Fermion mass hierarchy and CP violation in modular symmetry

Morimitsu Tanimoto

Niigata University

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Collaborated with
Serguey Petcov

1 Introduction

Flavor problem of quarks and leptons

- Mass hierarchy
- Flavor mixing
- CP violation

What is the mechanism of fermion mass hierarchy ?

Well known one is U(1) Froggatt-Nielsen

Modular forms meet the flavor problem

What is Modular form ?

$$f(x) = \sin 2\pi x\,, \qquad \mathrm{T}: x \to x+1 \Rightarrow f(x+1) = f(x)$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (a,b,c,d) are integer and $ad - bc = 1$

$$\gamma : z \to \frac{az+b}{cz+d}$$

z is complex

(Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{T}: z \to z+1$$

Modular form f(z) is defined by imposing three conditions

① $f(z)$ is holomorphic @ Im Z > 0

② $f(z)$ is holomorphic @ $z \to i\infty$

③ ~~$f\left(\frac{az+b}{cz+d}\right) = f(z)$~~

Modular function only constant

k: weight

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

Automorphy factor

Modular group

Three matrices construct γ (Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \; f(z+1) = f(z)$$ $z \rightarrow z+1$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \; f\left(\frac{1}{-z}\right) = (-z)^k f(z)$$ $z \rightarrow -1/z$

$$I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : \; f\left(\frac{-z}{-1}\right) = (-1)^k f(z)$$ $\Rightarrow$ k=even

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$

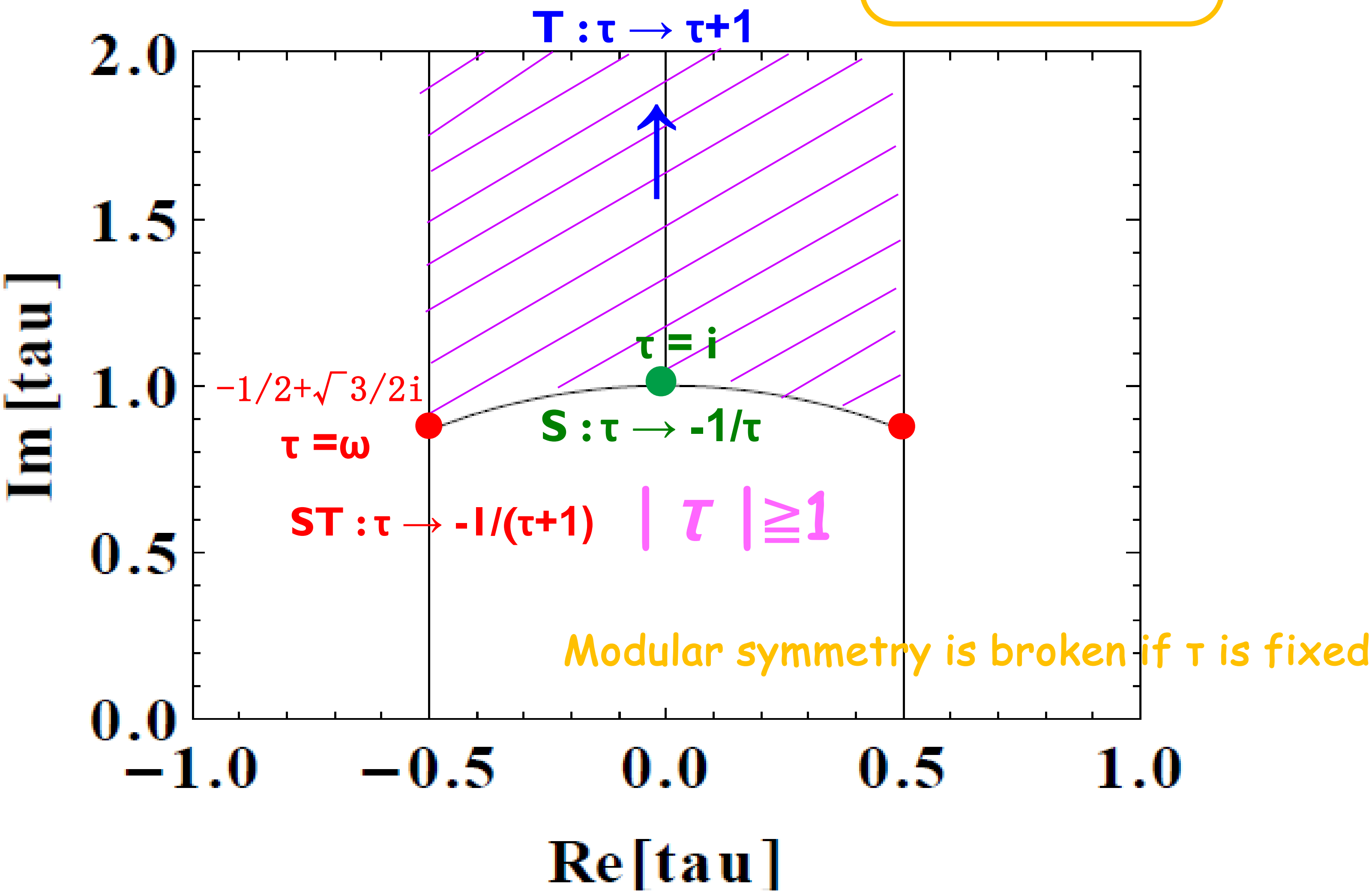
τ : modulus

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group PSL(2,Z)

Fundamental Domain of τ

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$



● ● **Fixed point of τ (Residual symmetry)**

Modular forms in favor

An example of mass matrix in terms of modular forms

$Y_3\, q^c\, Q_L\, H$

Modular form

like spurion

SM $y\, q^c\, Q_L\, H$

constant

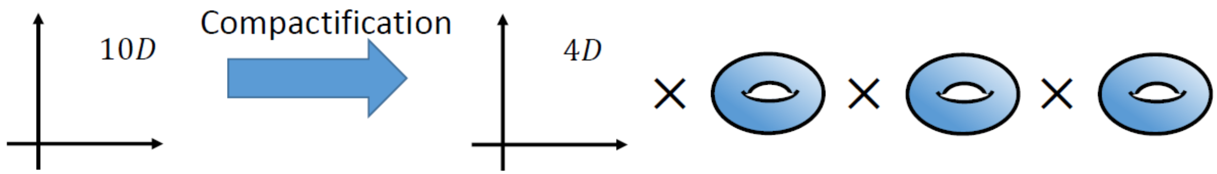
$$\mathbf{Y_3}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$M_q = \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

A triplet rep. of discrete group

$\mathbf{Y}_i$ are given by using

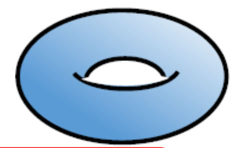
$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$



Modular forms appear naturally in top-down scenarios based on a class of string compactifications

We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \to \int d^4x \, \mathcal{L}_{\text{eff}}$$

$\Rightarrow$ $\mathcal{L}_{\text{eff}}$ depends on the structure of 

- $4D$ effective theory depends on internal space

Modular group **infinite group**

$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$

Modular group has subgroups

Γ_N **finite modular group of level N**

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_2 \simeq S_3 \quad \Gamma_3 \simeq A_4 \quad \Gamma_4 \simeq S_4 \quad \Gamma_5 \simeq A_5$

2 Modular forms for N=3

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_3 \simeq A_4$ group

Number of modular forms depend on weight k (even)

k+1 for A_4 (2k+1 for S_4)

For k=0, the modular form is constant (modular function)

For k=2, there are 3 linealy independent modular forms,

which form a A_4 triplet.

Modular transformation is the transformation of modulus τ

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}, \qquad T : \tau \longrightarrow \tau + 1.$$

weight 2; k=2

3 modular forms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

S transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

T transformation

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$(c\tau + d)^k$ cτ+d = -τ

$(c\tau + d)^k$ cτ+d = 1

$$\rho(\mathrm{S}) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(i\frac{2}{3}\pi)$$

Flavor symmetry acts non-linealy (Modular forms).

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular forms with weight 2

$$
\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned}
$$

$$Y_2^2 + 2Y_1Y_3 = 0$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Dedekind eta-function

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

Modular forms have hierarchy at nearby fixed points

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$$\varepsilon = 6\,|q|^{1/3}$$

$\tau \rightarrow \infty i$ $(Y_1, Y_2, Y_3)^T \longrightarrow (1, -\varepsilon, -1/2\,\varepsilon^2)^T$

A_4 triplet

$|\varepsilon| \ll 1$

Modular forms are hierarchical at $\tau = i\infty$ and ω !

$\tau = \omega$

$$ST = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Unitary transformation

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = Y_0 \begin{pmatrix} 1 \\ \omega \\ -\frac{1}{2}\omega^2 \end{pmatrix} \longrightarrow \mathbf{Y}_{\mathbf{3}}^{(2)} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

3 Mass hierarchy in modular invariance

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP 04(2021)206, arXiv:2102.07488

We can construct the mass matrix with hierarchical masses by using the hierarchical modular forms at nearby $\tau = \infty i$ and ω

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}_{RL}$$

This hierarchical structure is not accidental.
Thanks to Residual symmetry Z_3 (N=3)

F. Feruglio, V. Gherardi, A. Romanino,A. Titov,
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamurab,T. Kobayashi,
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamura ,T. Kobayashi

Modular invariance

$$M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau)\rho(\gamma)^\dagger \qquad K = k^c + k$$

τ=i∞ **ɤ=T: τ → τ+1** cτ+d = 1 $M_{ij}(T\tau) = (\rho_i^c\,\rho_j)^*\,M_{ij}(\tau)$

$$q \xrightarrow{T} q\,\xi$$

$$M_{ij}(\xi\ddot{q}) = (\rho_i^c\,\rho_j)^* M_{ij}(\ddot{q})$$

$q \equiv exp(i2\pi\tau/N)$ $\xi = \exp(i\,2\pi/N)$

n-th derivative

$$\xi^n M_{ij}^{(n)}(0) = (\rho_i^c\,\rho_j)^* M_{ij}^{(n)}(0)$$

$$M_{ij}(q) = a_0\,q^\ell + a_1\,q^{\ell+N} + a_2\,q^{\ell+2N} + \ldots\,, \qquad \ell = 0, 1, 2, \ldots, N-1,$$

For N=3 $M(\tau) \sim \mathcal{O}(\epsilon^\ell)$ $\ell = 0, 1, 2$ |q|= ε

Observed Yukawa ratios at GUT scale with tanβ=10

S. Antusch, V. Maurer, JHEP 1311 (2013) 115 [arXiv:1306.6879].

$$\frac{y_d}{y_b} = 9.21 \times 10^{-4}\ (1 \pm 0.111)\,, \qquad \frac{y_s}{y_b} = 1.82 \times 10^{-2}\ (1 \pm 0.055)$$

$$\frac{y_u}{y_t} = 5.39 \times 10^{-6}\ (1 \pm 0.311)\,, \qquad \frac{y_c}{y_t} = 2.80 \times 10^{-3}\ (1 \pm 0.043)$$

$$m_{b\,(t)} : m_{s\,(c)} : m_{d\,(u)} \sim 1 : |\epsilon| : |\epsilon|^2$$

For down quark sector $\varepsilon_d = 0.02 \sim 0.03$

For up quark sector $\varepsilon_u = 0.002 \sim 0.003$

We have only one parameter $|q| = \varepsilon$

4 A Model of Quark Mass Matrices with A_4 (N=3)

S.T.Petcov, M.Tanimoto, JHEP 08 (2023)086 [arXiv:2306.05730],
Eur. Phys. J. C 83(2023)579 [arXiv:2212.13336]

	Q	(d^c, s^c, b^c) , (u^c, c^c, t^c)		H_u	H_d
$SU(2)$	2	1		2	2
A_4	3	$(1', 1', 1')$	$(1', 1', 1')$	1	1
k	2	$(4, 2, 0)$	$(6, 2, 0)$	0	0

Irreducible representations

A_4 : 1, 1', 1", 3

Weight k is set to vanish

automorphy factor $(c\tau + d)^k$

$$W_d = \left[\alpha_d(\mathbf{Y_3^{(6)}}Q)_1 d_1^c + \alpha'_d(\mathbf{Y_{3'}^{(6)}}Q)_1 d_1^c + \beta_d(\mathbf{Y_3^{(4)}}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y_3^{(2)}}Q)_{1''} b_{1'}^c\right] H_d$$

$$M_d = v_d \begin{pmatrix} \hat\alpha'_d & 0 & 0 \\ 0 & \hat\beta_d & 0 \\ 0 & 0 & \hat\gamma_d \end{pmatrix} \begin{pmatrix} \tilde Y_3^{(6)} & \tilde Y_2^{(6)} & \tilde Y_1^{(6)} \\ \tilde Y_3^{(4)} & \tilde Y_2^{(4)} & \tilde Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \quad M_u = v_u \begin{pmatrix} \hat\alpha'_u & 0 & 0 \\ 0 & \hat\beta_u & 0 \\ 0 & 0 & \hat\gamma_u \end{pmatrix} \begin{pmatrix} \tilde Y_3^{(8)} & \tilde Y_2^{(8)} & \tilde Y_1^{(8)} \\ \tilde Y_3^{(4)} & \tilde Y_2^{(4)} & \tilde Y_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$\tilde Y_i^{(6)} = g_d Y_i^{(6)} + Y_i'^{(6)}, \qquad \tilde Y_i^{(8)} = f_u Y_i^{(8)} + Y_i'^{(8)}, \qquad g_d \equiv \alpha_d/\alpha'_d \qquad f_u \equiv \alpha_u/\alpha'_u$$

$$\mathrm{Det}\,[\mathcal{M}_u^2] = 0$$ due to $$\mathbf{Y}^{(8)} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y}^{(4)}$$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} , \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix} , \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} \equiv \begin{pmatrix} Y_1'^{(8)} \\ Y_2'^{(8)} \\ Y_3'^{(8)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_2^2 - Y_1 Y_3 \\ Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} = (Y_1^2 + 2Y_2 Y_3)\mathbf{Y}_{\mathbf{3}}^{(4)}$$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$q \equiv \exp\left(2i\pi\tau\right) = \left(p\,\epsilon\right)^3$$

$$\epsilon = \exp\left(-\frac{2}{3}\pi\,\mathrm{Im}\left[\tau\right]\right)\,, \qquad p = \exp\left(\frac{2}{3}\pi\, i\,\mathrm{Re}\left[\tau\right]\right)$$

τ=i∞

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0 \qquad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0$$

Superpotential

$$W_d = \left[\alpha_d(\mathbf{Y}_3^{(6)}Q)_1 d_1^c + \alpha'_d(\mathbf{Y}_{3'}^{(6)}Q)_1 d_1^c + \beta_d(\mathbf{Y}_3^{(4)}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y}_3^{(2)}Q)_{1''} b_{1'}^c\right] H_d$$

Kinetic terms

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

We renormalize superfields to get canonical kinetic terms

$$\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\,\psi^{(I)}$$

$$\alpha_u \to \hat{\alpha}_u = \alpha_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u\,(2\mathrm{Im}\tau)^4, \qquad \alpha'_u \to \hat{\alpha}'_u = \alpha'_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha'_u\,(2\mathrm{Im}\tau)^4\,,$$

$$\beta_u \to \hat{\beta}_u = \beta_u \sqrt{(2\mathrm{Im}\tau)^4} = \beta_u\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_u \to \hat{\gamma}_u = \gamma_u \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_u\,(2\mathrm{Im}\tau)\,,$$

$$\alpha_d \to \hat{\alpha}_d = \alpha_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d\,(2\mathrm{Im}\tau)^3, \qquad \alpha'_d \to \hat{\alpha}'_d = \alpha'_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha'_d\,(2\mathrm{Im}\tau)^3\,,$$

$$\beta_d \to \hat{\beta}_d = \beta_d \sqrt{(2\mathrm{Im}\tau)^4} = \beta_d\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_d \to \hat{\gamma}_d = \gamma_d \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_d\,(2\mathrm{Im}\tau)\,.$$

$2\mathrm{Im}\,\tau$ is large

Down type quark mass matrix

At τ=i∞

$$M_q = v_q \begin{pmatrix} g_q\,\hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{RL}$$

rank one

$$\mathcal{M}_q^{2(0)} \equiv M_q^\dagger M_q = v_q^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |g_q|^2\,\hat\alpha'^2_q + \hat\beta_q^2 + \hat\gamma_q^2 \end{pmatrix}$$

In the vicinity of τ=i∞ $\quad |\alpha'_q| \sim |\beta_q| \sim |\gamma_q| \qquad \hat\alpha'_q = \alpha'_q\,(2\mathrm{Im}\tau_q)^3$

$$\mathcal{M}_q = v_q \begin{pmatrix} \hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 18\,(\epsilon\,p)^2(4-g_q) & -6\,(\epsilon\,p)(2+g_q) & g_q \\ 54\,(\epsilon\,p)^2 & 6\,(\epsilon\,p) & 1 \\ -18\,(\epsilon\,p)^2 & -6\,(\epsilon\,p) & 1 \end{pmatrix}$$

$$\mathcal{M}_q^2 \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 p^* & \epsilon^2 p^{*2} \\ \epsilon^3 p & \epsilon^2 & \epsilon p^* \\ \epsilon^2 p^2 & \epsilon p & 1 \end{pmatrix}$$

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : \left|\frac{12\epsilon}{I_\tau g_q}\right| : \left|\frac{12\epsilon}{I_\tau g_q}\right|^2$$

$I_\tau = 2\mathrm{Im}\,\tau$

$g_q > \mathcal{O}(1)$

Up type quark mass matrix

In order to protect a massless quark, we can consider dimesuion 6 mass operator

$(u^c Q H_u)(H_u H_d)/\Lambda^2$ with $k_Q = 2 - k_{Hd}\,, \quad k_{u^c} = 6 + k_{Hd} - k_{Hu}$

or SUSY breaking by F term F/Λ^2

$$M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)}(1+C_{u1}) & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)}(1+C_{u2}) & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)}(1+C_{u3}) & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$m_t : m_c : m_u \simeq \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u$$

$$C_u = 3 f_u\,(C_{u1} - C_{u2}) + (-4C_{u1} + 3C_{u2} + C_{u3}) \qquad I_\tau = 2\mathrm{Im}\,\tau$$

I_τ is a overall normalization factor for canonical kinetic terms

A successful numerical result

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\|f_u\|$	$\arg[f_u]$	C_{u1}
$-0.3952 + i\,2.4039$	3.82	1.17	-0.677	1.72	3.21	1.68	$127.3°$	-0.07147

8 real parameters + 2 phase

!! Order 1 parameters, β_q/α_q , γ_q/α_q , g_d , f_u

$C_{u1} \sim (F/\Lambda^2)/(v_u\,\varepsilon^2)$

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\|V_{us}\|$	$\|V_{cb}\|$	$\|V_{ub}\|$	$\|J_{CP}\|$	δ_{CP}
Fit	1.89	8.78	2.81	5.52	0.2251	0.0390	0.00364	2.94×10^{-5}	$70.7°$
Exp 1σ	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	$66.2°$ $^{+3.4°}_{-3.6°}$

8 output $N\sigma = 2.0$

5 Summary

- Quark mass hierarchy is realized at nearby fixed point of $\tau=i\infty$ (and ω) thanks to the residual symmetry Z_3.

- Is Modulus τ common in both quarks and leptons ?
  One modulus or multi-modulei ?

- Spontaneous CP violation (origin of CP is τ) is challenging

Flavor theory with modular forms is developing !

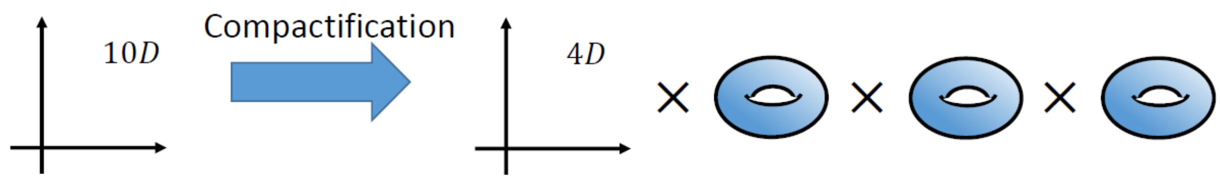
2 Modular Symmetry

2.1 Modular group

Superstring theory 10D
Our universe is 4D

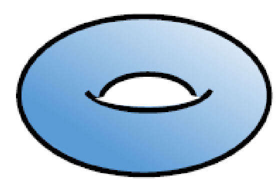
The extra 6D
should be compactified.

Torus compactification



We get 4D effective Lagrangian by integrating out over 6D.

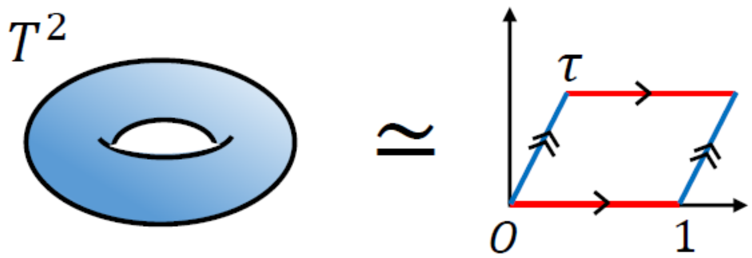
$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

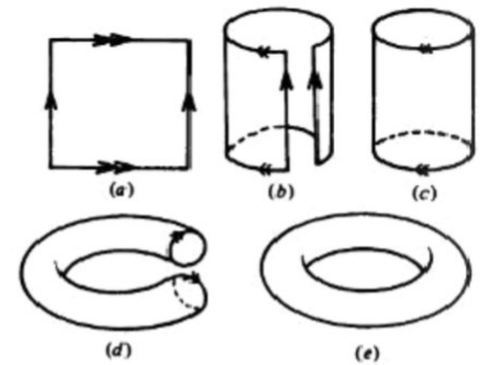
$\mathcal{L}_{\mathrm{eff}}$ depends on the structure of 

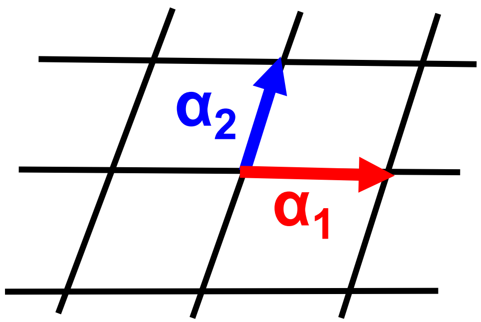
- $4D$ effective theory depends on internal space

$2D$ torus (T^2) is equivalent to
parallelogram with identification of confronted sides.

by Feruglio







$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice,
which is spanned by two lattice vectors

$\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

ad-bc=1
a,b,c,d are integer SL(2,Z)

The modular transformation is generated by S and T .

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

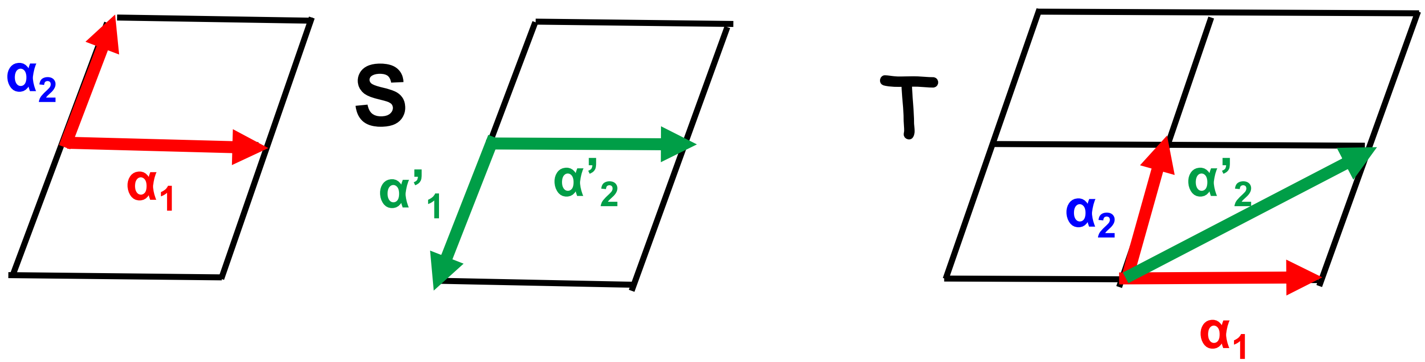
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$$\tau = \alpha_2 / \alpha_1$$

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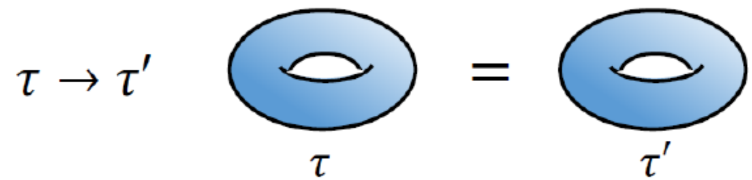
Modular transformation



Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

e.g.) $\mathcal{L}_{\rm eff} \supset Y(\tau)_{ij} \phi \overline{\psi_i} \psi_j$

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP 04(2021)206, arXiv:2102.07488

Hierarchical fermion mass matrices arise due to the proximity of the modulus τ to a fixed point, in which a residual symmetery remainds.

$$M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \qquad K = k^c + k$$

At τ=i ∞, mass matrix is invariant under T transformation (Z_N symmetry)

$$M_{ij}(T\tau) = (\rho_i^c \rho_j)^* M_{ij}(\tau)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

redefine $q \equiv exp(i2\pi\tau/N)$ ε=|q| $q \xrightarrow{T} \xi q$, with $\xi = exp(i\,2\pi/N)$

$$\xi^n M_{ij}^{(n)}(0) = (\rho_i^c \rho_j)^* M_{ij}^{(n)}(0) \quad \rightarrow \quad (\rho_i^c \rho_j)^* = \xi^n$$

ν_L ν_R

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}}$$

$$\oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}}$$

$$\oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}}.$$

symmetric × 3_Y anti-symmetric × 3_Y

We can consider effective theories with Γ_N symmetry.

$$\mathcal{L}_{\rm eff} \in f(\tau)\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

Modular form of Level N (N=2,3,4,5) (S_3, A_4, S_4, A_5)

$$\tau \longrightarrow \tau' = \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Modular transformation

Automorphy factor

$$f_i(\tau) \longrightarrow f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

modular form of weight k (Level N)

representation matrix of finite group

K is weight

Chiral superfields $(\phi^{(I)})_i(x) \longrightarrow (c\tau + d)^{-k_I}\rho(\gamma)_{ij}(\phi^{(I)})_j(x)$

$f_i(\tau)\ \phi^{(I)}\phi^{(J)}H$ Automorphy factor $(c\tau + d)^k(c\tau + d)^{-k_I}(c\tau + d)^{-k_J} = (c\tau + d)^{k-k_I-k_J}$

$\mathcal{L}_{\rm eff}$ is modular invariant if sum of weights satisfy $\sum k_I = k$.

Modular forms meet the flavor problem

Yukawa couplings (masses) are modular forms ?

Modular form

Holomorphic function of z ,

which under modular transformations $z \to \frac{az+b}{cz+d}$

obeys $f(z) \to f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

ad-bc=1
a,b,c,d are integer
SL(2,Z)