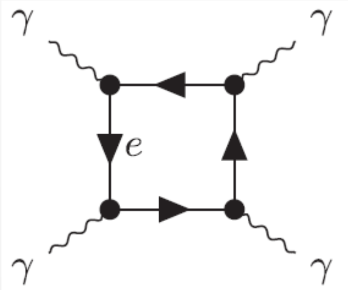
➡ $\boldsymbol{\mathcal{M}(s,t)}$ **behaves well** at high energies.

mass



QED

$$\mathcal{L}_{\mathrm{QED}}[\gamma, e^-]$$



$$\sim \alpha^2 \log^2(m_e^2/s)$$

Euler-Heisenberg

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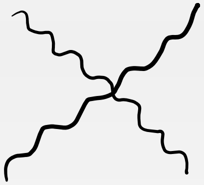

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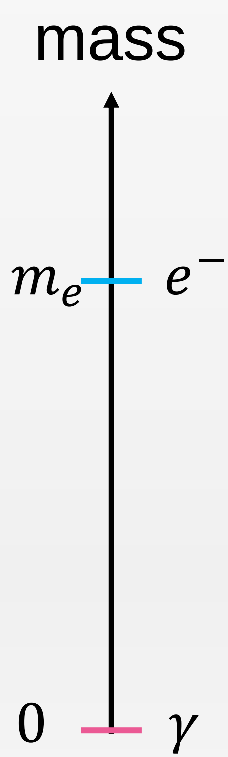

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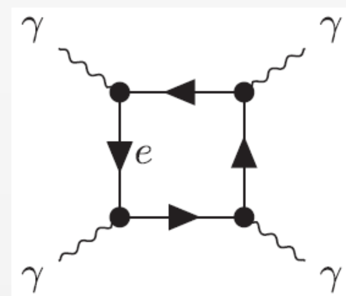

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Embedding is possible
if and only if $\boldsymbol{c_2 > 0}$.

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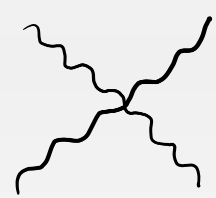

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Positivity bound without gravity (2/3)

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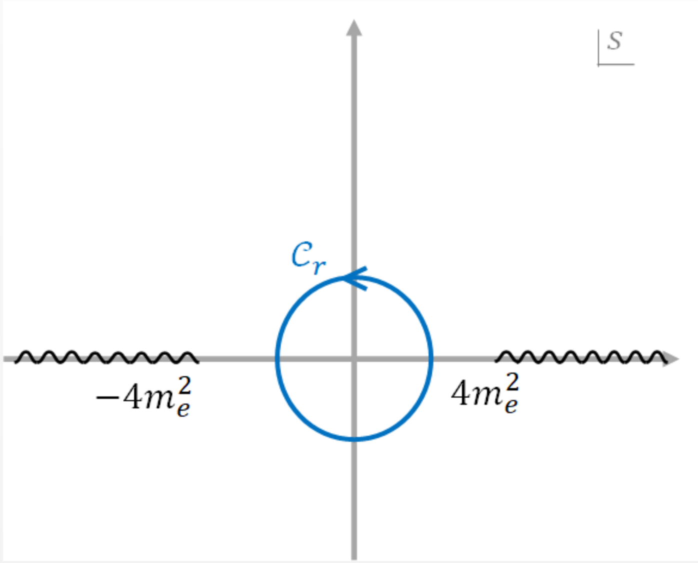

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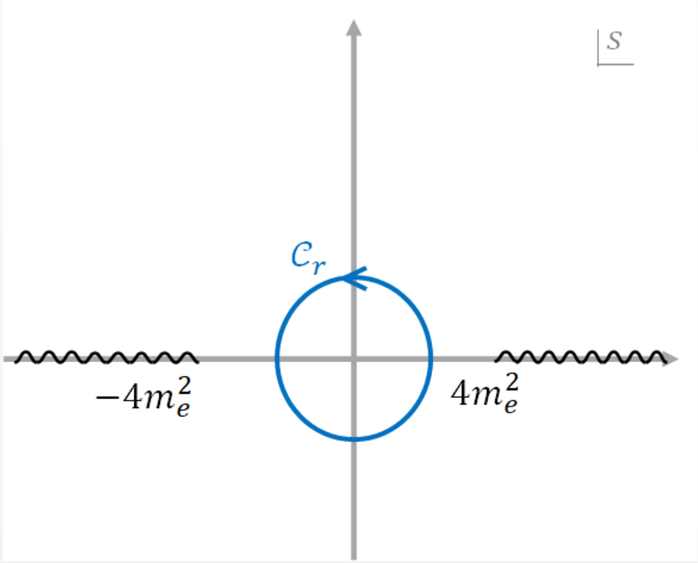

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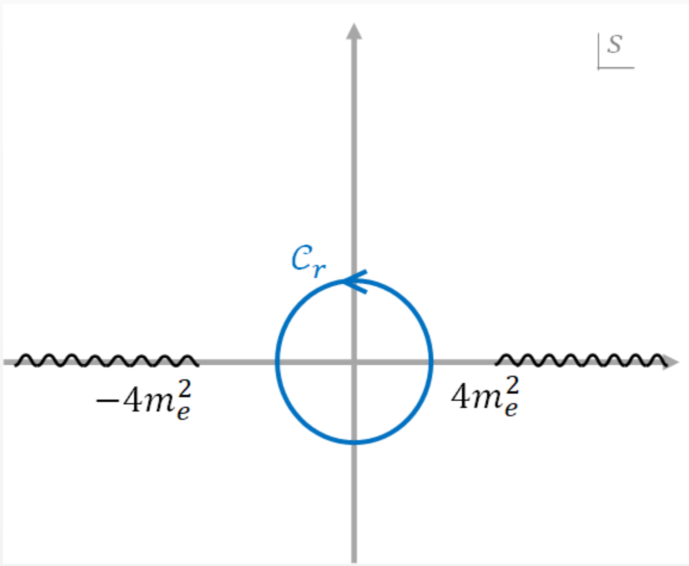
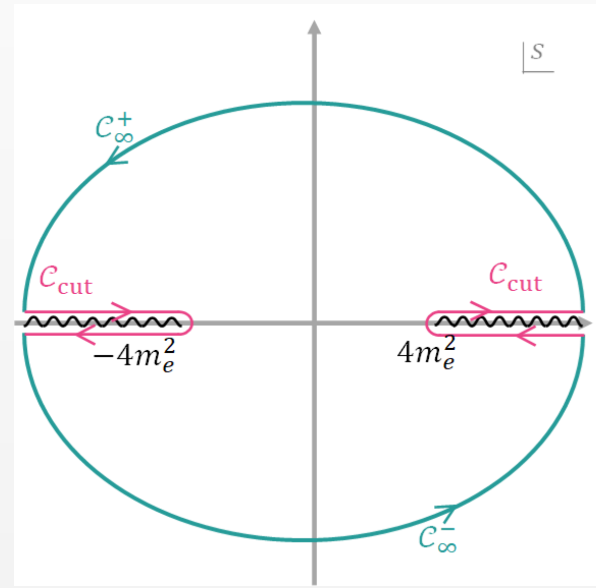


Contour deformation



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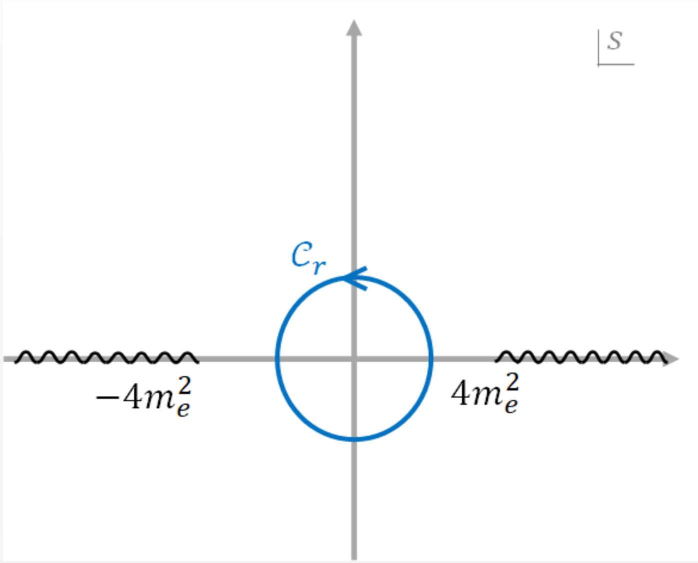

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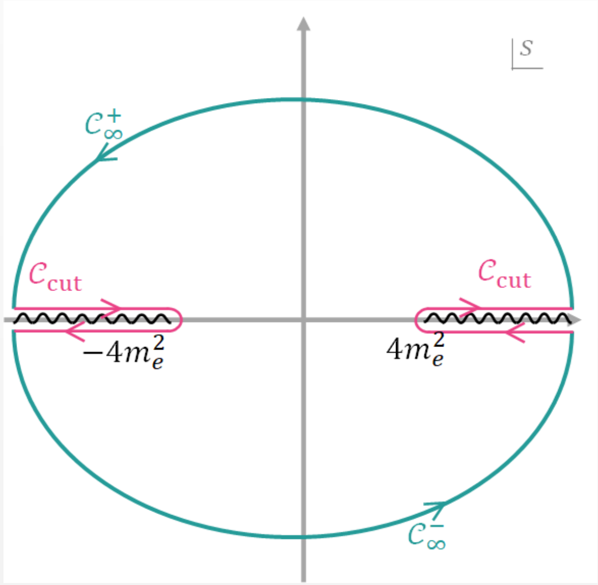

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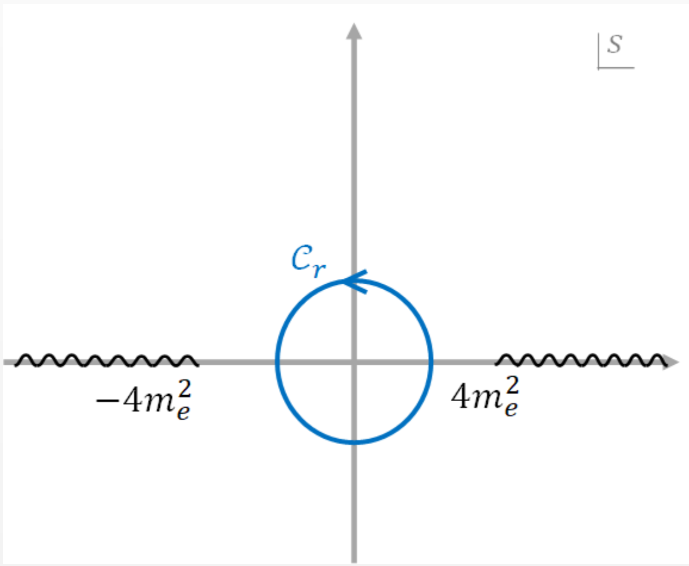

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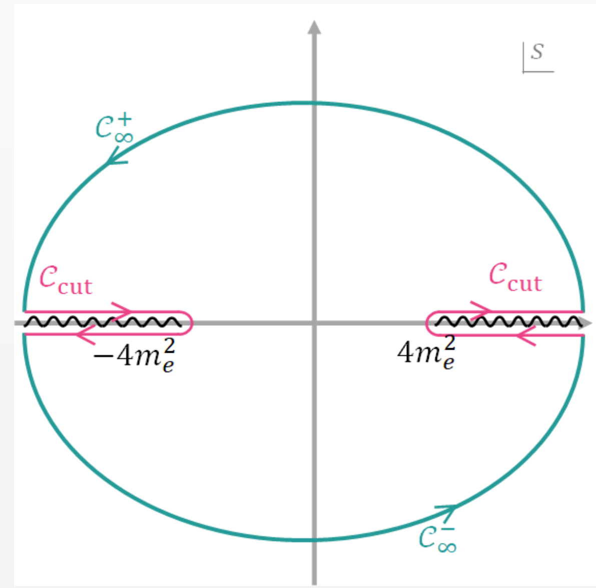

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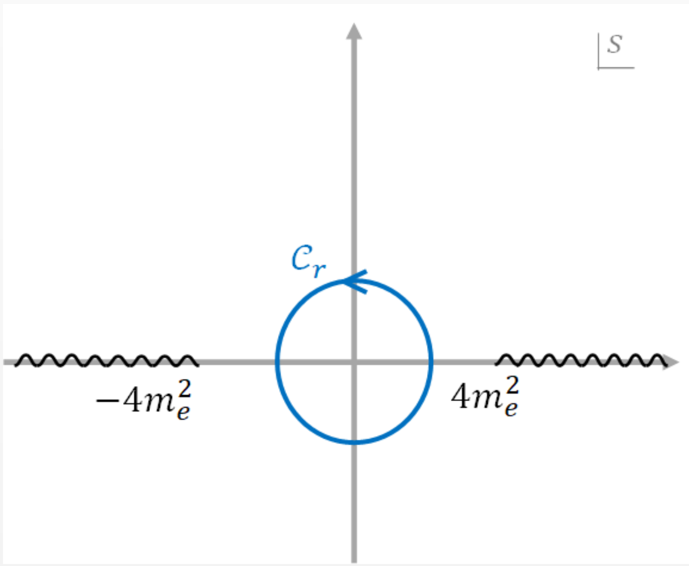

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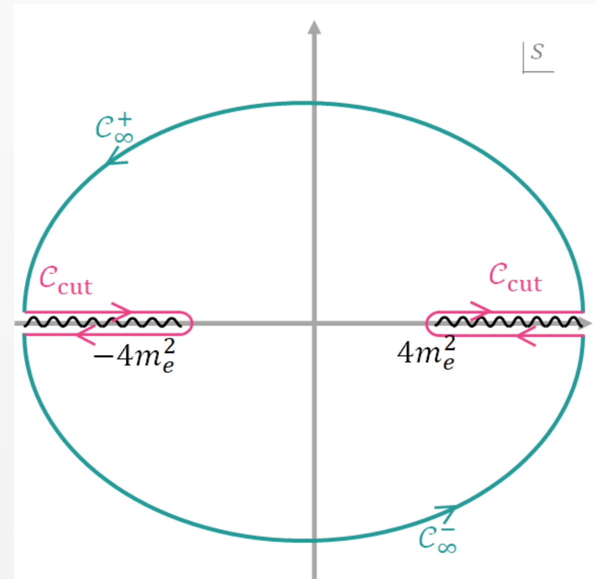

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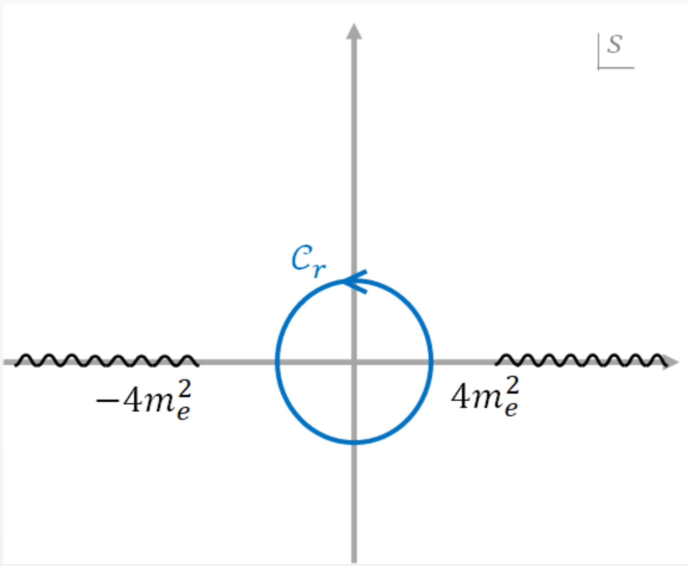

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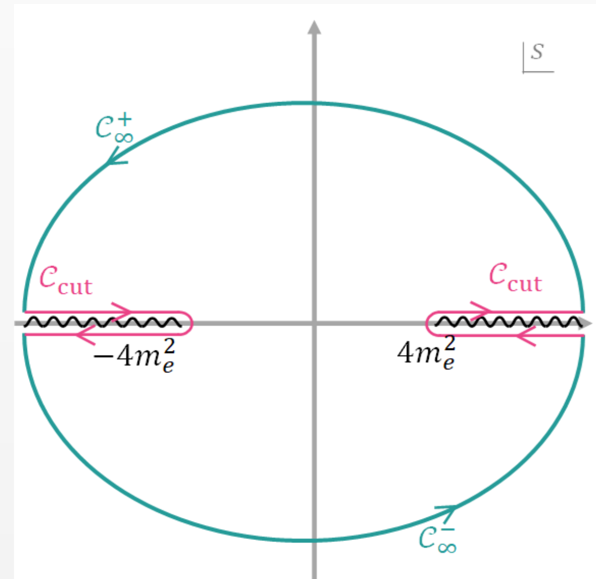

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- **"Positivity bounds" (without gravity)**

$$\boldsymbol{c_2} = \frac{2}{\pi}\int_{4m_e^2}^{\infty} \mathrm{d}s \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3} > \mathbf{0}.$$

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$$c_2(\Lambda) \coloneqq c_2 - \frac{2}{\pi}\int_{4m_e^2}^{\Lambda^2} \mathrm{d}s \, \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3}$$

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“Improved positivity bounds”

[Bellazzini(’16), de Rham+(’17)]

tin)

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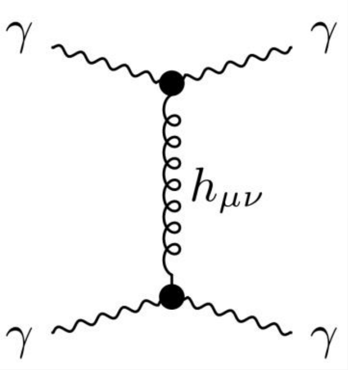
§ II. Implications

Subtlety with Gravity

- Quantum gravity S-matrix: **not fully understood.** c.f.) [Haring+('22)]
- Feature: t-channel graviton exchange **grows as fast as s^2,**

$\mathcal{M}(s,t) \ni$  $\sim \dfrac{s^2}{M_{\mathrm{pl}}^2 t}$

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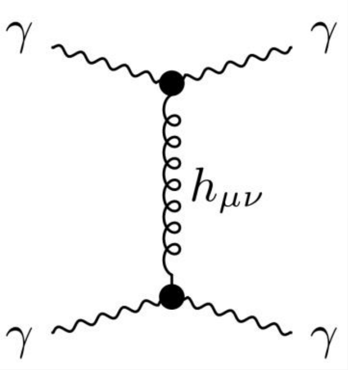

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- UV behavior is softened by tower of higher-spin states in tree-level string amplitude, $\mathcal{M} \sim \frac{1}{M_{\rm pl}^2 t} s^{2+\alpha' t} < s^2$: **Regge behavior.**

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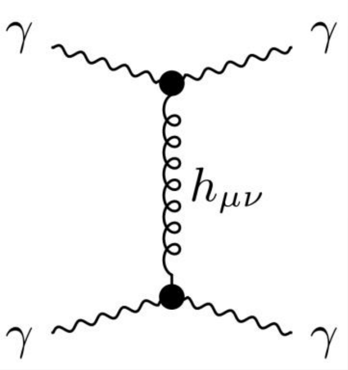

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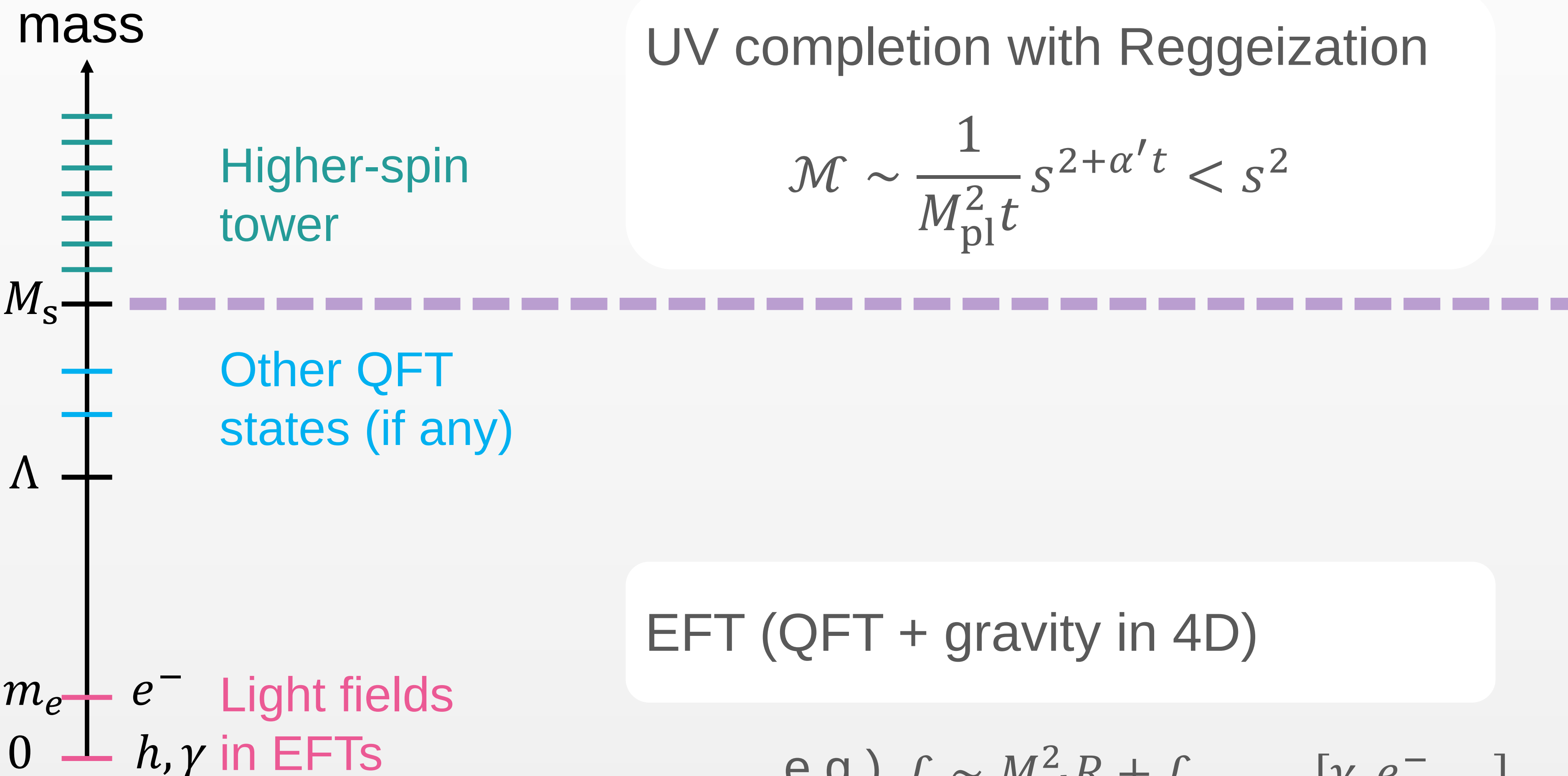
- We ***assume*** this property. **Can we derive positivity bound?**

The setup we consider



UV completion with Reggeization

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EFT (QFT + gravity in 4D)

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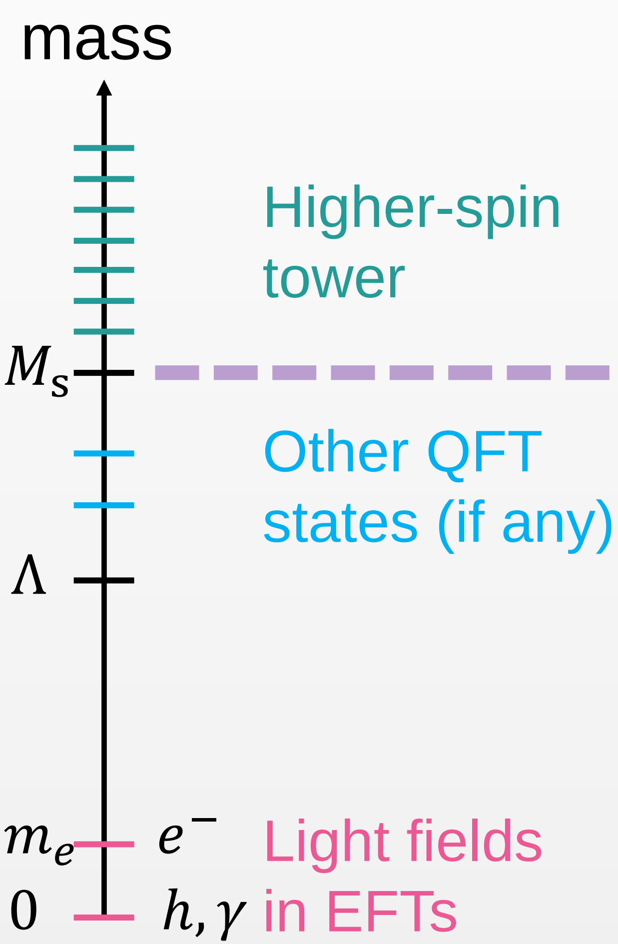

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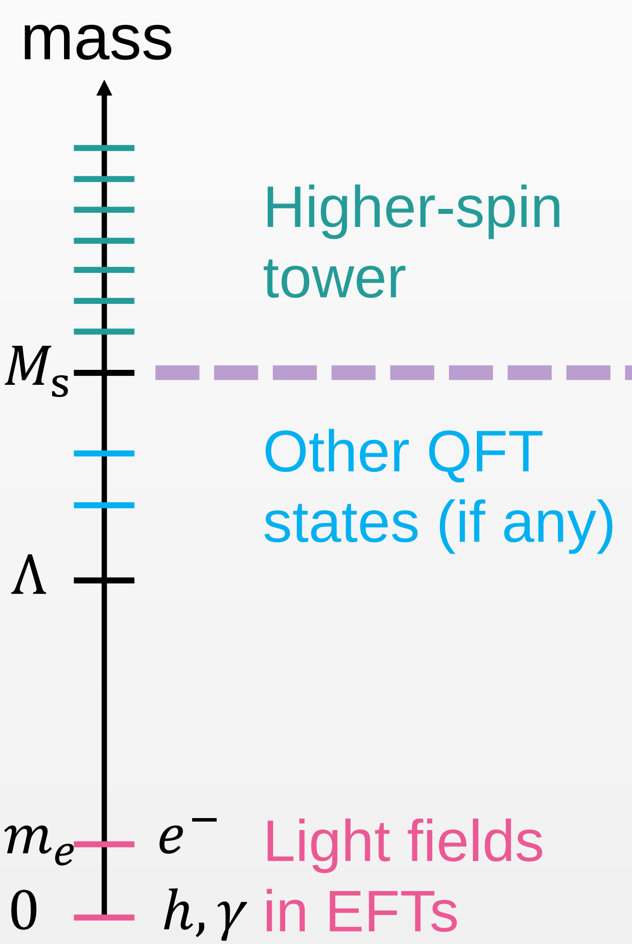

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- We work in **weakly-coupled regime of gravity** up to $\mathcal{O}\left(M_{\rm pl}^{-2}\right)$.

Positivity bounds with Gravity (1/2)

- The sum rule for $c_2(\Lambda)$ contains **graviton t-channel pole:**

$$c_2(\Lambda) = \lim_{t\to 0^-} \left\{ \int_{\Lambda^2}^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{s^3} + \frac{\mathbf{1}}{\boldsymbol{M_{\mathrm{pl}}^2 t}} \right\} = \text{"}\infty - \infty\text{"} \overset{?}{>} \mathbf{0}$$

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- Key: **The graviton t^{-1} pole is canceled** by the first term **due to the Regge behavior** $\mathrm{Im}\,\mathcal{M} \simeq f(t) s^{2+\alpha' t+\alpha'' t^2/2\cdots}$ @ $s \gg M_s^2$.

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Positivity bounds with Gravity (1/2)

- The sum rule for $c_2(\Lambda)$ contains **graviton t-channel pole:**

$$c_2(\Lambda) = \lim_{t\to 0^-}\left\{\underline{\int_{\Lambda^2}^{\infty} ds \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{s^3}} + \frac{\mathbf{1}}{\boldsymbol{M_{\mathrm{pl}}^2 t}}\right\} = \text{"}\infty-\infty\text{"} \overset{?}{>} 0$$

$$\int_{s_*\gg M_s^2}^{\infty} ds \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{s^3} \sim \int_{s_*\gg M_s^2}^{\infty} ds \frac{s^{2+\alpha' t+\cdots}}{s^3} = \frac{\boldsymbol{-s_*}}{\boldsymbol{\alpha' t}} + \mathcal{O}(t^0).$$

- Key: **The graviton t^{-1} pole is canceled** by the first term **due to the Regge behavior** $\mathrm{Im}\,\mathcal{M} \simeq f(t)s^{2+\alpha' t+\alpha'' t^2/2\cdots}$ @ $s \gg M_s^2$.

- After confirming the cancellation, we compute the $\mathcal{O}(t^0)$ term.

Positivity bounds **with** Gravity (2/2)

(Related discussions: [Hamada+('18)] [Herrero-Valea+('20)] [Bellazzini+('19)] [Alberte+('20,'21)] [Caron-Huot+('21)])

- **Gravitational positivity bound:** [JT-Aoki-Hirano ('20)]

$$c_2(\Lambda) = \underbrace{\int_{\Lambda^2}^{s_*} \mathrm{d}s \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3}}_{>0 \text{ (unitarity)}} + \frac{1}{M_{\mathrm{pl}}^2}\left[\underbrace{-\frac{2\partial_t f(t)|_{t=0}}{f(0)}}_{<0 \text{ (unitarity!)}} + \frac{\alpha''}{\alpha'}\right]$$

- **Negative term = Details of the Regge behavior.**

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- **$c_2(\Lambda) = 0$ is allowed.** ✓ Virasoro-Shapiro amplitude: $c_2 = 0$.

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- **Negative term = Details of the Regge behavior.**
- $\boldsymbol{c_2(\Lambda) = 0}$ **is allowed.** ✓ Virasoro-Shapiro amplitude: $c_2 = 0$.

- If we ***assume*** $\partial_t f/f,\ \alpha''/\alpha' \sim \mathcal{O}\left(M_s^{-2}\right)$, (*see also [Alberte+ ('21)])

$c_2(\Lambda) > -\mathcal{O}\left(M_{\rm pl}^{-2}\boldsymbol{M_s^{-2}}\right)$. ➡ **Interesting implications!**

§ I. Formulation

1. Positivity bounds without gravity (Review)

2. Gravitational positivity bounds

§ II. Implications

- QED coupled to gravity (Review)

QED + GR

[Alberte-de Rham-Jaitly-Tolley. ('20)]
(See also [Cheung+('14), Andriolo+('18), Chen+('19)])

- We focus on the $\gamma\gamma \to \gamma\gamma$ process.

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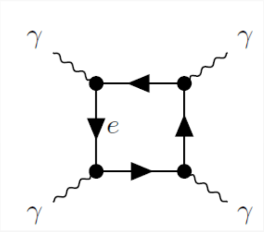
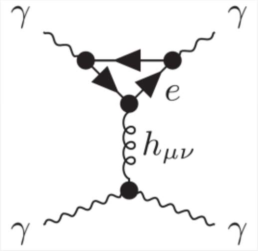
- We focus on the $\gamma\gamma \to \gamma\gamma$ process.

$$c_2(\Lambda) = (\mathbf{nongrav}) + (\mathbf{grav}) > -\mathcal{O}\left(M_{\rm pl}^{-2} M_s^{-2}\right).$$

$(\mathbf{nongrav})$: 

$$\sim \frac{e^4}{\Lambda^4} > 0$$

$(\mathbf{grav})$: 

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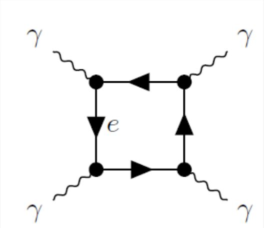
$(\mathbf{nongrav})$: 

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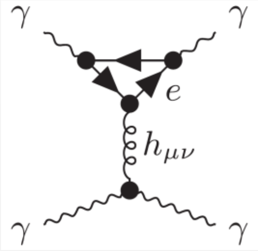
$(\mathbf{grav})$: 

$\sim -\dfrac{e^2}{M_{\rm pl}^2 m_e^2} < 0$

$$\boldsymbol{\Lambda \lesssim \sqrt{e M_{\rm pl} m_e}}$$
$$\approx \mathcal{O}(\mathbf{10^8})\textbf{GeV.}$$

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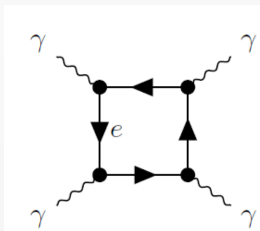
$(\mathbf{nongrav})$: 

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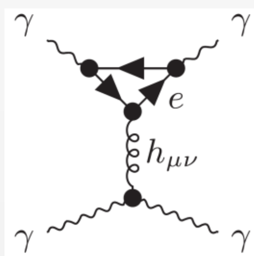
$(\mathbf{grav})$: 

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A cutoff is too low??
$m_e \to 0$ limit is prohibited??

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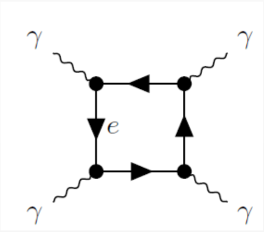

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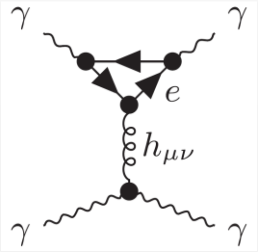
(**grav**):



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- **Renormalizable couplings can be constrained!**

QED + GR

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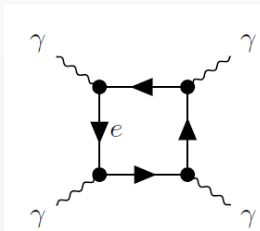

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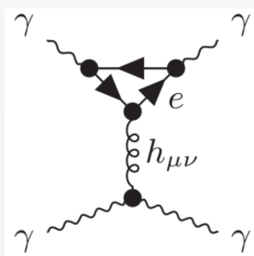

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A cutoff is too low??
$m_e \to 0$ limit is prohibited??

- **Renormalizable couplings can be constrained!**

- Bounds on **Standard Model** → **Noumi-san's** talk (tomorrow!)
  Dark photon → **Sato-san's** talk (next talk!)

Summary

- We derived **positivity bounds on gravitational EFTs.**
- We imposed assumptions on quantum gravity S-matrix.

➡ A bound on low-energy S-matrix.

= “**Correlations** between IR and UV physics”

- **Renormalizable couplings can be constrained!**

  c.f.) talks by **Sato-san** (next talk) and **Noumi-san** (tomorrow)

- How robust are the properties we imposed?

backup

Summary & Prospects

- **How robust are the properties we imposed?** (Amplitude)

  e.g.) Can we prove the expected scaling $\partial_t f/f,\ |\alpha''/\alpha'| \lesssim \mathcal{O}(M_s^{-2})$?

  Explicit examples?

- **Applications to other models.** (Particle pheno, Cosmology)

  e.g.) QCD axion? Dark photon models? …

- **Any suggestions from other considerations?** (String pheno)

  e.g.) [Reece ('18)] suggests a lower bounds on dark photon mass.

- It is surprising that **these bounds are implied by only several general properties of S-matrix.**

Analyticity & Causality

c.f.) [Camanho-Edelstein-Maldacena-Zhiboedov+('14)]

- To get some intuition, let's consider **a signal model.**
- We have an initial signal $f_{\rm in}(t)$ and an out-signal $f_{\rm out}(t)$ with

$$f_{\rm out}(t) = \int_{-\infty}^{\infty} {\rm d}t' \, S(t-t') f_{\rm in}(t').$$

$$\Leftrightarrow \ \tilde{f}_{\rm out}(\omega) = \tilde{S}(\omega)\tilde{f}_{\rm in}(\omega), \qquad S(t) = \int_{-\infty}^{\infty} \frac{{\rm d}\omega}{2\pi} \tilde{S}(\omega) e^{-i\omega t}.$$

- $\tilde{S}(\omega)$: S-matrix element.
- **Causality** implies: $S(t) = 0$ for $t < 0$.

$$\Rightarrow \ \tilde{S}(\omega) = \int_{0}^{\infty} \frac{{\rm d}t}{2\pi} S(t) e^{i\omega t}$$

Analytic in the upper half plane.

Mild behavior from Locality & Unitarity

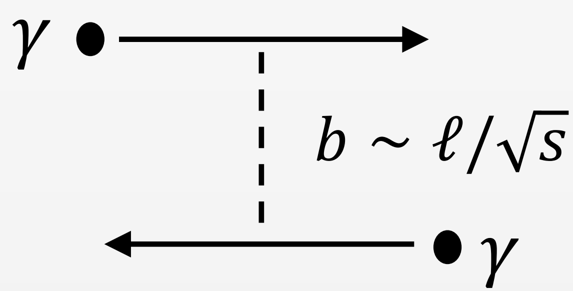
- Key: **polynomial boundedness** (PB)

$$\boldsymbol{\mathcal{M}(s,t) < s^N} \quad \text{as } s \to \infty, t\text{: fixed}.$$

$s^N \sim \partial^\# \sim$ **Locality**

- Consider the partial-wave expansion

$$\mathcal{M} \sim \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell(\cos\theta)$$



  - **Unitarity:** $|f_\ell(s)| \leq 1$
  - Short-range force: $|f_\ell(s)| < s^N \exp\left(-m_e \ell/\sqrt{s}\right)$ @ large ℓ

$$|\mathcal{M}(s,0)| < \sum_{\ell=0}^{\sqrt{s}\ln s} (2\ell+1) \sim s(\ln s)^2 \quad \text{as } s \to \infty.$$

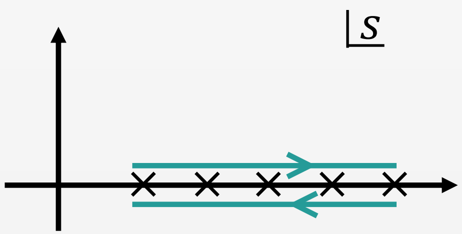
Froissart-Martin bound.

Example of positivity violation

- e.g) type-Ⅱ superstring amplitude of identical massless boson

$$\mathcal{M}(s,t) = -\left(s^2u^2 + t^2u^2 + s^2t^2\right)\frac{\Gamma\left(-\frac{\alpha' s}{4}\right)\Gamma\left(-\frac{\alpha' t}{4}\right)\Gamma\left(-\frac{\alpha' u}{4}\right)}{\Gamma\left(1+\frac{\alpha' s}{4}\right)\Gamma\left(1+\frac{\alpha' t}{4}\right)\Gamma\left(1+\frac{\alpha' u}{4}\right)}$$

An infinite number of higher-spin states Reggeizes the amplitude.



$$s = \frac{4N}{\alpha'}\,(N = 1,2,\cdots)$$

spin $J = 2N + 2$

$$\mathrm{Im}\,\mathcal{M}\left(se^{i\varepsilon},t\right) \approx f(t)\left(\frac{\alpha' s}{4}\right)^{2+\alpha' t/2}$$ for $s \gg \alpha'^{-1}, 0 < \varepsilon \ll 1.$

$$f(0) = \frac{256}{\alpha'^4},\ \ \partial_t f(t)|_{t=0} = \frac{128}{\alpha'^3}.$$

$$c_2(\Lambda) = \lim_{t\to -0}\left\{\frac{4}{\pi}\int_{\Lambda^2}^{\infty} \mathrm{d}s'\,\frac{\mathrm{Im}\,\mathcal{M}(s',t)}{\left(s'+\frac{t}{2}\right)^3} + \frac{2}{M_{\mathrm{pl}}^2 t}\right\} = 0$$

Strict positivity is **violated,** due to **the exact cancellation.**

(Regge states) – (graviton t-pole) =0

Remark(1/2)

- Why is the (nongrav) term is positive?

$$c_2(\Lambda) = \underbrace{(\mathbf{nongrav})}_{>\,\mathbf{0}\ \text{(why?)}} + (\mathbf{grav}) > -O\left(M_{\rm pl}^{-2} M_s^{-2}\right).$$

- This is because,

We have $(\mathbf{nongrav}) = \lim\limits_{M_{\rm pl}\to\infty} c_2(\Lambda).$

In the $M_{\rm pl} \to \infty$ limit, a model becomes renormalizable.

A condition $c_2(\Lambda) > 0$ is satisfied in renormalizable theory.

Remark(2/2)

- More technically, $\mathcal{M}_{\rm nongrav} = \lim_{M_{\rm pl}\to\infty} \mathcal{M}$ satisfies the twice-subtracted dispersion relation. Hence,

$$(\textbf{nongrav}) = \int_{\Lambda^2}^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\, \mathcal{M}(s,0)}{s^3} \sim \int_{\Lambda^2}^{\infty} \mathrm{d}s \, \frac{\sigma_{\rm tot}(s)}{s^2} > 0.$$

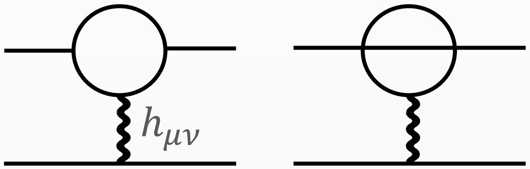
$\sigma_{\rm tot}$: total cross section

- Typically, a particle with mass M contributes to $\sigma_{\rm tot}$ as

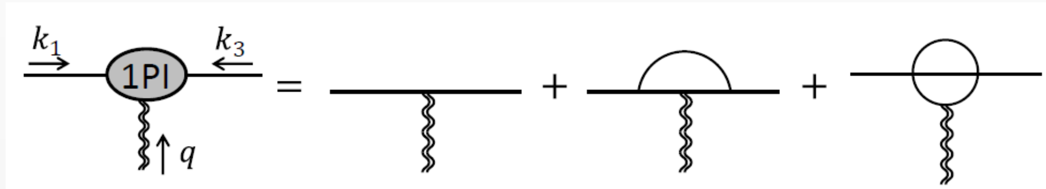
$$\sigma_{\rm tot}(s) \sim M^{-2} \quad \left(@\, \mathrm{s} \sim M^2\right).$$

- Contributions from **light particles are important.**

Computation of c_{grav}

- $c_{\mathrm{grav}} =$ 

$+\cdots$

- 1PI vertices $V^{\mu\nu}(k_1, k_3) =$ 

$$V^{\mu\nu}(k_1, k_3)|_{k_1^2 = k_3^2 = -m^2} \ni R(q^2)(k_1 - k_3)^\mu (k_1 - k_3)^\nu, \qquad R_{\mathrm{tree}}(q^2) = 1/2.$$

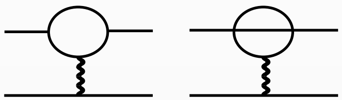
$$\Rightarrow \mathcal{M}(s,t)\Big|_{\mathrm{grav}} \sim \frac{4R^2(-t)su}{M_{\mathrm{pl}}^2 t} \sim \frac{4R'(0)}{M_{\mathrm{pl}}^2} s^2$$

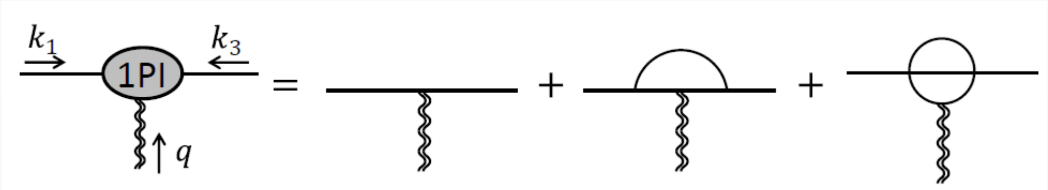
$$\Rightarrow c_{\mathrm{grav}} \simeq \frac{8R'(0)}{M_{\mathrm{pl}}^2} \simeq -\frac{1}{M_{\mathrm{pl}}^2}\left(\frac{45 - 8\pi\sqrt{3}}{1296\pi^2}\frac{g^2}{m^4} + \frac{10 - \pi^2}{4608\pi^4}\frac{\lambda^2}{m^2}\right) < 0.$$

Negative!!

- Negative term arises as a result of expanding $R(q^2)$ around $q^2 = 0$.

Sign of c_{grav} and superluminality

- Consider scalar theory. $c_{\text{grav}} =$  $+\cdots$

- Relevant 1PI vertices: 

$$\Rightarrow \quad \mathcal{L} = M_{\text{pl}}^2 R - \frac{1}{2}(\partial\phi)^2 - V(\phi) + \alpha R_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi)$$

$\alpha < 0$

$\Leftrightarrow c_{\text{grav}} < 0$

Effective metric for ϕ: $\tilde{g}_{\mu\nu} = g_{\mu\nu} - 2\alpha R_{\mu\nu}$

e.g.) FLRW metric with $\dot{H} < 0$

Dispersion relation: $\omega^2 \simeq \left(1 + 4\alpha\dot{H}\right)k^2 > k^2$

Superluminal relative to the speed of GW!

$c_{\text{grav}} < 0 \sim$ Superluminal propagation in b.g. satisfying null-E condition.