Entropy constraints on effective field theory

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Based on **arXiv:2201.00931** with **Qing-Hong Cao (Peking University)**

arXiv:2211.08065 with **Qing-Hong Cao (Peking University)**, and **Naoto Kan (Osaka University)**

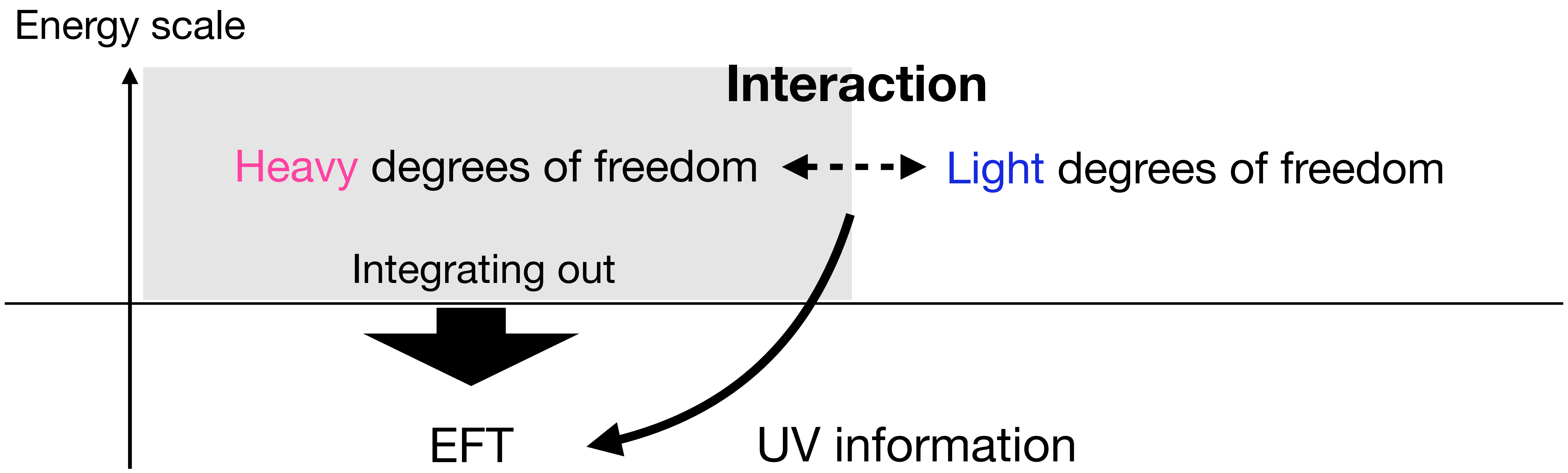
Introduction

Introduction

- Effective Field Theory (EFT):

  - EFT is generated by integrating out dynamical degrees of freedom

  - Information on UV theory is transferred through interaction b/w heavy and light degrees of freedom



Differences between theories with and without interaction characterize UV information

⇒ Relative entropy characterizes their difference

Relative entropy

※ $\mathrm{Tr}[\tilde{\rho}] = \mathrm{Tr}[\rho] = 1, \ \ \tilde{\rho} = \tilde{\rho}^{\dagger}, \ \ \rho = \rho^{\dagger}$

- Definition of relative entropy b/w two probability distribution functions $\tilde{\rho}$ and ρ

$$S(\tilde{\rho}\,||\,\rho) \equiv \mathrm{Tr}\left[\tilde{\rho}\ln\tilde{\rho} - \tilde{\rho}\ln\rho\right]$$

- relative entropy is **non-negative**

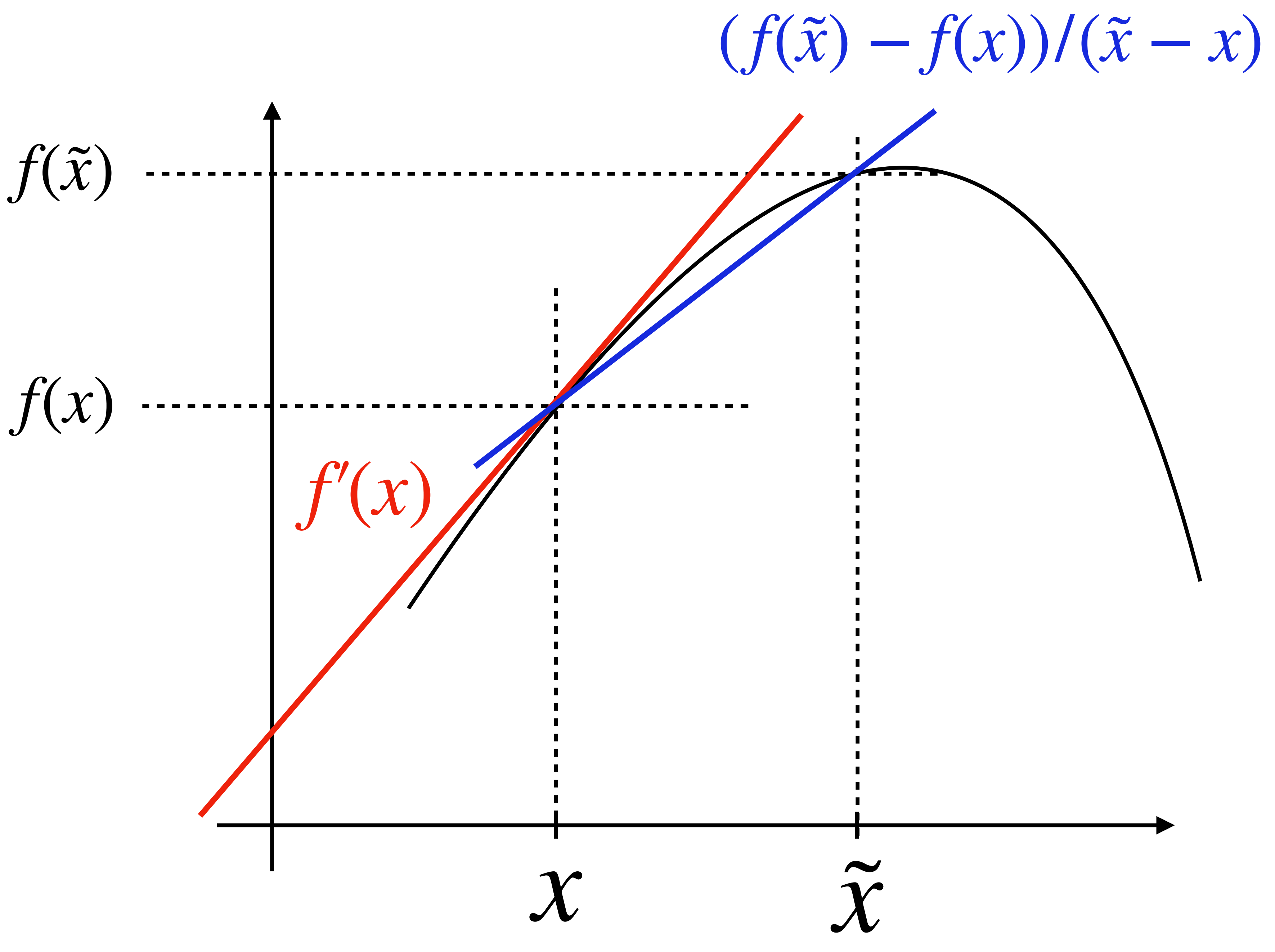
A proof:

$f(x)$: a convex function $\Rightarrow \mathrm{Tr}[f(\tilde{\rho}) - f(\rho) - (\tilde{\rho} - \rho)f'(\rho)] \leq 0$

⬇ $f(x) \to x\ln x$ (convex function)

$$S(\tilde{\rho}\,||\,\rho) \geq 0$$

※ equality holds if and only if $\tilde{\rho} = \rho$



Definition of convex function: $\dfrac{f(\tilde{x}) - f(x)}{\tilde{x} - x} \leq f'(x)$

Relative entropy characterizes difference between two probability distributions

Our idea

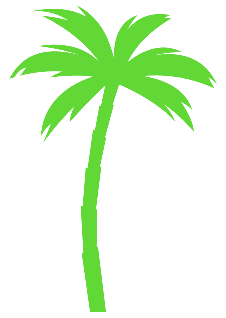
- Relative entropy characterizes difference between two probability distributions

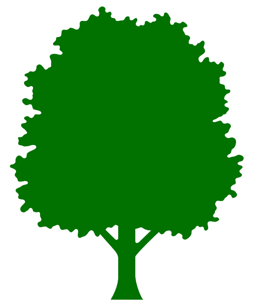
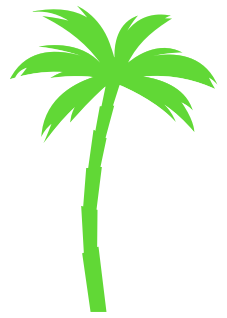
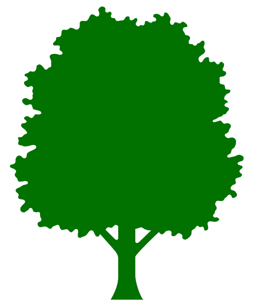
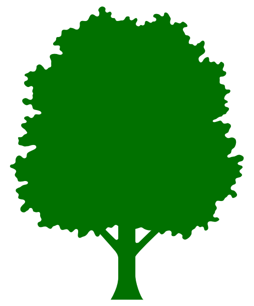
$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right] \geq 0$$

※ equality holds if and only if $\rho_A = \rho_B$

- Relative entropy provides quantitative difference between two things defined by probability distribution functions

Ex.

 $\mapsto \rho_A$  $\mapsto \rho_B$

$S($  $||$  $) > 0$ $S($  $||$  $) = 0$

What about relative entropy b/w theories with and without interaction?

⇒ We have to define probability distribution for each theory.

Probability distributions of theories

- We define probability distributions of theory described by Euclidean action I as follows:

Probability distribution function: $P[\phi, \Phi] = e^{-I[\phi,\Phi]}/Z$

Partition function: $Z = \int d[\phi]d[\Phi]e^{-I[\phi,\Phi]}$

where I: Euclidean action, ϕ: light fields, Φ: heavy fields

- Relative entropy between two theories

$$S(P_A || P_B) \equiv \int d[\phi]d[\Phi]\left(P_A \ln P_A - P_A \ln P_B\right) \geq 0$$

where $P_A = e^{-I_A}/Z_A, \quad P_B = e^{-I_B}/Z_B$

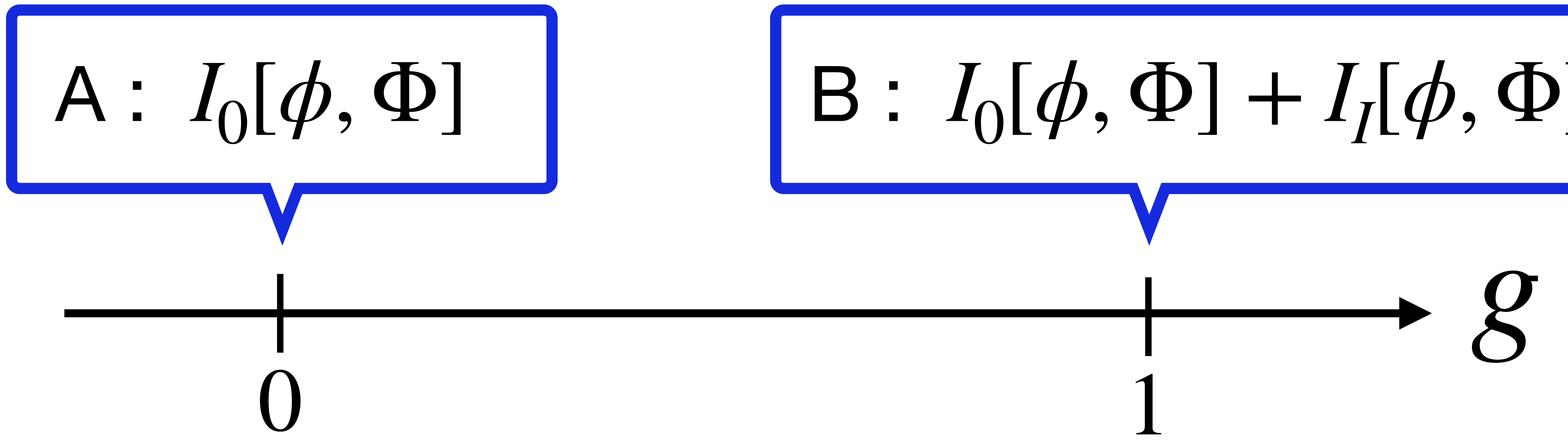
Definition of two theories

No interaction b/w ϕ and Φ | Interaction b/w ϕ and Φ

- We consider theories described by $$I_0[\phi, \Phi] + I_I[\phi, \Phi]$$

※ Φ: heavy fields, ϕ: light fields

- We define $I_0[\phi, \Phi] + g \cdot I_I[\phi, \Phi]$ by introducing parameter g



We consider relative entropy $S(P_A || P_B)$

※ (Φ, ϕ) of A is the same as that of B

Relative entropy between two theories

$$S(P_A||P_B) = \int d[\phi]d[\Phi]\left[P_A \ln P_A - P_A \ln P_B\right]$$

$$P_A = e^{-I_0[\phi,\Phi]}/Z_0 \qquad P_B = e^{-(I_0[\phi,\Phi]+gI_I[\phi,\Phi])}/Z_g$$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

Effective actions: $W_g = -\ln Z_g, \quad W_0 = -\ln Z_0$

$S(P_A||P_B)$ yields constraints on the Euclidean effective actions

even in quantum mechanical system

$$S(P_A||P_B) \to \mathrm{tr}\left[P_A \ln P_A - P_A \ln P_B\right]$$

$$P_A \to e^{-H_0}/Z_0 \qquad P_B \to e^{-(H_0+gH_I)}/Z_g$$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

$$W_g = -\ln Z_g, \; W_0 = -\ln Z_0$$

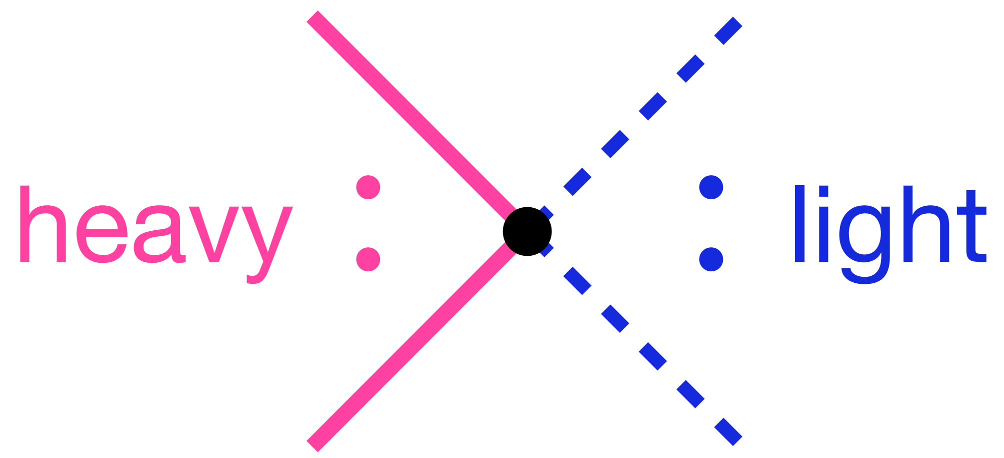
Bottom-up approach

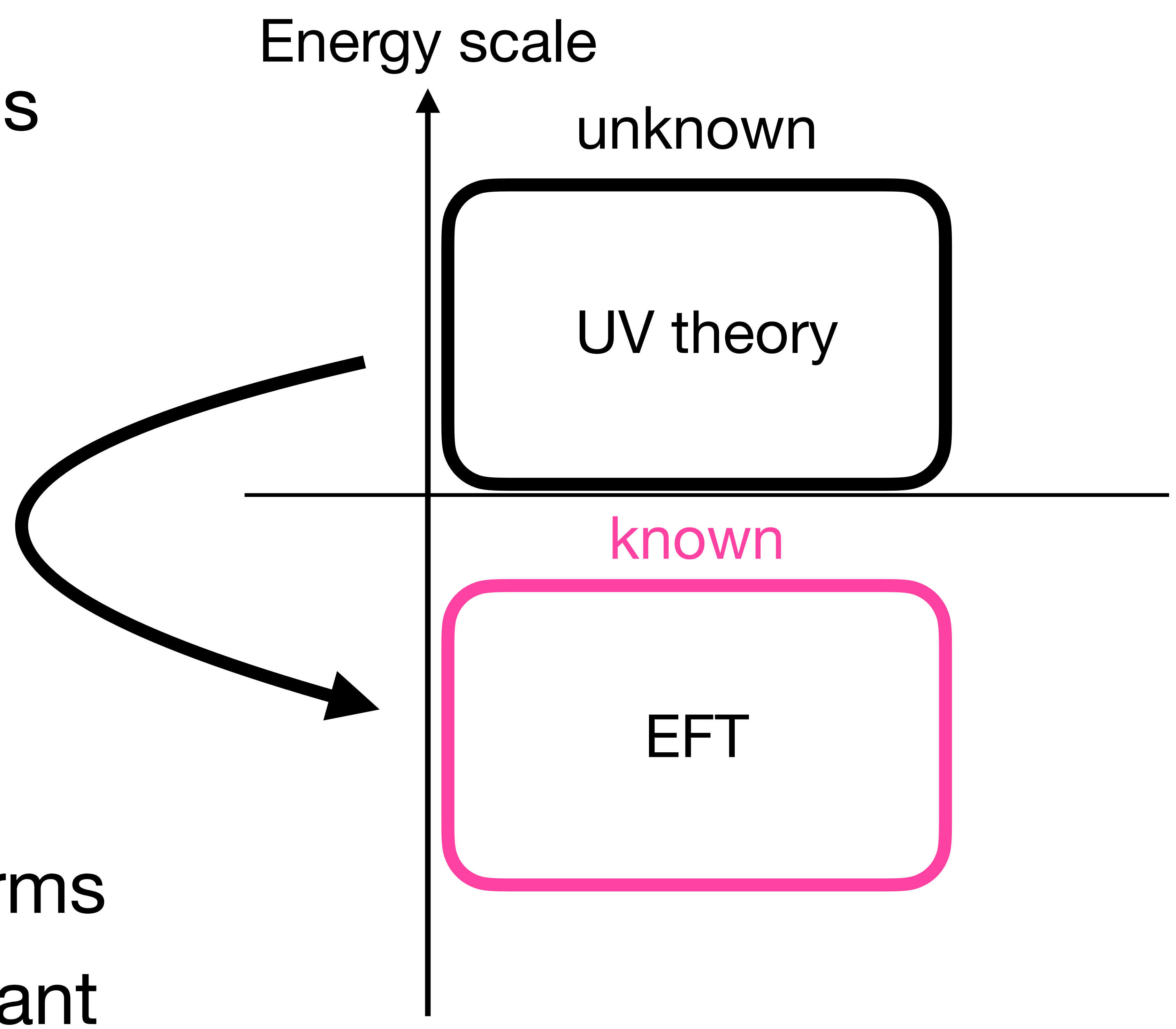
Bottom-up approach

- **Assumptions:**

  - EFT is generated through interaction b/w heavy and light fields

$$I_I[\phi, \Phi] = \int (d^4x)_{\mathrm{E}}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$



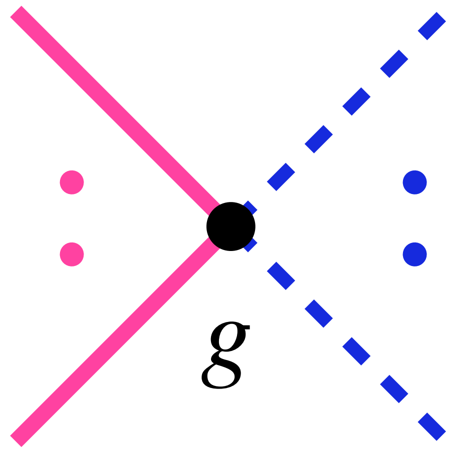


where we assume $J[\phi]$ does not involve higher-derivative terms

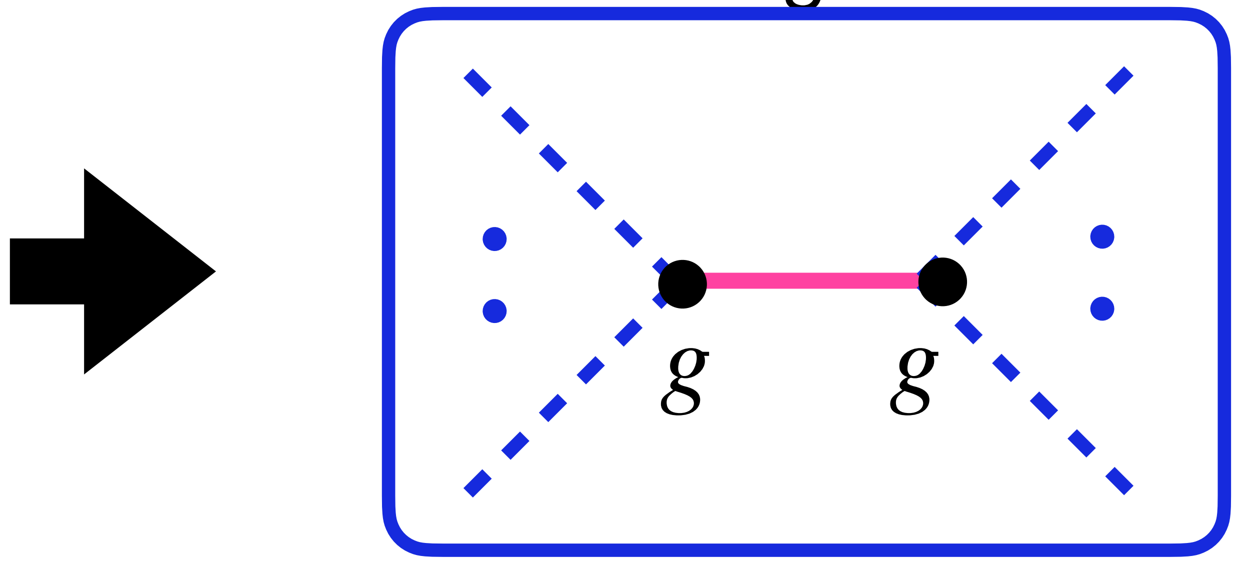
※ it would be subdominant

What is the consequence of the non-negativity of relative entropy in the bottom-up approach?

Tree-level UV theory

$$I_I[\phi, \Phi] = \int (d^4x)_{\mathrm{E}}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$ heavy :  : light

Second or higher order of g



※ Linear terms of heavy field can be removed by field redefinition

- Ex. Single mass less field theory with shift symmetry

Effective action:

$$W_g[\widetilde{\phi}\,] = \int (d^4x)_{\mathrm{E}} \left(-\frac{1}{2}(1 + a_2^{\mathrm{tree}})(\partial_\mu \widetilde{\phi'} \partial^\mu \widetilde{\phi'}) - \frac{c_2^{\mathrm{tree}}}{M^4}(\partial_\mu \widetilde{\phi'} \partial^\mu \widetilde{\phi'})^2 \right)$$ where $a_2^{\mathrm{tree}}, c_2^{\mathrm{tree}}$: second or higher order of g

$$= \int (d^4x)_{\mathrm{E}} \left(-\frac{1}{2}(\partial_\mu \widetilde{\phi}\, \partial^\mu \widetilde{\phi}\,) - \frac{c_2^{\mathrm{tree}}}{M^4}(1 + a_2^{\mathrm{tree}})^{-2}(\partial_\mu \widetilde{\phi}\, \partial^\mu \widetilde{\phi}\,)^2 \right)$$ where $\widetilde{\phi} = (1 + a_2^{\mathrm{tree}})^{1/2} \cdot \widetilde{\phi'},\ \partial \widetilde{\phi} = \mathrm{const}\,.$

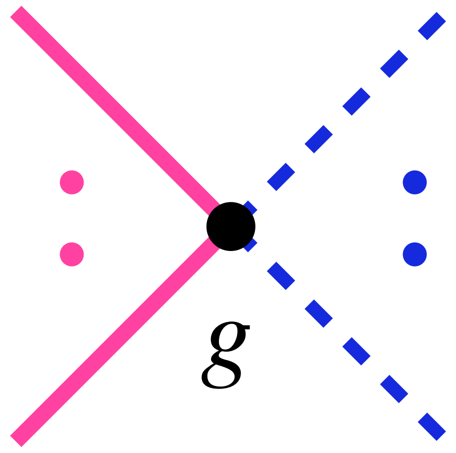
※ To remove dim-6 terms

- Relative entropy

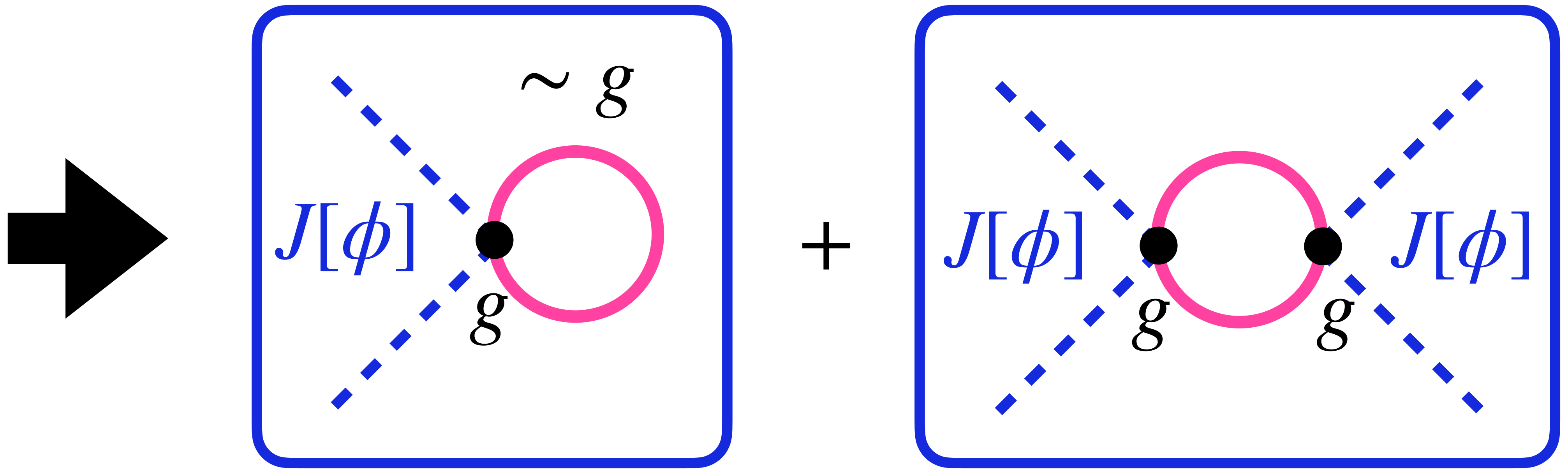
$$S(P_0||P_g) = W_0[\widetilde{\phi}\,] - W_g[\widetilde{\phi}\,] + g\left(\partial W_g/\partial g\right)_{g=0} = \frac{c_2^{\mathrm{tree}}}{M^4}(1 + a_2^{\mathrm{tree}})^{-2} \int (d^4x)_{\mathrm{E}} (\partial_\mu \widetilde{\phi}\, \partial^\mu \widetilde{\phi}\,)^2 \geq 0 \Rightarrow c_2^{\mathrm{tree}} \geq 0$$

Relative entropy constrains Wilson coefficient of dim-8 operator

One-loop level UV theory

$$I_I[\phi, \Phi] = \int (d^4x)_{\mathrm{E}}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$ heavy :  : light

First order of g — Second or higher order of g



- Example of EFT: Single massless field theory with shift symmetry($\phi \to \phi + \mathrm{const}.$)

Effective action:

$$W_g[\widetilde{\phi}] = \int (d^4x)_{\mathrm{E}} \left(-\frac{1}{2}(1 + a_1^{\mathrm{loop}} + a_2^{\mathrm{loop}})(\partial_\mu \widetilde{\phi}' \partial^\mu \widetilde{\phi}') - \frac{c_2^{\mathrm{loop}}}{M^4}(\partial_\mu \widetilde{\phi}' \partial^\mu \widetilde{\phi}')^2 \right)$$

where a_1^{loop} : first order of g and $a_2^{\mathrm{loop}}, c_2^{\mathrm{loop}}$: second or higher order of g

Relative entropy constrains Wilson coefficient of dim-8 operator

$$S(P_0 || P_g) = W_0[\widetilde{\phi}] - W_g[\widetilde{\phi}] + g \left(\partial W_g / \partial g \right)_{g=0} = \frac{c_2^{\mathrm{loop}}}{M^4} \int (d^4x)_{\mathrm{E}} (\partial_\mu \widetilde{\phi} \partial^\mu \widetilde{\phi})^2 \geq 0 \;\Rightarrow\; c_2^{\mathrm{loop}} \geq 0$$

Class of theories

- Reason why bounds on higher-derivative terms arise: $\int (d^4x)_E \left(-\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) - \frac{c}{M^4}(\partial_\mu\phi\partial^\mu\phi)^2 \right)$

⇒ corrections to non-higher derivative term can be removed by field redefinition

$$\phi \to \phi + \delta\phi,\ \ A^a_\mu \to A^a_\mu + \delta A^a_\mu,\ \ g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

Ex.

SMEFT SU(N) gauge bosonic operators

$$\int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + \frac{1}{\Lambda^4}\sum_i c_i \mathcal{O}_i \right)$$

Einstein-Maxwell theory with higher-derivative terms

$$\int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha_1}{4M_{\rm Pl}^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{\alpha_2}{4M_{\rm Pl}^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \frac{\alpha_3}{2M_{\rm Pl}^2}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma}\right]$$

Relative entropy yields constraints on the above higher-derivative terms in the bottom-up approach.

⇒ The same procedures as a single massless field with shift symmetry work well

Entropy constraints on SMEFT gauge bosonic operators

- Non-negativity of relative entropy:

$$S(P_0||P_g) = W_0 - W_g + g \cdot (\partial W_g/\partial g)_{g=0} = \int d^4x \frac{1}{\Lambda^4} \sum_i c_i \mathcal{O}_i \geq 0$$

- Classical solution of $\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu,b} F^c_{\mu\nu} = 0$: $A^a_\mu = u^a_1 \epsilon_{1\mu} w_1 + u^a_2 \epsilon_{2\mu} w_2$ with $f^{abc} u^a_1 u^b_2 = 0$, $\partial_\mu w_1 = l_\mu$, and $\partial_\mu w_2 = k_\mu$

※ l_μ, k_μ : constant vectors

- $U(1)_Y$: $c_1^{B^4} \geq 0, \quad c_2^{B^4} \geq 0, \quad 4c_1^{B^4} c_2^{B^4} \geq (\tilde{c}_1^{B^4})^2,$
- $SU(2)_L$: $c_1^{W^4} + c_3^{W^4} \geq 0, \quad c_2^{W^4} + c_4^{W^4} \geq 0, \quad 4(c_1^{W^4} + c_3^{W^4})(c_2^{W^4} + c_4^{W^4}) \geq (\tilde{c}_1^{W^4} + \tilde{c}_2^{W^4})^2,$
- $SU(3)_C$: $2c_1^{G^4} + c_3^{G^4} \geq 0, \quad 3c_2^{G^4} + 2c_5^{G^4} \geq 0, \quad 3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4} \geq 0, \quad 3c_4^{G^4} + 2c_6^{G^4} \geq 0,$

$$4(3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4})(3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4}) \geq (3\tilde{c}_1^{G^4} + 3\tilde{c}_2^{G^4} + \tilde{c}_3^{G^4})^2$$

$$4(3c_3^{G^4} + 2c_5^{G^4})(3c_4^{G^4} + 2c_6^{G^4}) \geq (3\tilde{c}_2^{G^4} + 2\tilde{c}_3^{G^4})^2$$

These bounds are consistent with positivity bounds from unitarity and causality

[G.N. Remmen, and N.L. Rodd, arXiv:1908.09845]

Entropy constraints on Einstein-Maxwell theory

- Non-negativity of relative entropy:

$$S(P_0||P_g) = \int (d^4x)_E \sqrt{g} \left(\frac{\alpha_1}{4M_{\rm Pl}^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{\alpha_2}{4M_{\rm Pl}^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \frac{\alpha_3}{2M_{\rm Pl}^2}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma} \right) \geq 0$$

- For charged BH background field, thermodynamic relations yield

$$(\Delta M_{\rm ext})_Q \propto -S(P_0||P_g) \leq 0$$ where Q is U(1) charge of BH

Extremal BH mass shift at fixed charge by higher derivative terms

Shift by higher-derivative terms

charge-to-mass ratio of extremal BH: $\frac{Q}{M_{\rm ext}/\sqrt{2}M_{\rm Pl}} = 1 \quad \Rightarrow \quad \frac{Q}{(M_{\rm ext} + (\Delta M_{\rm ext})_Q)/\sqrt{2}M_{\rm Pl}} \geq 1$

Extremal BH can behave as a state with charge-to-mass ratio larger than one

⇒ Mild Weak Gravity Conjecture

※ This argument is based on a field theory approach and may not apply to theories with stringy particles.

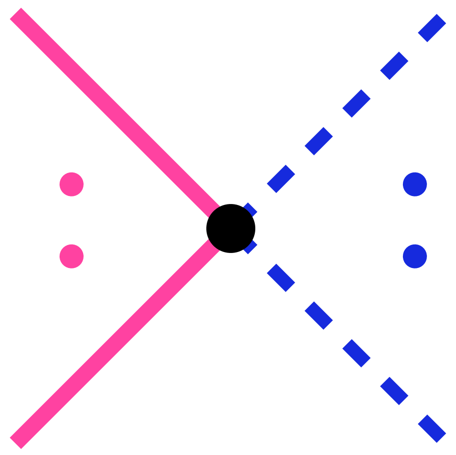
Summary

- Differences between theories with and without interaction characterize UV information

- We quantified their differences by relative entropy

- In the bottom-up approach, i.e.,

$$I_I[\phi, \Phi] = \int (d^4x)_{\mathrm{E}}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$

heavy :  : light

where we assume $J[\phi]$ does not involve higher-derivative terms

we found that the non-negativity of relative entropy constrains EFTs, e.g.,

SMEFT SU(N) gauge bosonic operators

Einstein-Maxwell theory with higher-derivative terms

- Relative entropy provides a new approach to constraining EFTs.