Foundations of Linear Electron Accelerators

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Outline

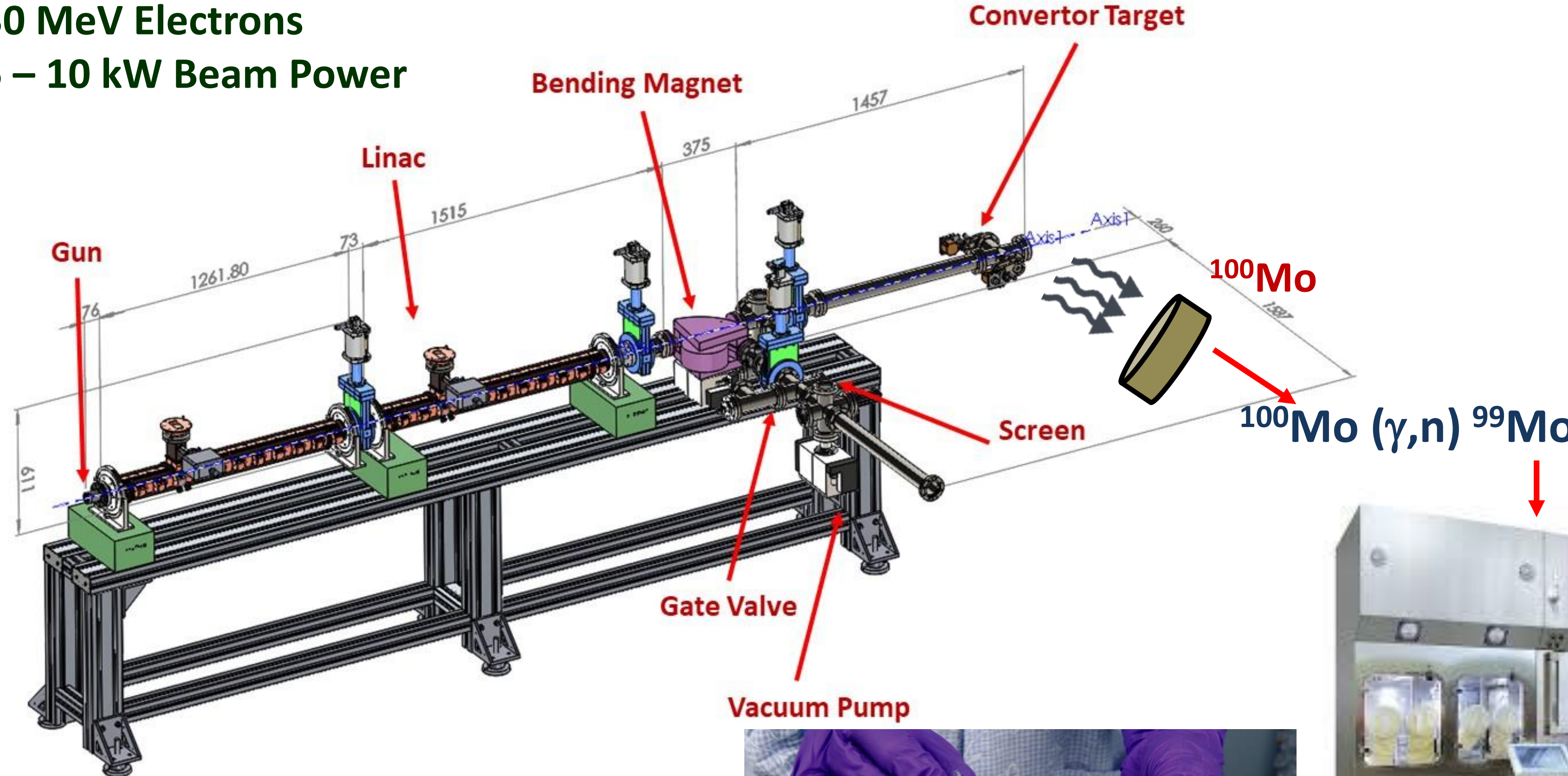
- Introduction
- Pill Box cavity
- Cavity design parameters
- Coupled linac
- Modes of linac
- Travelling Wave Accelerators
- Standing Wave Accelerators
- Longitudinal beam dynamics

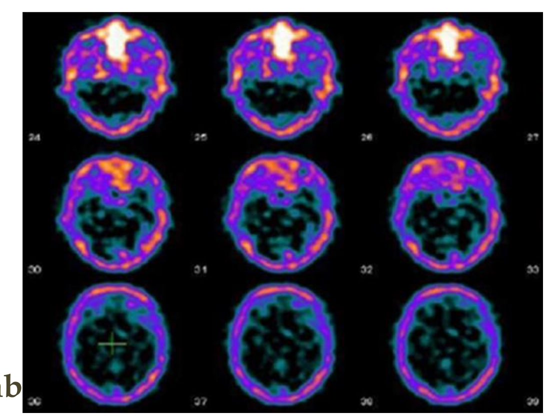
Radiotherapy System

Parameter	Value
Energy	6 MeV
Dose Rate	240 (flattened) Rads per min at 1 m
Beam current average	120 μA
R F Power	Magnetron 2.6 MW
Flatness	<2%
Symmetry	<3%
Field size	0x0 to 35x35 cm^2

LINAC BASED ^{99m}Tc GENERATION

30 MeV Electrons
5 – 10 kW Beam Power





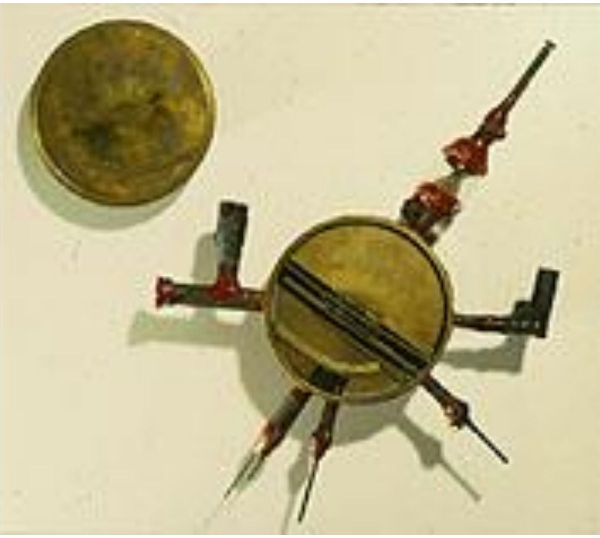
Introduction

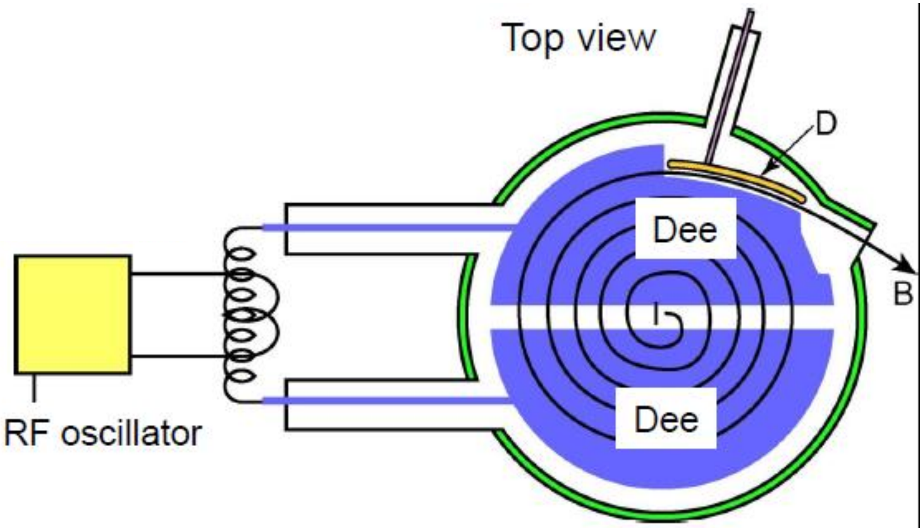
- The purpose of linac is to accelerate particles, for example electrons, to very high energy
- The role of accelerator designer is to achieve:
  - As high energy as possible for as small length as possible. This parameter can be quantified as gradient in MV/m
  - To achieve high gradient at as low input power as possible. Thus the RF to beam efficiency is important
  - The 'beam quality' should not be degraded or minimally degraded during linear acceleration
  - The structure should be easy to construct, easy to 'tune' and easy to 'use'. This needs elegant designing skills !
- Designer has to optimize on various fronts to ensure that the end goal is achieved

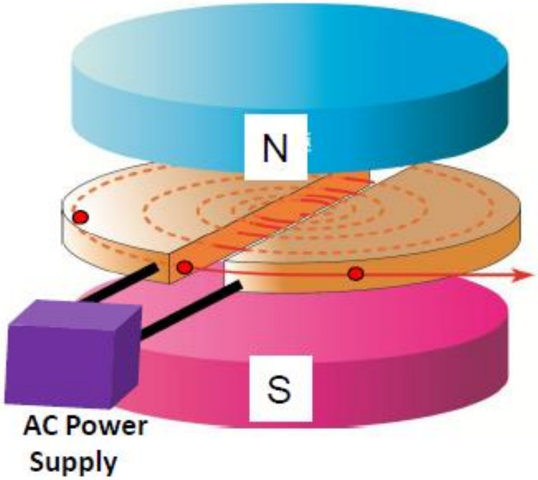
Variety of Accelerators !

Type	Energy	Gradient	Scheme
Cockcroft Walton	1 MeV	1 MV/m	DC
Van DeGraff	10 MeV	10 MV/m	DC
Cyclotron	100 MeV	1 MV/ Dee	Cyclic
Synchrotron	100 GeV	100 MV/turn	Cyclic
Linear	1 TeV	100 MV/m	Periodic
Plasma	10 TeV ?	> 10GV/m	Plasma

Circular: Cyclotrons



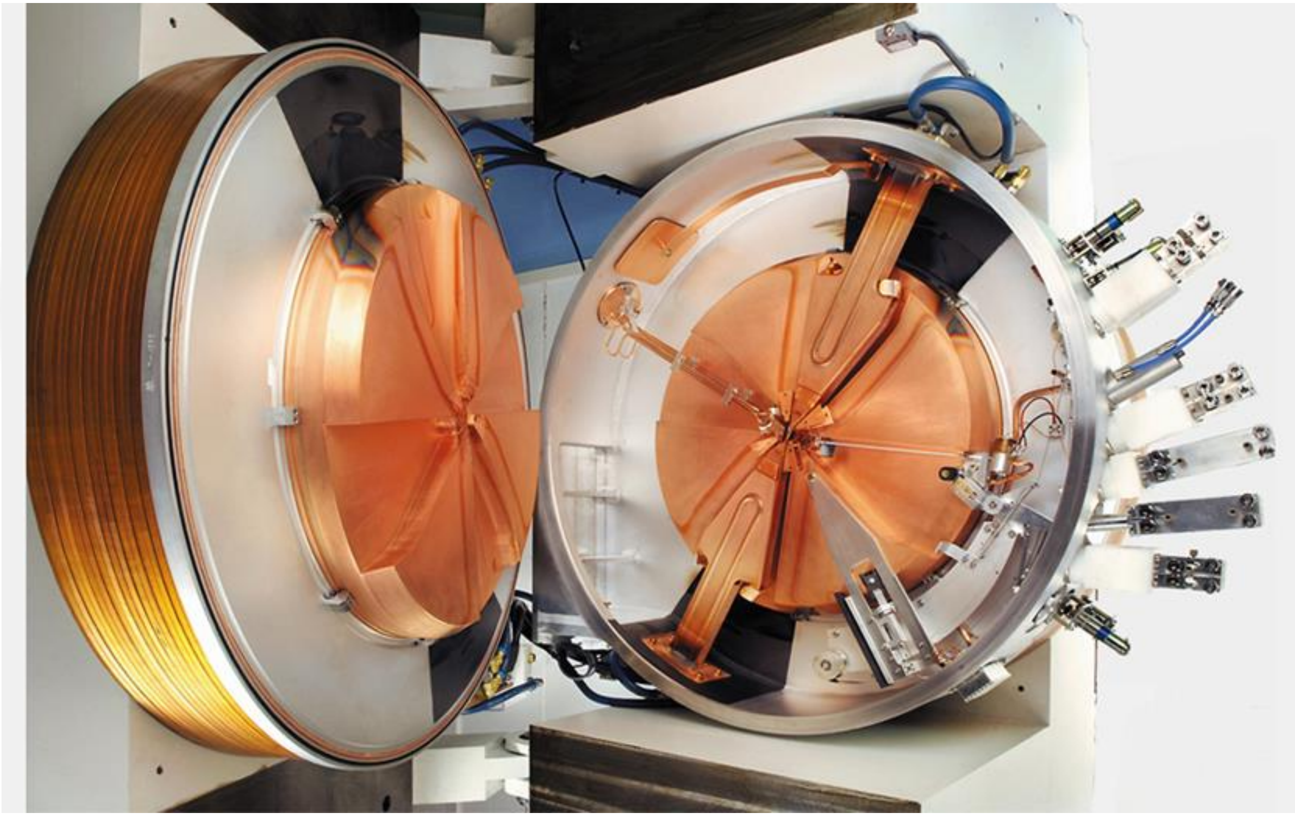




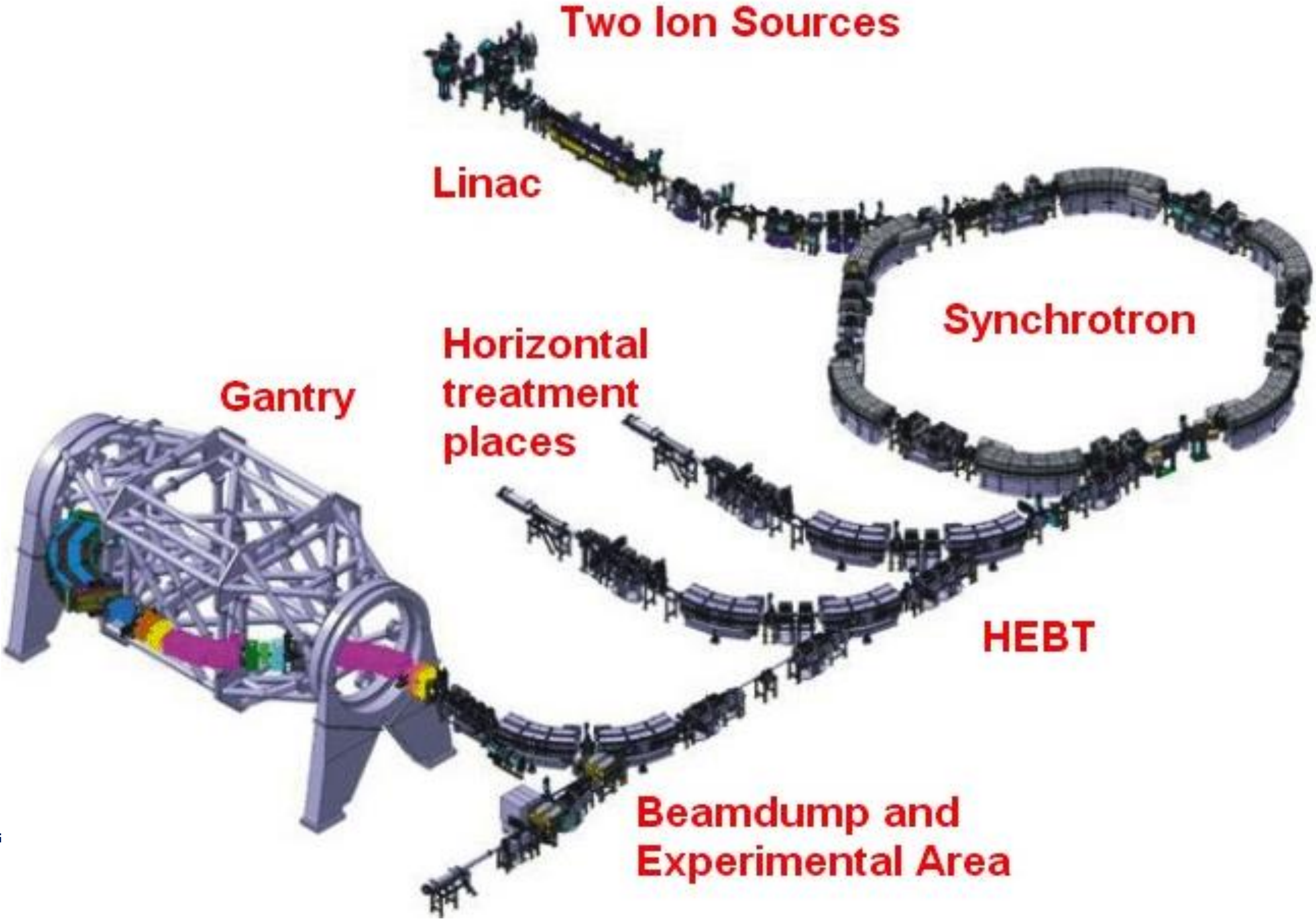
Slide courtesy: Prof. Ken Takayama, KEK

Circular: Cyclotrons

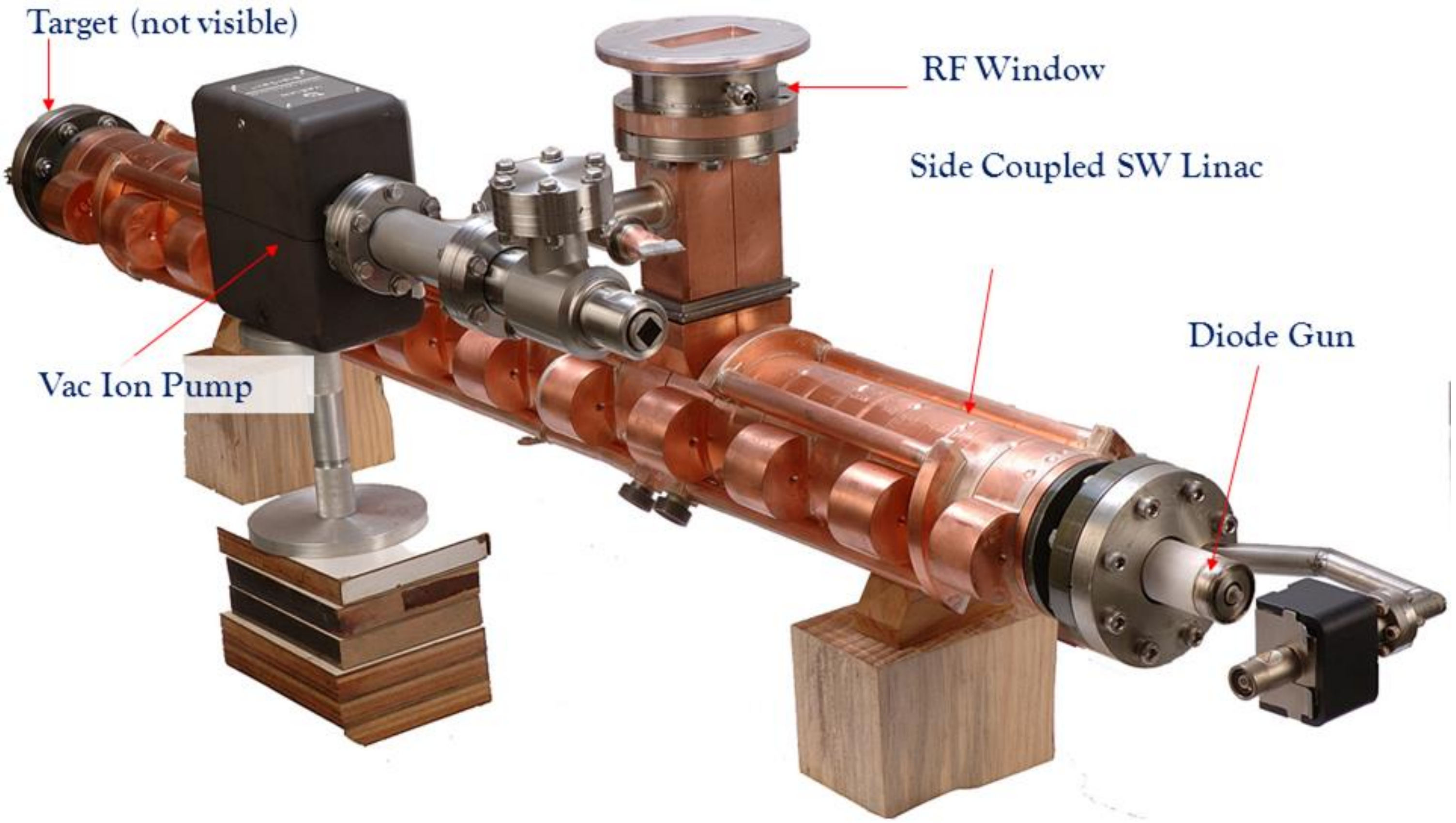
Commercial Cyclotron



Synchrotron



Linac

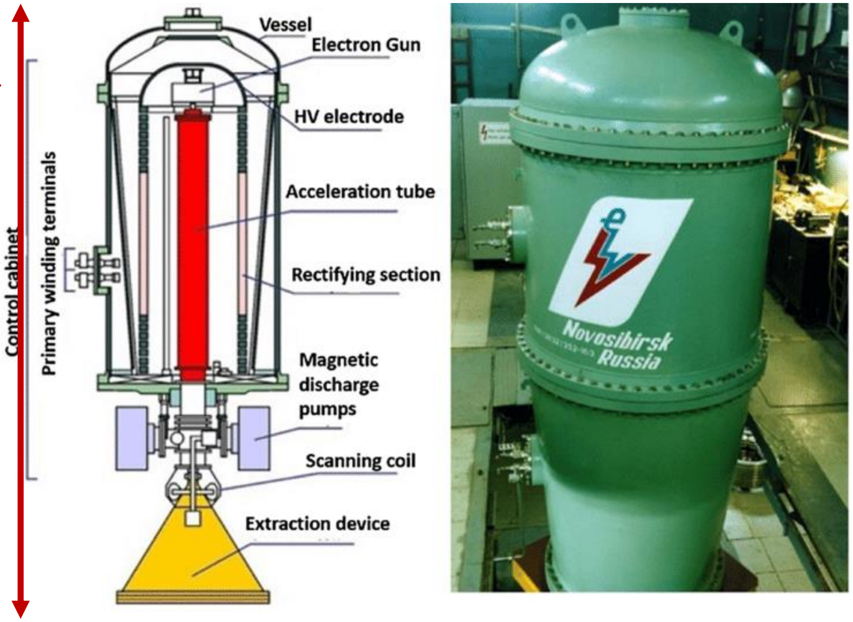

Target (not visible)
RF Window
Side Coupled SW Linac
Vac Ion Pump
Diode Gun

Linear Accelerator

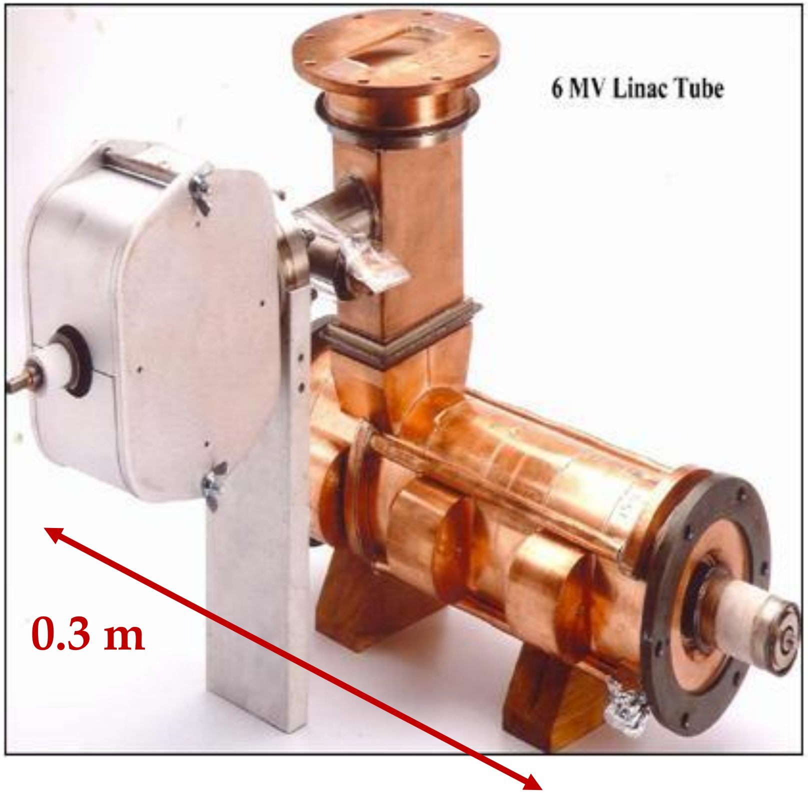
- The most basic mechanism is to use a DC potential to accelerate the particles.
- A high voltage DC potential is applied across a gap from which particles are passed. The particles can get accelerated and the maximum energy gain is limited by the maximum potential available across the gap.
- The DC acceleration is very slow acceleration scheme.
- For example, to achieve 1 MV you may need 1 MV potential.
- This can be achieved by passing particle through multiple gaps with achievable potential per gap

DC electron vs RF electron Linac

5 m



6 MeV ELV



0.3 m

Gradient, Current and issues

- For electrons to accelerate we need voltage across the gaps
  - For DC the voltage is a limiting factor
  - So we design RF structures which can offer high voltage across smaller gaps
- For RF structures, the beams get bunched and hence there is a limit to the current that one can draw.
- So DC structures are used for low energy high current applications where as the RF structures are used for high energy with low current applications
- To overcome the problem of high current applications in RF accelerators, novel methods were devised. This included CW accelerators and superconducting RF accelerators.

Gradient related issues

- Energy gain = Gradient x Length
- Hence, high gradient ensures small length per structure. Thereafter having multiple structures we can achieve high energy with less footprint
- Issues
  - To achieve high gradient, we need higher power from source. This is a big issue. Very high power sources are difficult to handle and create variety of problems.
  - RF breakdown can occur while using high power sources
  - Coupling the power from Source to Accelerator poses other major problem
  - Dark current are very high
  - Thermal issues may disturb the operation

Current and Applications

Application	Energy	Average Current	Beam Power
Food irradiation	6 MeV	2 mA	12 kW
Cancer Therapy	6 MeV	150 μA	0.9 kW
Water Purification	1 MeV	100 mA	100 kW
Isotope Generation	30 MeV	400 μA	12 kW

In contrast, the facilities like ATF (KEK) will accelerate not more than 1nC per bunch charge up to energy like 1.5 GeV. Light sources like Swiss Light Source will accelerate few tens of pC charge with 1 bunch configuration to 6 GeV energy.

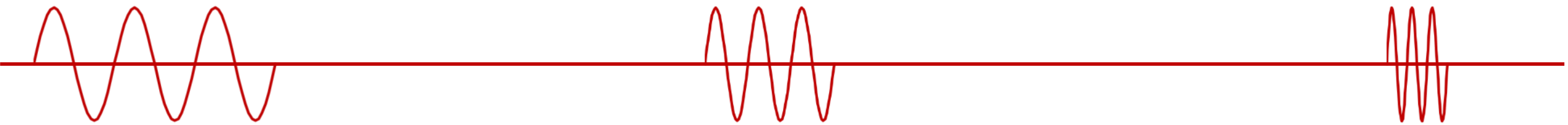
Hence, the choice of design depends strongly on the research or the application for which the accelerator is made.

Accelerator designers then try and optimize as many parameters as they can to ensure the researchers get the best outcome.

Let us focus on S-band Normal Conducting Linear Accelerator and evolve schemes to match various applications

Frequency Selection

S-band | **C-band** | **X-band**



F= 2998 / 2856 MHz
λ = 0.1 / 0.104 m
R_c = 38.27 / 40.17 mm

F= 5996 / 5712 MHz
λ = 0.05 / 0.052 m
R_c = 19.13 / 20.08 mm

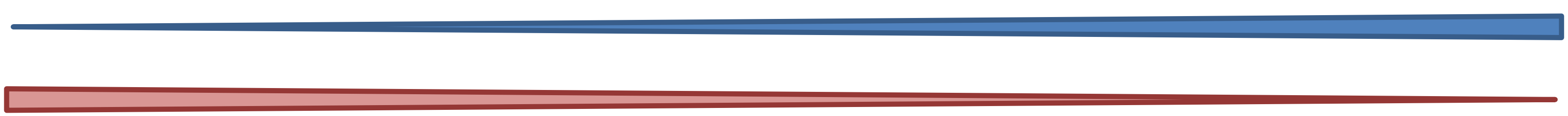
F= 11992/ 11424 MHz
λ = 0.025 / 0.026 m
R_c = 9.56 / 10.044 mm

$$R_c = \frac{J' c}{2\pi f}$$

Where J′ is modified Bessel and for TM_{01} it is 2.405

Gradient

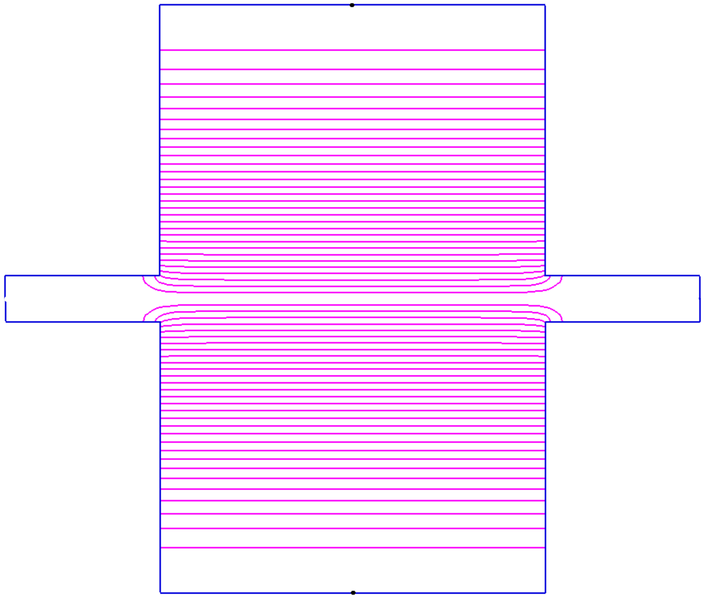
Size

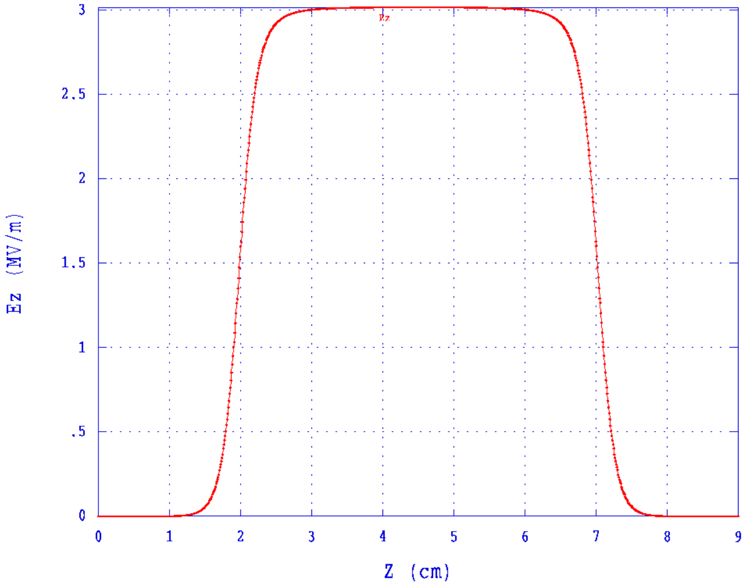


PILL BOX CAVITY

Pill Box cavity

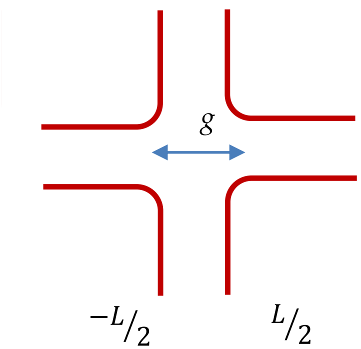
- Cavity is a hollow structure with a particular frequency f
- RF fields are passed in the cavity which form a field pattern inside
- Particles can borrow some energy from the fields and energize themselves to higher energies !
- A typical Pill Box structure is shown below:

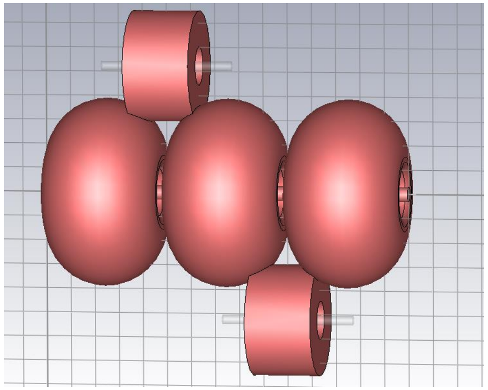


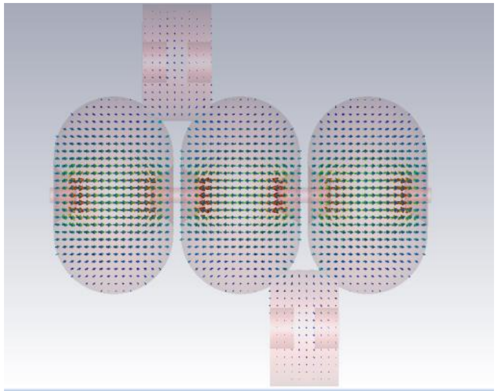


Transit Time Factor

- In a cavity, if the gap is narrow and the beam aperture opening is very small then the field will be almost same for the particles travelling across the gap.
- If the gap increases, then the particle sees varying field along its path. Therefore it sees less of the 'peak field' and subsequently gets lesser acceleration.
- Transit time factor accounts for this reduction due to variable field across the gap







Transit time (contd)

- Let the field in the gap vary with frequency *f*

$$\boldsymbol{E(r=0,z,t)=E(r=0,z)\cos(wt+\varphi)}$$

- We will choose it so that the particle crosses the center of the gap at t=0 where the field is at a maximum value.
- For a particle traveling at a velocity β, and the field varies with angular frequency ω,

$$\boldsymbol{z=\beta ct \quad \lambda=\frac{c}{f}=\frac{2\pi c}{\omega}}$$

Hence,

$$\boldsymbol{E(r,z,t)=E(r,z)\cos\left(\frac{wz}{\beta c}+\varphi\right)=E(r,z)\cos\left(\frac{2\pi z}{\beta\lambda}+\varphi\right)}$$

Transit time (contd)

Then energy gain in the gap will be

$$\Delta W = q\int_{-L/2}^{L/2} E(r,z)\cos\left(\frac{wz}{\beta c}+\varphi\right)dz$$

If we use:

$E(z)= V_0/L$ when $-L/2 < z < L/2$

$E(z)= 0$ for all other regions

$$\Rightarrow \Delta W = q\int_{-L/2}^{L/2} \frac{V}{g}\cos\left(\frac{wz}{\beta c}\right)dz = qV_0\underbrace{\frac{\sin(\pi L/\beta\lambda)}{\pi L/\beta\lambda}}\cos\varphi$$

$$= qV_0 T\cos\varphi$$

Where we define T as Transit time factor.

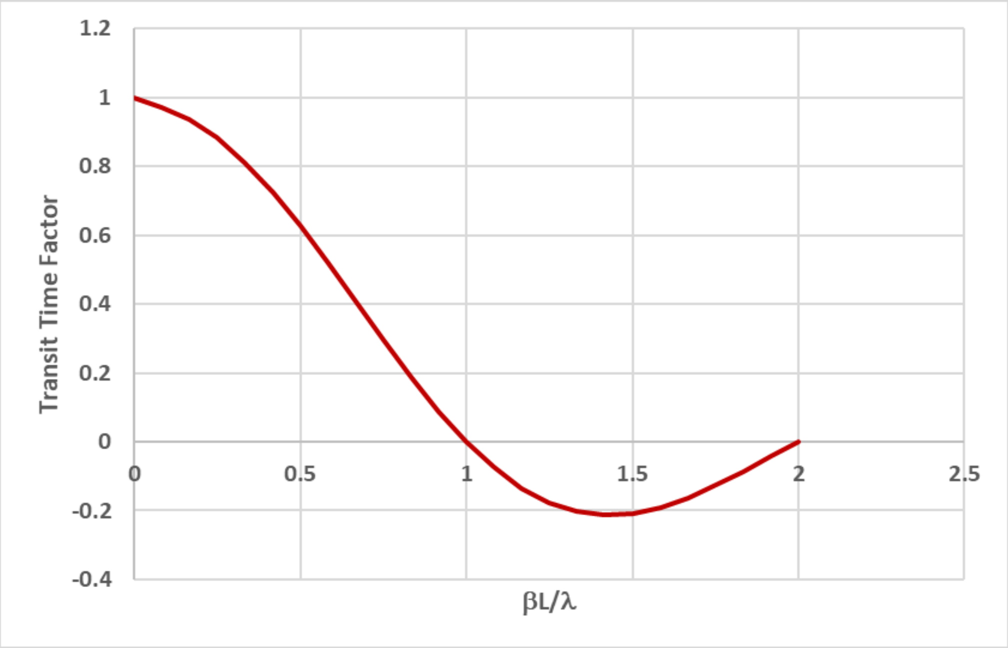
In general it can be expressed as: $T = \dfrac{\int_{-L/2}^{L/2} E(0,z)\cos\left(\frac{wz}{\beta c}\right)dz}{\int_{-L/2}^{L/2} E(0,z)dz}$

Transit Time Factor (contd)

As seen before

$$\Delta W = qV_0 T\cos\varphi$$

As $E_0 = \frac{V_0}{L}$ => $\Delta W = qE_0 T\cos\varphi L$



As seen V_0 is the voltage gain of a particle passing through a constant DC field equal in the gap at time t = 0. The value of average field E_0 depends on length L.

E_0T is thus the accelerating gradient.

Also, note that the energy gain in a RF field is always less than the corresponding DC field at the center of the gap. This is how the Transit Time factor is manifested

Skin Depth

- At high frequencies, current passing through conductor is seen concentrating near surface. We define skin depth at the depth at which the current density reduces to 1/e or ~37% of original value

$$\boldsymbol{\delta = \sqrt{\frac{1}{\pi\sigma\mu_0 f}} = \sqrt{\frac{2}{\sigma\mu_0\omega}}}$$

where μ_0 is the permeability in free space and is given as
$\mu_0 = 1.25663706 \times 10^{-6}$ m kg/ s^{-2} A^{-2}

- The skin depth causes losses in the cavity
- For a good conductor like Copper, the resistivity is very low. From the datasheet we find **$\rho = 1.7 \times 10^{-6}$ Ω-m**.

Skin Depth

Hence $\boldsymbol{\sigma = 1/\rho} = 5.814 \times \mathbf{10^7}$

$$\boldsymbol{\delta} = \sqrt{\frac{\mathbf{1}}{\boldsymbol{\pi\sigma\mu_0 f}}} = \frac{\mathbf{0.066}}{\sqrt{\mathbf{f}}}$$

At frequency of 3 GHz, this means **$\boldsymbol{\delta}$= 1.2055 microns**

Next we define Surface Resistance $\boldsymbol{R_s = {}^{1}/_{\sigma\delta} = \sqrt{\frac{\mu_0 w}{2\sigma}}}$

Hence, at 3GHz, $\mathbf{R_s = 14.272\ m\Omega}$

Surface Resistance for SC structure

We can calculate Rs for superconducting structure as:

$$R_s(\Omega) = 9 \times 10^{-5} \frac{f^2(GHz)}{T(o_K)} exp\left[-\alpha \frac{T_c}{T}\right] + R_{res}$$

Hence at 1.3 GHz and 2 K temperature. T_c is 9K, α=1.83

$\Rightarrow$ **R_s= 20.16 n Ω**

Clearly, the surface resistance in this case scales as square of frequency.

Quality Factor Q

- Quality Factor 'Q' is a measure of merit of RF Cavity as resonator. It can be expressed as: $\boldsymbol{Q = \frac{wU}{\left(\frac{-dP}{dz}\right)}}$
  - Where U is the stored energy
  - w is the frequency and P is the power dissipated per unit length
- The Power loss $\boldsymbol{P_c = \frac{R_s}{2}\int |H^2| ds}$

Where as the stored energy goes as Volume Integral

$$\boldsymbol{U = \frac{\mu_0}{2}\int |H^2| dv = \frac{\varepsilon_0}{2}\int |E^2| dv}$$

- Hence $\boldsymbol{Q = \frac{\omega \frac{\mu_0}{2}\int |H^2| dv}{\frac{R_s}{2}\int |H^2| ds} = \frac{\omega\, \mu_0 \int |H^2| dv}{R_s \int |H^2| ds} = \frac{G}{R_s}}$

G termed as Geometrical Factor.

Therefore Q can be enhanced by reducing the surface losses or by increasing the geometrical factor

Quality Factor Q

- **ILC type SC cavity:**
  - Wall losses are essentially surface integrals:

$$P_c = \frac{R_s}{2}\int |H^2| ds$$

  - Hence to reduce the losses, H should be optimized
  - This can be done by choice of larger outer diameter and choice of Geometry
- **NC Disc loaded structure:**
  - Q can be increased by proper choice of geometry
  - In reality by maintaining good surface finish it can be optimized
  - As it depends on frequency, choice of frequency also makes a difference

Shunt Impedance

We define Shunt Impedance as :

$$r_s = -\frac{V_0^2}{P}$$

– Where V_0 is the axial voltage gain and P indicates the RF power dissipated in the structure

- It can be shown that the shunt impedance scales as : $r_s \propto w^{1/2}$
  Hence, for conserving RF power higher frequency is desirable
- Naturally for given structure one desires as much high energy as possible and hence the dissipated power should be less.
- Thus if the shunt impedance is higher, it means more energy for same source.

Shunt Impedance (contd)

As seen before, maximum energy gain can occur at ϕ =0 and there is the factor of transit time as well,

Knowing: $\boldsymbol{E_0 L = V_0}$

$$r = \frac{{V_0}^2 T^2}{P} = r_s T^2$$

- We can also define shunt impedance per unit length

$$Z = \frac{r_s}{L} = \frac{{V_o}^2}{PL} = \frac{{E_o}^2}{P/L}$$

And finally effective shunt impedance per unit length

$$ZT^2 = \frac{E_0^2 T^2}{P/L} = \frac{(V_0 T)^2}{PL}$$

R over Q

We can define r/Q as

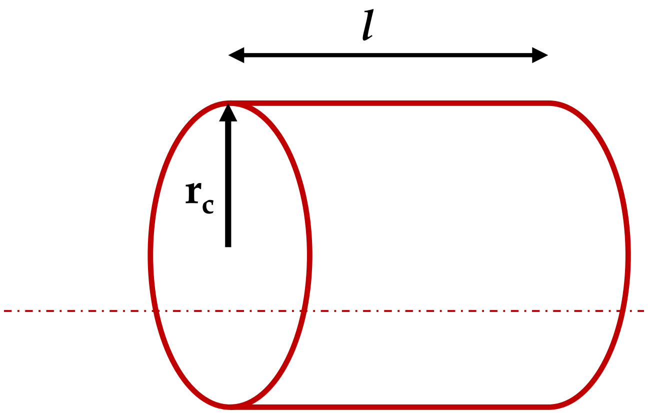
$$\frac{\boldsymbol{r}}{\boldsymbol{Q}} = \frac{(\boldsymbol{V_0 T})^2}{\boldsymbol{\omega U}}$$

r/Q is measure of efficiency of acceleration and it is a function of cavity geometry only.

Re-visit to Pill Box

Now, lets make use of Pill Box Cavity to further our discussion.

The cavity is made by placing conducting Plates at **z=0** and ***z=l*** and having radius $\mathbf{r_c}$ The wave equation in cylindrical coordinates is given as:



$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{\partial^2 E_z}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

The non zero solution is:

$$E_z = E_o J_0(k_r r)\cos(\omega t) \qquad B_\theta = -\frac{E_0}{c} J_1(k_r r)\sin(\omega t)$$

Resonant Frequency is given as $\omega = k_r c = \frac{2.405c}{r_c}$

Frequency and Radius

Band	Frequency (GHz)	r_c (mm)	length (cm)
X	11.992	9.569	2.5
	9.43	12.169	3.179
C	5.996	19.138	5
	5.712	20.089	5.248
S	2.998	38.276	10
	2.856	40.179	10.49
L	1.3	88.272	23.0

Pill box analysis (contd)

To find out the accelerating voltage we assume that the speed of a particle is near to that of light, and it starts at ***t=0*** and leaves the cavity at ***t=d/c***. To see maximum gain, the time taken should be equal to half RF period.

$$t = \frac{d}{c} = \frac{t_{RF}}{2} = \frac{\pi}{\omega}$$

If v~ c; then:

$$V_{acc} = \int_{0}^{z=d} E_z(r=0,z)\exp\left(\frac{i\omega z}{c}\right)dz$$

$$V_{acc} = dE_0 \frac{\sin\left({}^{\omega d}/_{2c}\right)}{{}^{\omega d}/_{2c}} = \mathrm{d}E_0 T$$

At maximum

$$V_{acc} = \frac{2dE_0}{\pi}$$

Pill box analysis (contd)

Thus: $\mathbf{E_{acc} = \frac{V_{acc}}{d} = \frac{2E_0}{\pi}}$

In order to get maximum accelerating field, peak fields should be minimized. For TM_{010} mode, the peak field is

$$E_{pk} = E_0$$

And

$$\mathbf{H_{peak} = \sqrt{\frac{\varepsilon_0}{\mu_0}} J_1(1.841) E_0 = \frac{E_0}{647}}$$

$$\frac{E_{peak}}{E_{acc}} = \frac{\pi}{2} = 1.6 \text{ and } \frac{H_{peak}}{E_{acc}} = 2430 \quad \frac{A}{MV}$$

Pill box analysis (contd)

As seen earlier, $\mathbf{Q} = \frac{\mathbf{G}}{\mathbf{R_s}}$

For Pill box like cavity, G is 257 and at 3 GHz R_s was 14.272 mΩ and hence **Q=18007**

For SC cavity, R_s was 20 nΩ and hence **Q=1.285x10^{10}**

For a cavity with 3 GHz frequency cavity length= 5 cm and cavity radius = 3.82 cm. If we assume accelerating voltage as 1 MV, then

$$E_{acc} = 20\frac{\mathrm{MV}}{\mathrm{m}}$$

$$E_{peak} = 31.4MV/m$$

Pill box analysis (contd)

Further Stored energy is calculated as: $\boldsymbol{U = \frac{\pi\varepsilon_0 l r_c^2}{2} E_o^0 J_1^2(2.405)}$

Maximum $\mathbf{J_1(2.405) = 0.5191}$

Therefore, Stored Energy **U = 0.27 J**

Power dissipated: $Q = \frac{\omega U}{P}$ => **P= 284 kW**

For SC case: **P = 0.4 W**

RF WAVES IN CAVITIES

Review of accelerators

- The two essential conditions for acceleration are:
  - There should be an electric component along the direction of particle motion
  - This is achieved by passing the waves in uniform waveguides
  - The particle and wave should maintain synchronism
  - For a uniform infinite waveguide, this is not possible as phase velocity will be greater than velocity of light.
  - However, if discs with specific periodicity are introduced in the waveguide, then the waves slow down thereby making is possible to maintain synchronism
- In cylindrical system the electric field can be described as :

$$\boldsymbol{E_z = E_0 J_0 (k_c r) exp[j(\omega t - \beta z)]}$$

Where J_0 is the Bessel function, ω_c is cut-off frequency,

$$\boldsymbol{k = {}^{\omega}/_{c}\, ; \quad k_c = {}^{\omega_c}/_{c} \quad ; \quad \beta = {}^{\omega}/_{v_p}}$$

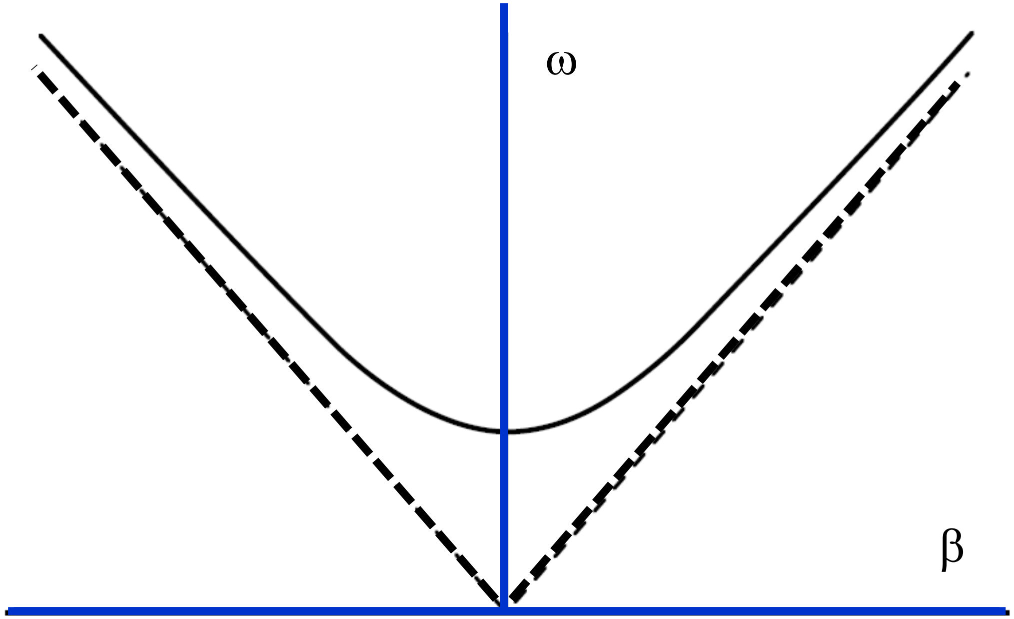
Dispersion

The dispersion relation is given as $\boldsymbol{\beta^2 = k^2 - k_c^2}$

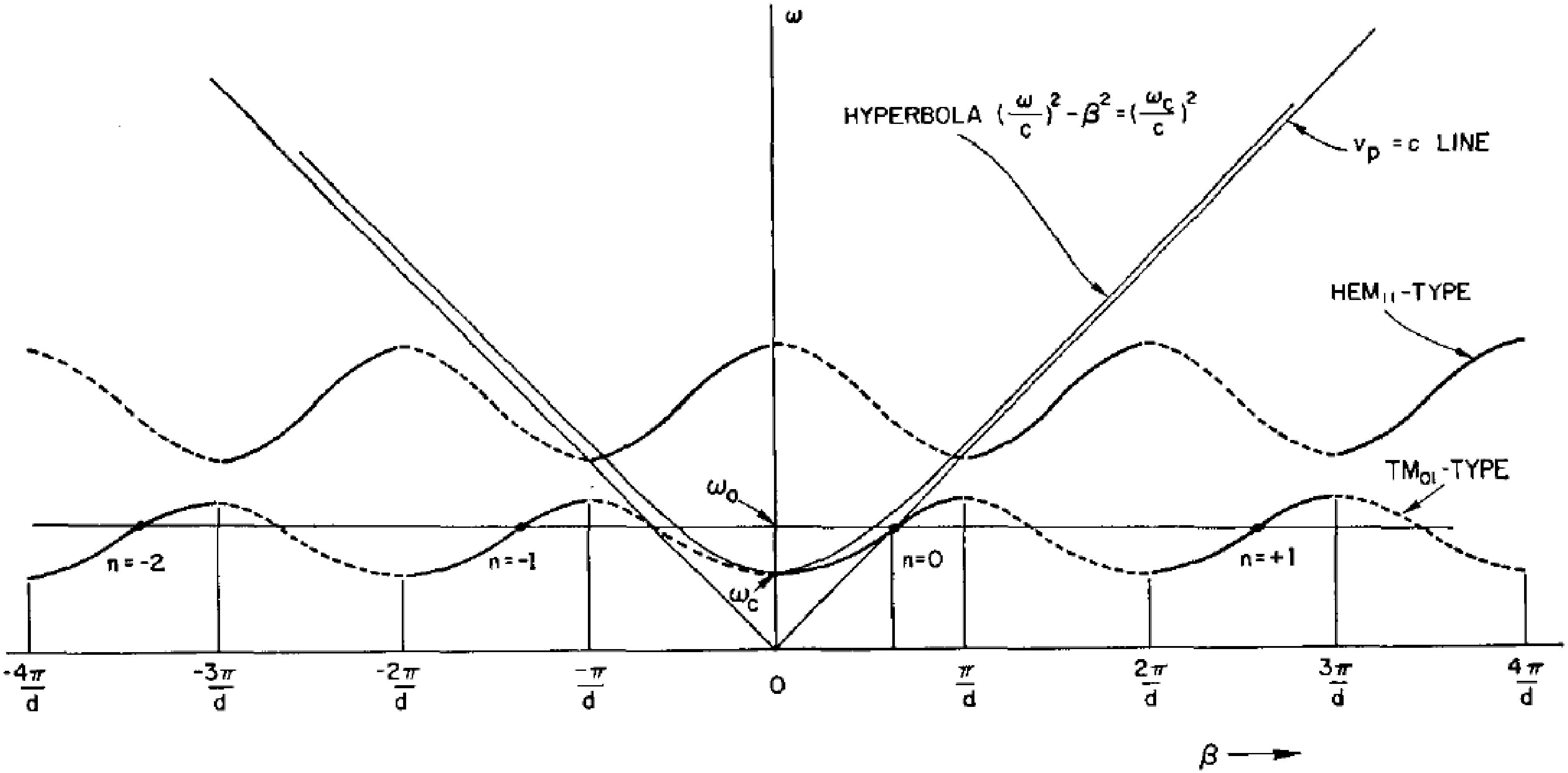
Hence:

$$\mathbf{v_p} = \frac{\mathbf{c}}{\sqrt{\mathbf{1} - (\frac{\boldsymbol{\omega_c^2}}{\boldsymbol{\omega^2}})}}$$

For an infinite long structure this can be true only if $\boldsymbol{v_p > c}$

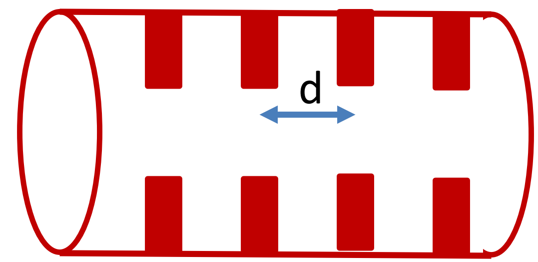


Dispersion (contd)



Slowing down of phase velocity

By adding 'discs' we slow down the wave:



$$\mathbf{E_z} = \sum_{n=-\infty}^{n=\infty} \mathbf{a_n J_0(k_{rn} r) \exp[j(\omega t - \beta_n z)]}$$

$$\beta_n = \beta_0 + \frac{2\pi n}{d}$$

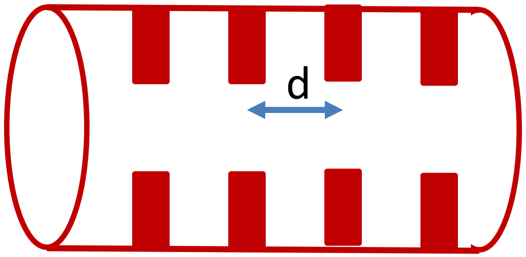
$$k_{rn}^2 = k^2 - \beta_n^2$$

If d=half wavelength, then reflection from each obstacle add up. These gaps slow down the phase velocity making real structures possible.

Floquet's Theorem

Floquet's Theorem:

For a given mode of an propagation at a given steady state frequency, the fields at two different cross sections that are separated by one period differ only by a complex constant factor



Consequences:

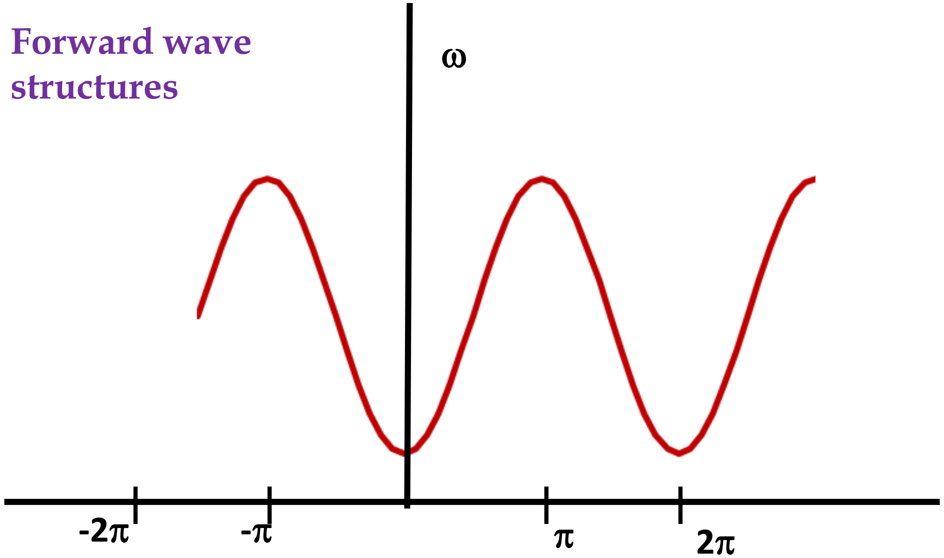
- Phase shift for fundamental mode n=0 is $\beta_0 d$ and for any other space harmonics it differs by integral number 2π
- Thus a_{-1} the term with the second largest amplitude will have a phase shift $\beta_{-1}d = \beta_0 d - 2\pi$ and a_{+1} the term with the third largest amplitude will have phase shift $\beta_{+1}d = \beta_0 d + 2\pi$
- As seen in plot earlier, at a given frequency ω_0 the slopes of are equal which implies that all space harmonics propagate with the same group velocity, ${}^{d\omega}/_{d\beta}$, for a given ω_0
- Each harmonic, however, has a different phase velocity and only one is made synchronous with the electrons and acts cumulatively during the acceleration process.

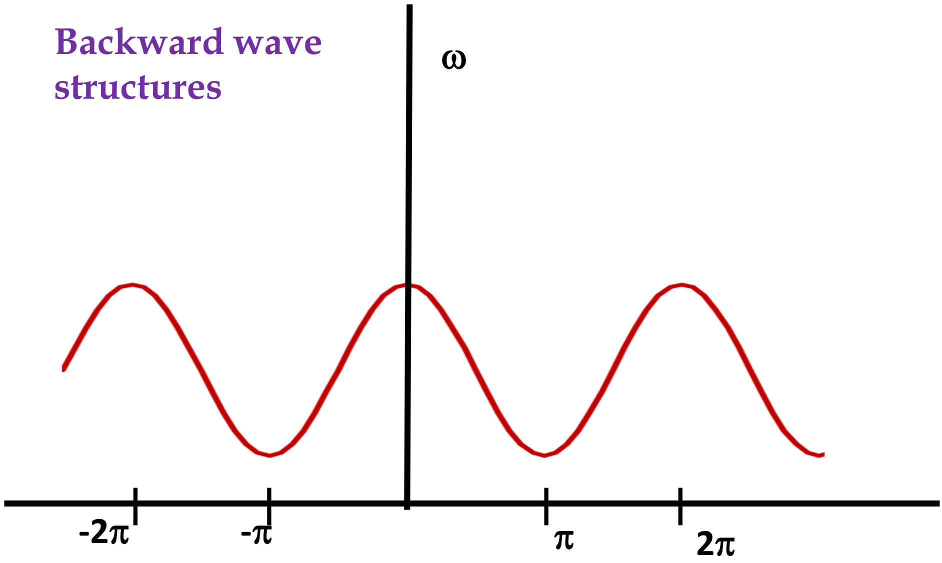
Forward and Backward Waves

Consequences:

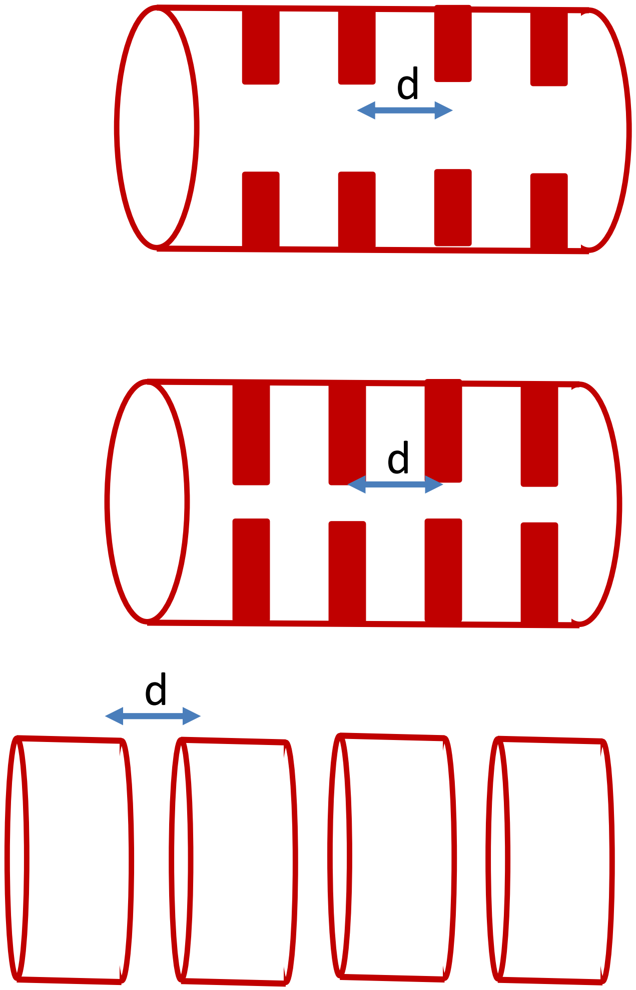
- Negative branches have positive group velocity but negative phase velocity.
  - At n=0, group velocity and phase velocity, both are positive. Such structures are forward wave structures.
  - If the signs are opposite, they are termed as backward wave structures

- At n=0, $v_p = c$; $\beta_0 = k$ and $k_{r0} = 0$. Hence, $J_0(k_{r0})$ is independent of r and Ez has no radial variations.
  - Thus, even for a beam of finite radius the acceleration seen is same.





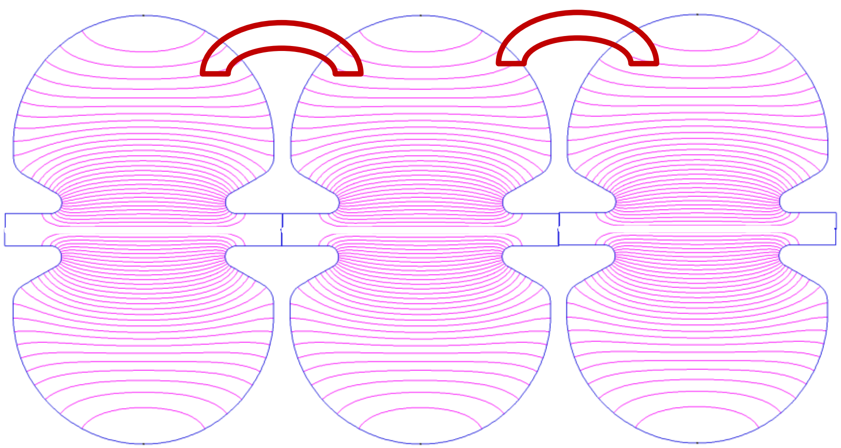
Disc loaded structures

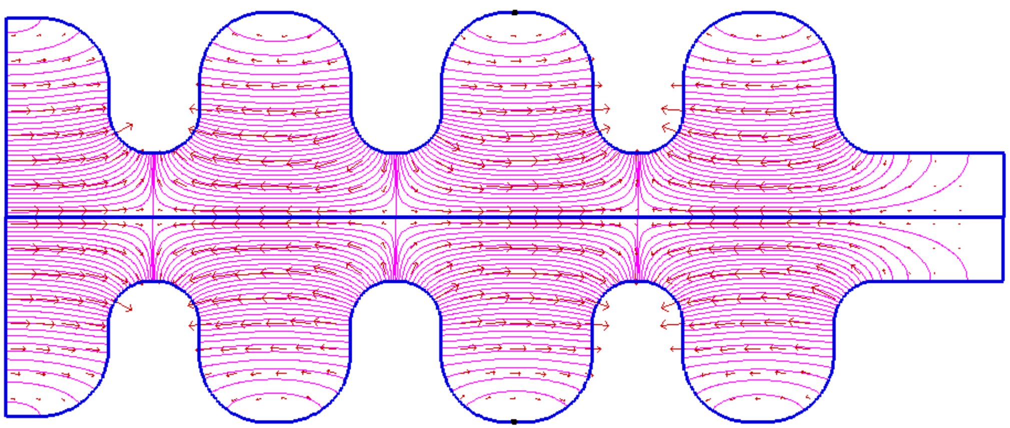


CIRCUIT EQUIVALENT FOR CAVITIES

Coupled cavities

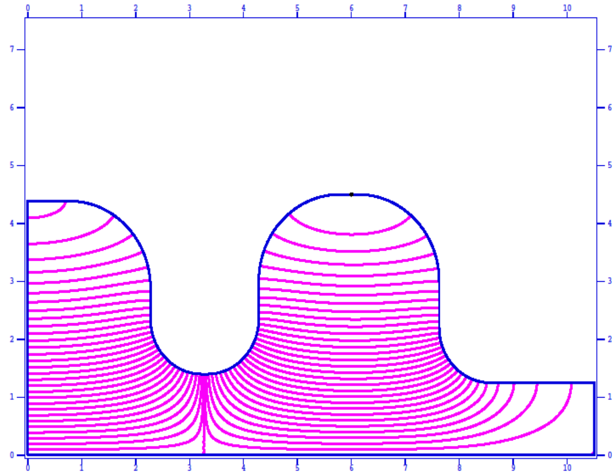
- Once a desired cavity shape is optimized and the shunt impedance is maximized, next stage the designer has to wonder how to enhance energy gain per meter.
- One of the easiest way is to couple multiple cells together and achieve successive acceleration along the entire length
- Such a structure is coupled cell structure

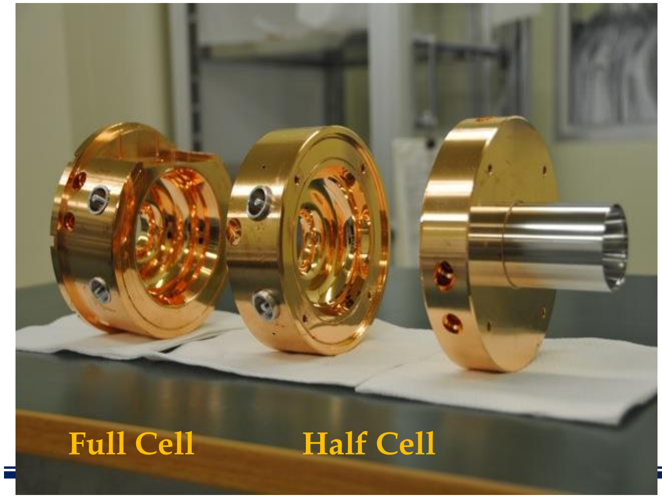


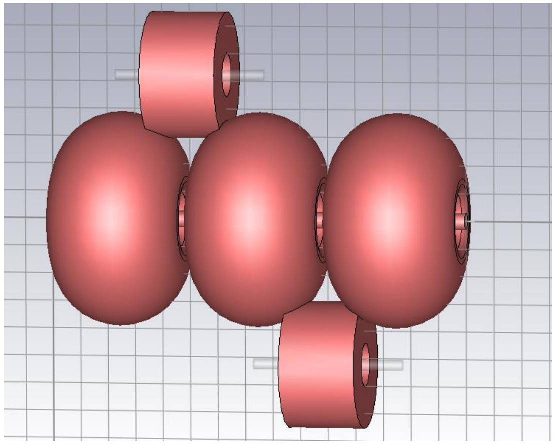


Coupled cavities

- For coupled cavities, a coupling mechanism is needed
- The central beam aperture can be increased and thereby the cavities can be coupled. This is a weak coupling scheme.
- The advantage being that distortions due to coupling apertures is avoided and the beam will get least disturbed
- RF gun, ILC linac structures are of this type
- Other option is to introduce coupling slots at edges where E field is minimum.
- High coupling values are possible

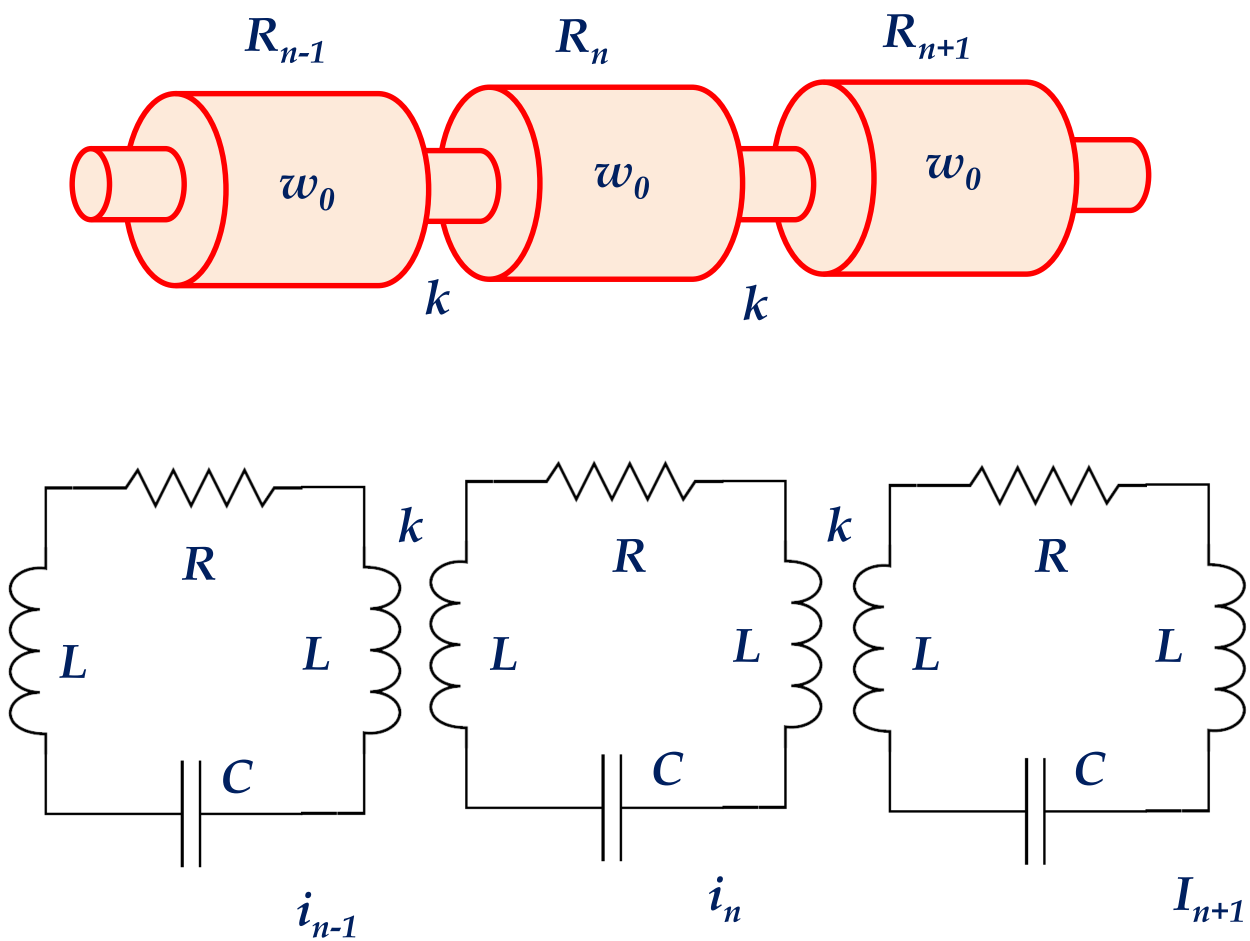






Circuit Model

The cavities can be seen as circuits coupled through a coupling constant k.



Circuit Model

Then the circuit equations can be written as:

$$I_0 = X_0\left(1 + \frac{\omega_0}{j\omega Q} - \frac{\omega_0^2}{\omega^2}\right) + kX_1$$

$$I_n = X_n\left(1 + \frac{\omega_0}{j\omega Q} - \frac{\omega_0^2}{\omega^2}\right) + \frac{k}{2}(X_{n-1} + X_{n+1})$$

$$I_N = X_N\left(1 + \frac{\omega_0}{j\omega Q} - \frac{\omega_0^2}{\omega^2}\right) + kX_{N-1}$$

Where

$$X_n^{(q)} = (const)\cos\left(\frac{\pi q n}{N}\right)\exp(j\omega q t)$$

&

$$\omega_q^2 = \frac{\omega_0^2}{1 + k\cos\left({}^{\pi q}/_{N}\right)}$$

Circuit Model

For 3 cell structure: q= 0, 1, 2 and N = 2

$$\boldsymbol{\omega_0 = \frac{\omega_0}{\sqrt{1+k}}} \qquad \boldsymbol{\omega_{\pi/2} = \omega_0} \qquad \boldsymbol{\omega_\pi = \frac{\omega_0}{\sqrt{1-k}}}$$

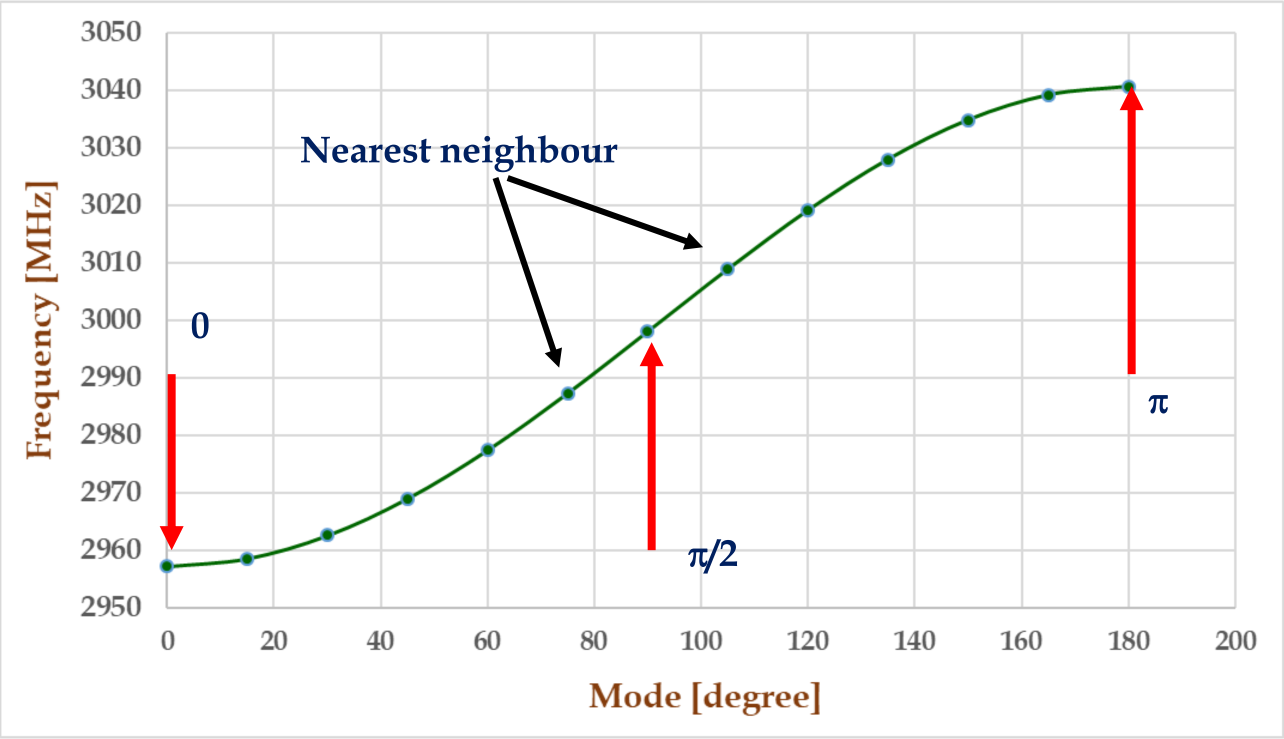
If k = 0.03 and f= 2998 MHz then, mode spectrum will be:

f_0 = 2954.01 MHz **$f_{\pi/2}$= 2998 MHz** **f_π = 3044.00 MHz**

Modes

For 13 cell structure: q= 0, 1…12 and N = 12; n=N+1 = 13
The modes will appear as

Mode	Frequency MHz
0	2957.18
15	2958.54
30	2962.55
45	2968.96
60	2977.38
75	2987.27
90	2998.00
105	3008.84
120	3019.06
135	3027.91
150	3034.75
165	3039.08
180	3040.56



Modes (contd)

- The coupling k is obtained as: $\boldsymbol{k = \frac{\Delta f}{f_0}}$ where Δf is the difference between frequency of π mode and 0 mode
- As the number of cells increase, more and more modes are packed in the plot.
- More modes are seen near 0 and π-mode frequency.
- However, the next neighbouring modes start coming close to the operational mode.
- The operation mode is related to number of cells wavelength 'n' as

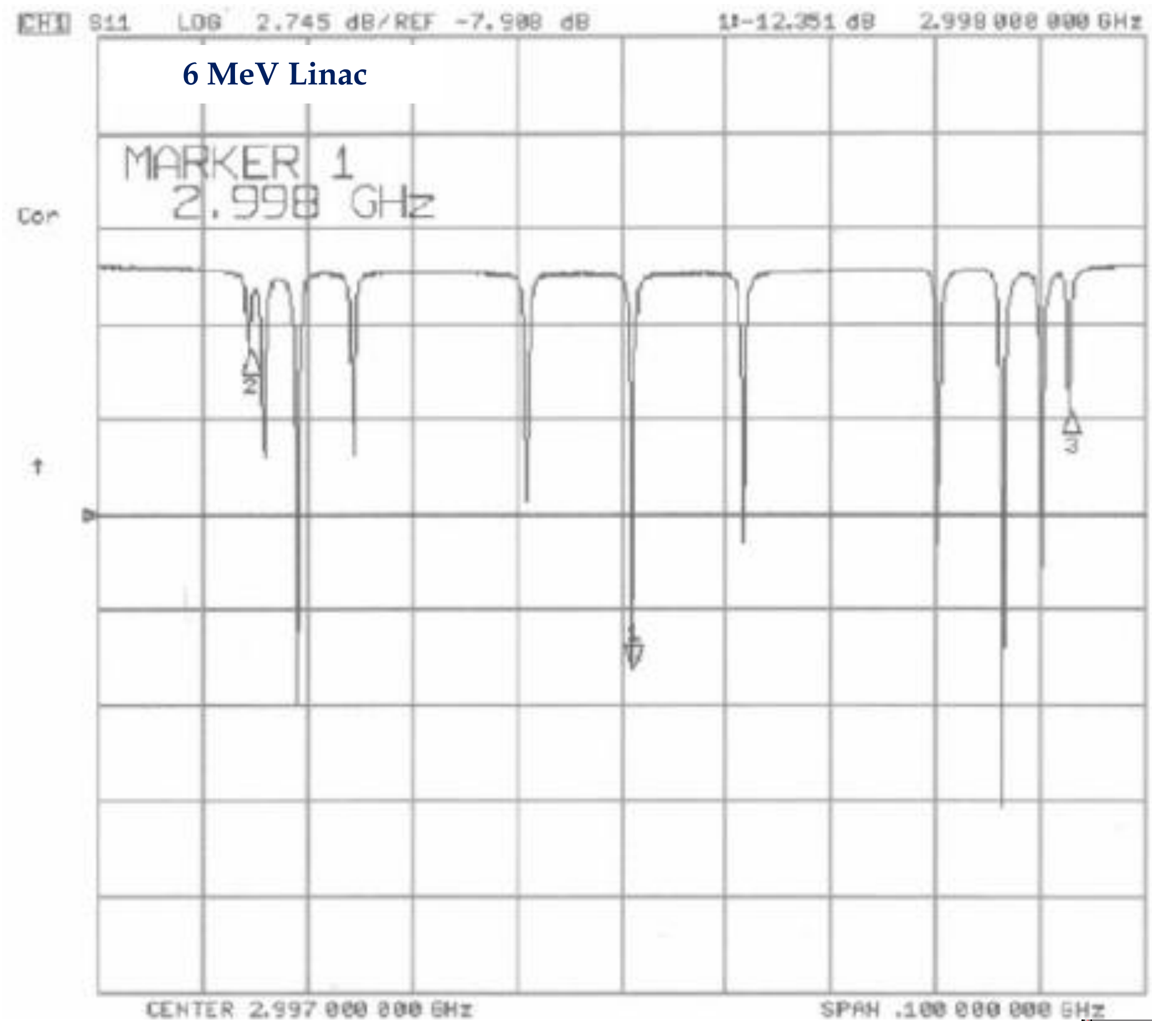
$$\boldsymbol{\Phi_0 = \frac{2\pi}{n}}$$

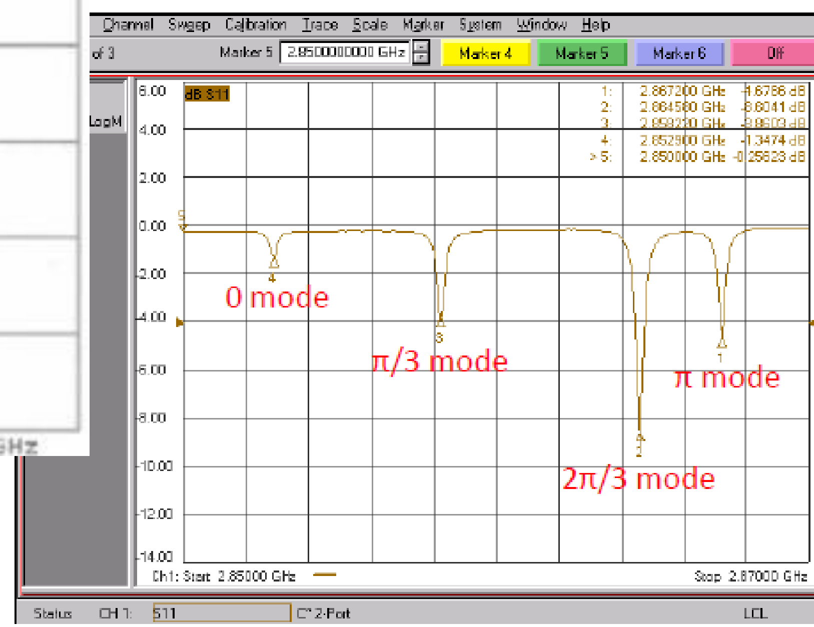
Thus for π mode, n = 2

π/2 mode, n = 4

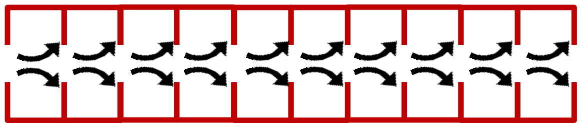
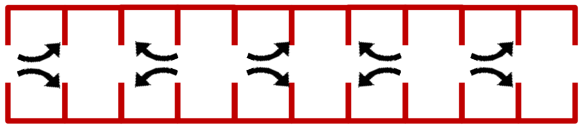
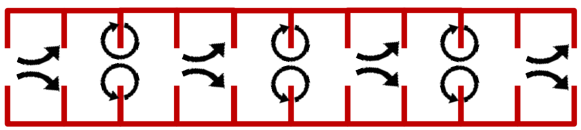
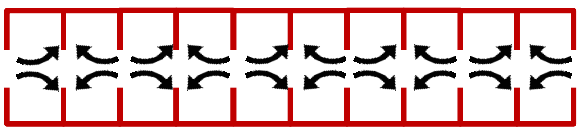
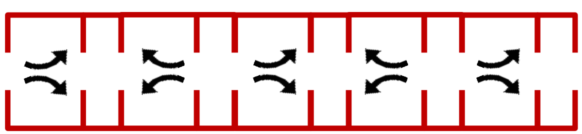
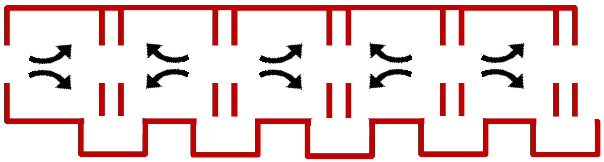
This means the phase advance is 180^0 for π-mode and there will be two cells per wavelength.

Measured Plots





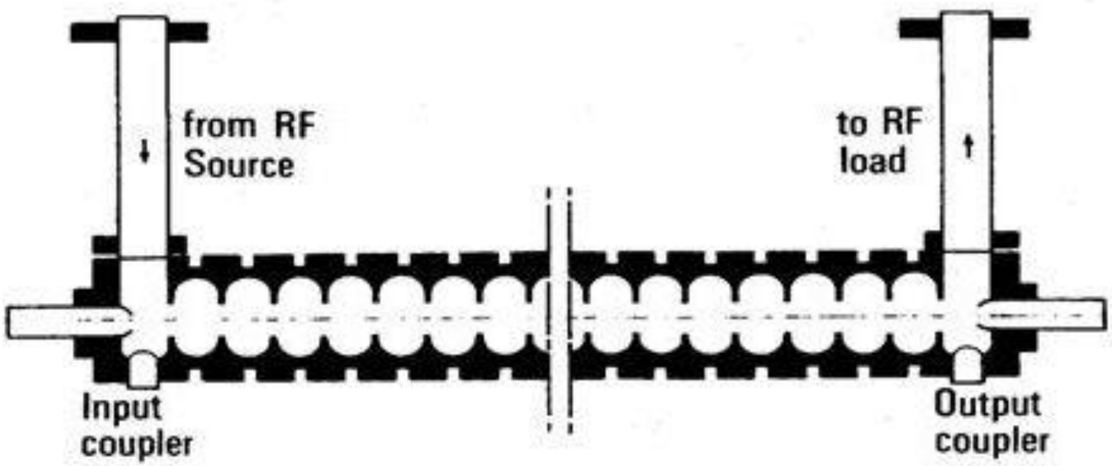
Modes (contd)

	0	No net acceleration
	π/2	Very stable operation Acceleration only for half cycle
	2π/3	In between. Mostly TW linac
	π	Most efficient Most sensitive to small deviations
	π/2	Coupling cell length reduced Efficiency comparable to π mode
	π/2	Electrically π-mode like Stability of π/2 mode

OVERVIEW OF TRAVELLING AND STANDING WAVE STRUCTURES

Travelling Wave Accelerator

- The power is flowing from one end to other end.
- There are two possibilities:
  - Constant Impedance: In this case the impedance is maintained and hence the gradient need not be constant
  - Constant Gradient: As name indicates, gradient should be same and hence energy gain will be same per cell. For this the geometry per cell needs to be modified



Constant Impedance TW

Let power P enter the accelerator and propagates through the accelerator with phase velocity same as electron velocity such that it is nearly c.

We define 'filling time' t_f as time required to fill the accelerator with RF power. It depends on group velocity v_g.

For an accelerator with diameter 2R and beam aperture 2b, group velocity can be approximated as

$$\mathbf{v_g \approx K\left(\frac{R}{b}\right)^4}$$

The power flowing and stored energy is related to group velocity as $\mathbf{v_g = \frac{P}{U}}$

Constant Impedance TW accelerator

Using definition of Q :

$$\frac{dP}{dz} = -\frac{\omega P}{v_g Q}$$

If $\boldsymbol{\alpha = \frac{\omega}{2v_g Q}}$ then $\frac{dP}{dz} = -2\alpha P$

The solution is : $P = P_0 \exp(-2\alpha z)$

Similarly for E we get $E = E_0 \exp(-\alpha z)$

If the length of structure is l then

The solution is : $P = P_0 \exp(-2\tau)$ and $E = E_0 \exp(-\tau)$

Where we have define attenuation parameter $\boldsymbol{\tau = \alpha l = \frac{\omega l}{2v_g Q}}$

Constant Impedance TW accelerator

Thus the total energy gain will be:

$$\boldsymbol{V_0 = \sqrt{2r_l P_0 l}\ \frac{(1-e^{-\tau})}{\sqrt{\tau}} cos\varphi}$$

This is maximized with $\tau = \frac{(e^{\tau}-1)}{2} \approx 1.26$

$$\boldsymbol{V_{0.max} \approx 0.903\sqrt{2r_l P_0 l}\ cos\varphi}$$

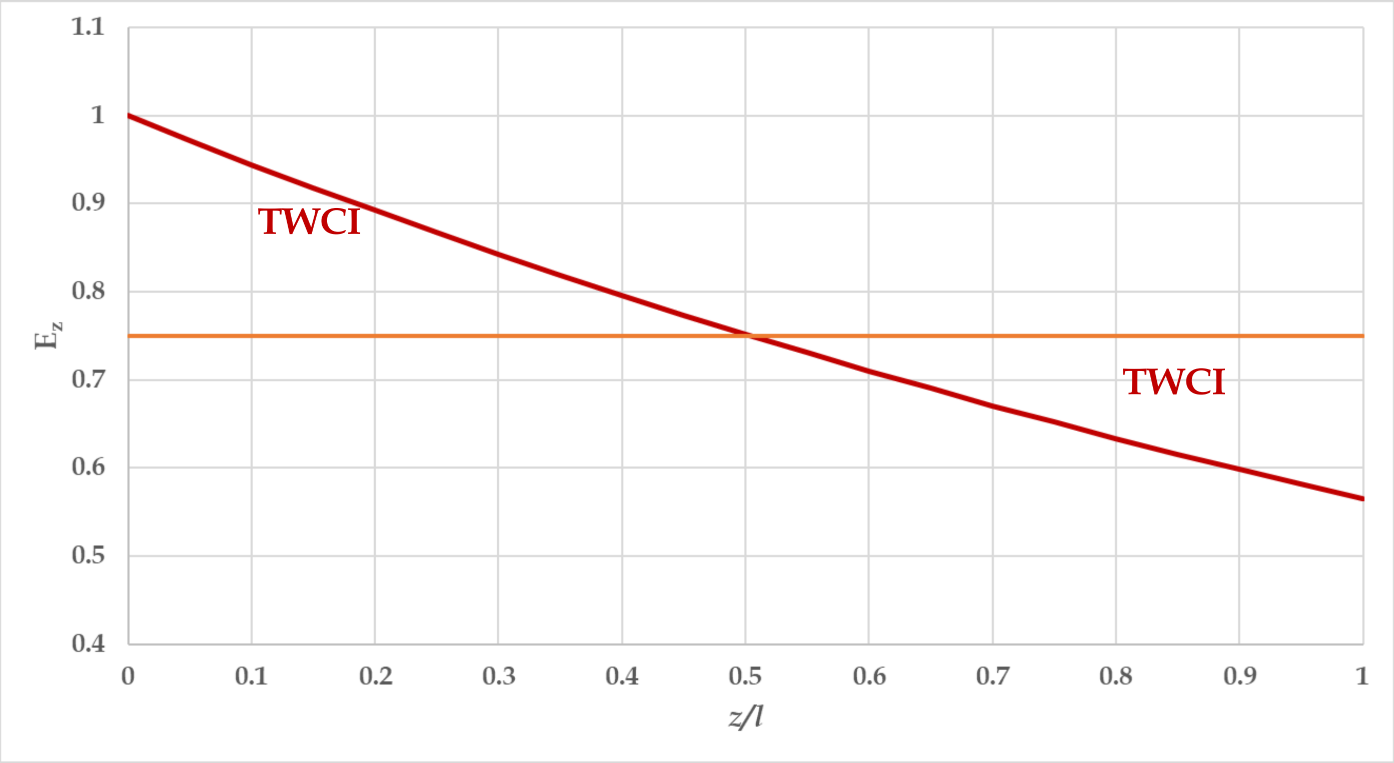
The time required to fill the accelerating structure with the power is called the filling time and is given as:

$$\mathrm{t_f} = \frac{l}{\mathrm{v_g}} = \frac{2\mathrm{Q}\tau}{\omega}$$

For the above value of τ, we get $\boldsymbol{t_f = \frac{2.52Q}{\omega}}$

Constant Gradient TW accelerator

- In above case, the electric field strength varies as the distance increases. To overcome this, we can vary the geometry of cells such that the field strength is maintained over the length.
- Such structures are called Constant Gradient Structures



Constant Gradient TW accelerator

$$\frac{\mathbf{dP}}{\mathbf{dz}} = \mathbf{constant}$$

If the power entering is P_o and power escaping at $z=l$ is P_l then

$$\mathbf{P = P_o - (P_o - P_l)\frac{z}{l}}$$

$$\frac{P_l}{P_0} = \mathbf{e^{-2\tau}}$$

=> $$\mathbf{P = P_0[1 - (1 - e^{-2\tau})\, {}^{z}/_{l}]}$$

CWTG accelerator (contd)

hence $\frac{dP}{dz} = -P_0\left(1 - e^{-2\tau}\right)/l$

Also: $\frac{dP}{dz} = -\frac{\omega P}{v_g Q}$

Therefore: $v_g = \frac{\omega l}{Q}\frac{\left[1-\left(1-e^{-2\tau}\right)^{z}/_{l}\right]}{\left(1-e^{-2\tau}\right)}$

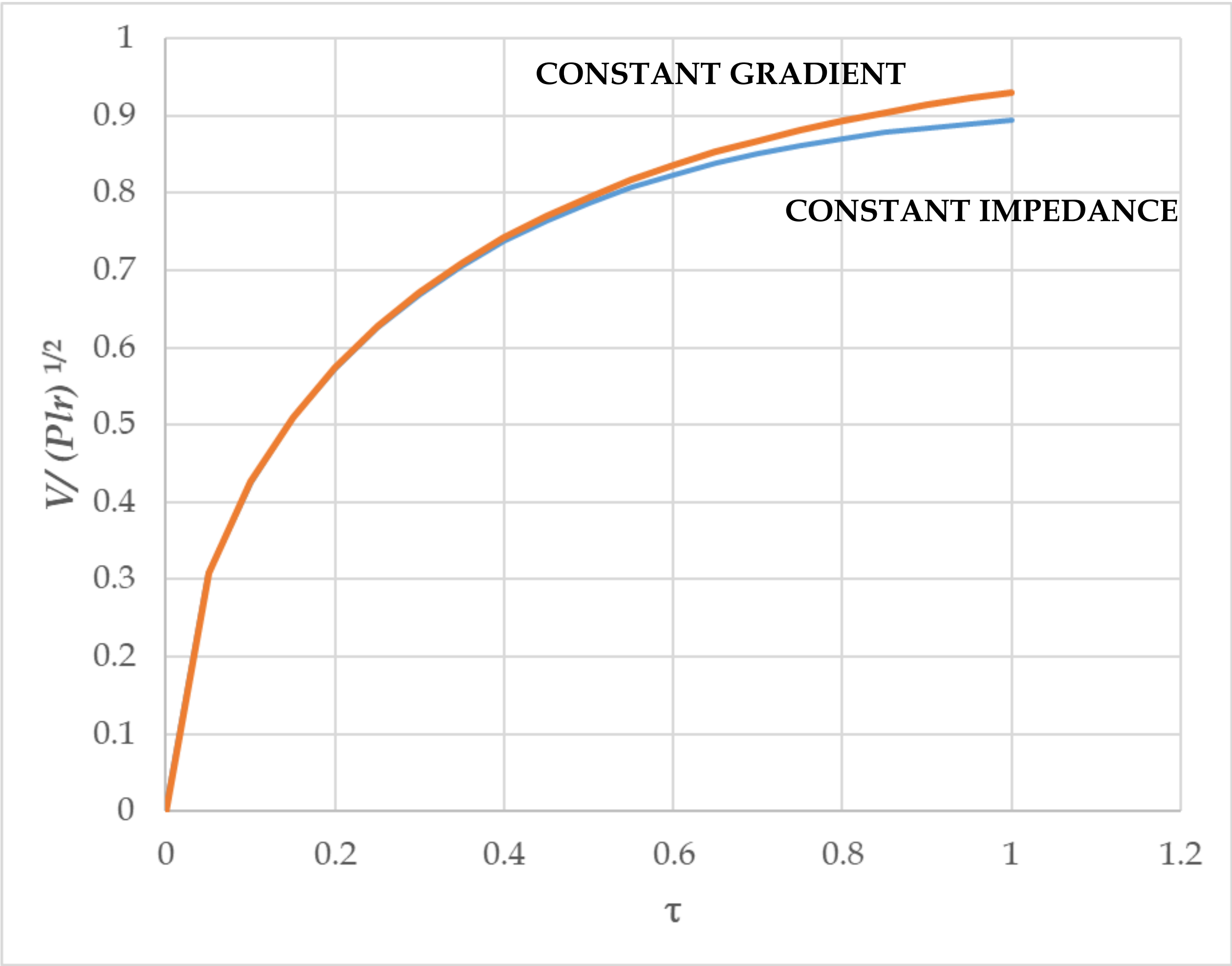
Clearly, P and v_g decrease in same manner from z=0 to z=l

Now, Energy Gain is $V_0 = E_0 l cos\varphi$

And $E_0^2 = -r_l\frac{dP}{dz} = +r_l P_0\left(1 - e^{-2\tau}\right)/l$

Hence: $$V_0 = \sqrt{r_l P_0 l(1 - e^{-2\tau})}cos\varphi$$

Comparative Energy Gain



Standing Wave Structure

- In this case, the entire waveguide is seen like a ‘resonant’ structure and the waves are reflected back.
- Thus there are waves in both the direction.
- Let a wave with Amplitude A be travelling in forward direction with velocity equal to phase velocity and a backward wave with amplitude B with negative phase velocity be travelling in same guide. $A \neq B$.

$$E_F = Asin(\omega t - \beta_0 z)$$

$$E_B = -Bsin(\omega t + \beta_0 z)$$

$$E = E_F + E_B = (A - B)(sin\omega tcos\beta_0 z) - (A + B)(cos\omega tsin\beta_0 z)$$

Standing Wave Structure

Differentiating and finding maxima and minima, we conclude

$\boldsymbol{E_{max} = |A + B|}$

$\boldsymbol{E_{min} = |A - B|}$

$\boldsymbol{E_{max}}: \beta_0 z = (2\mathrm{n} + 1)\,{}^{\pi}/_{2}$

$\boldsymbol{E_{min}}: \beta_0 z = \mathrm{n}\pi$ where n=0,1,2,3…

The steady state energy gain can be seen as:

$$\boldsymbol{V = \sqrt{\frac{tanh\frac{\tau}{2}}{\frac{\tau}{2}}]}\ [1 + e^{-2\tau}]^{-\frac{1}{2}}\sqrt{P_s r_0 l}}$$

Where P_s is power from the source; $P_s = P_0\,(1 - e^{-4\tau})$

Standing Wave Structure

Energy Gain can be seen as:

$$\mathbf{V} = \frac{2\sqrt{\beta P_0 (ZT^2) l}}{1+\beta}$$

Where P_0 is power from the source; β is coupling factor (usually 1.0)

Thus for a typical input power of 6 MW with effective shunt impedance of 80MW/m; ~21MeV per meter is expected. This is if there is no beam.

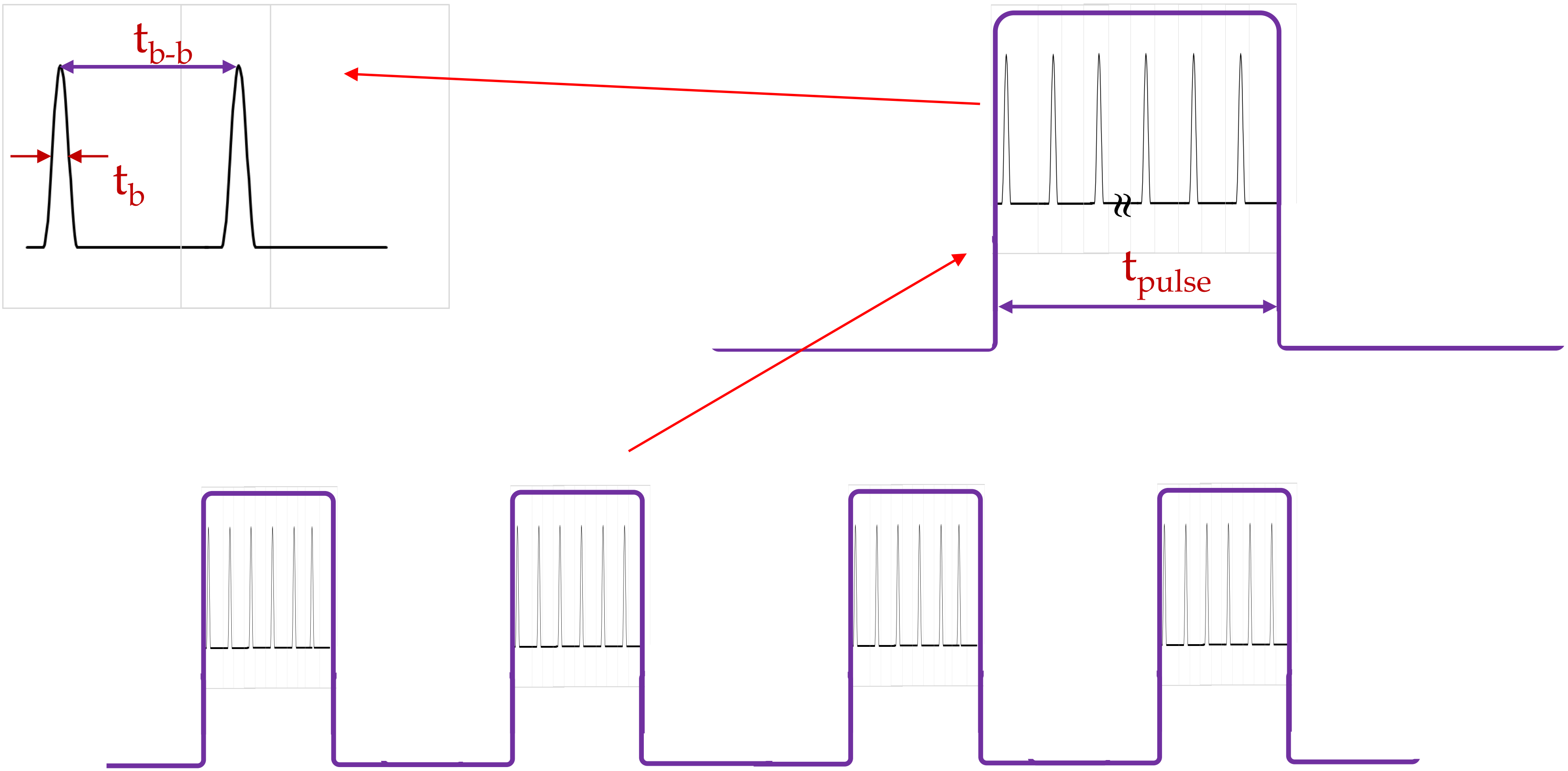
For same input parameters, SW structures give more energy gain as compared to TW structures.

Energy gain in presence of beam

- Now let us introduce bunch of electrons in the accelerator
- If electrons arrive in phase, they will derive some of the cavity power and accelerate. But in the process, the power in the cavity is slightly reduced.
- Hence, the next arriving bunch will have lesser power available and consequently the acceleration will be less.
- Hence, the energy will go down as a function of number of bunches arriving and the power available.
- This phenomenon is called beam loading and if not compensated it will lead to large deviation of energy from bunch to bunch

Energy gain in presence of beam

Now let us introduce bunch of electrons in the accelerator



Beam parameters

- For 2856 MHz, the minimum $t_{b\text{-}b}$ can be 350 ps
- The single bunch could be few tens of ps in length and can have some pC charge. As example, 8.75 pC and 10 ps can be a typical bunch distribution.
- If RF pulse width is 4 μs then at most 11424 bunches can be present.
- The operation can also be done by choosing to increase bunch-to-bunch spacing 8 times to 2.8ns. This means the RF frequency is still at 2856 MHz, but synchronization has to be done at 2856/8 = 357 MHz.
- The number of bunches now will go down 1428. Per bunch charge will go up to 70 pC.
- How much charge per bunch can be sustained without adversely affecting the energy can be calculated and optimized

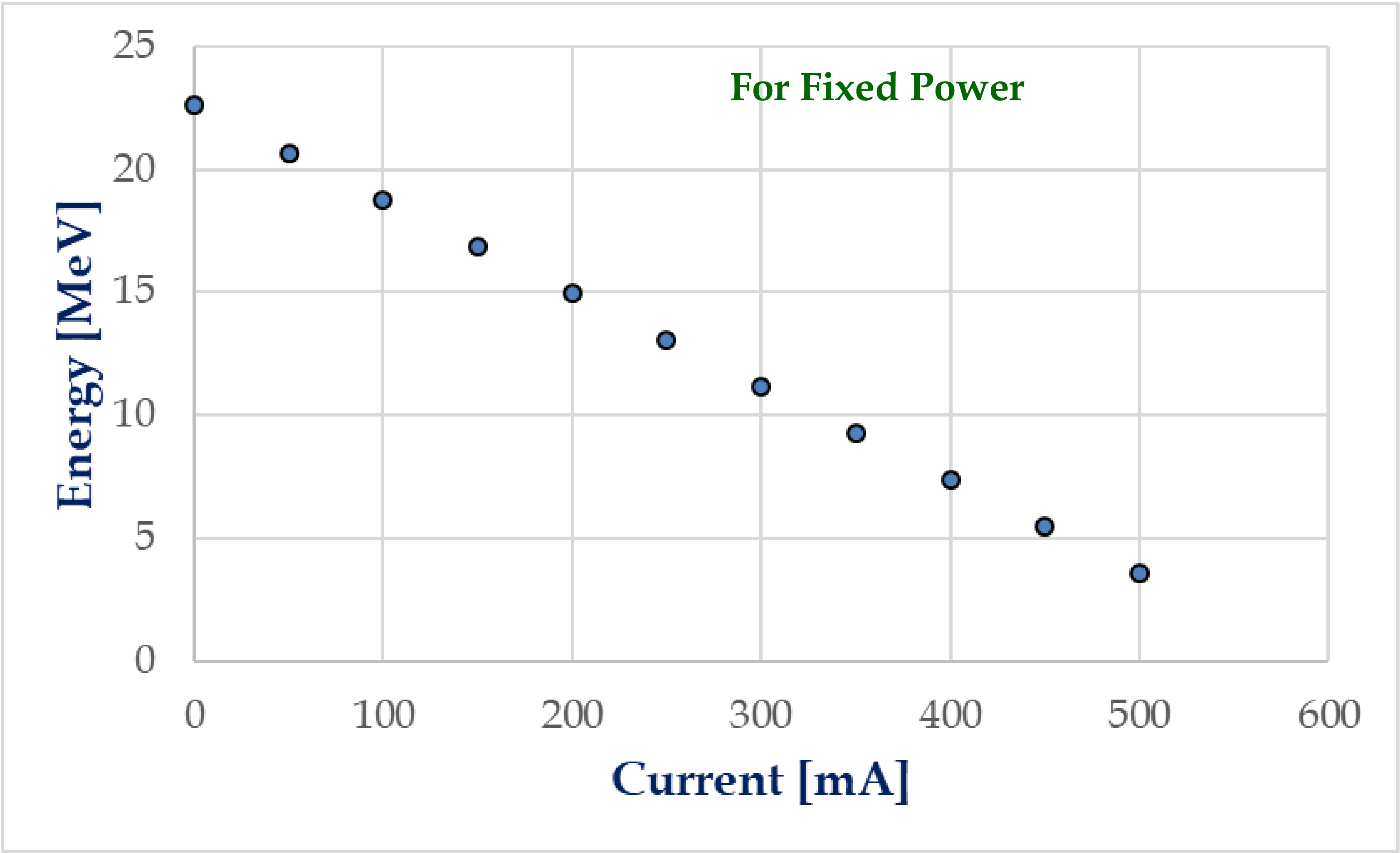
Beam Loading

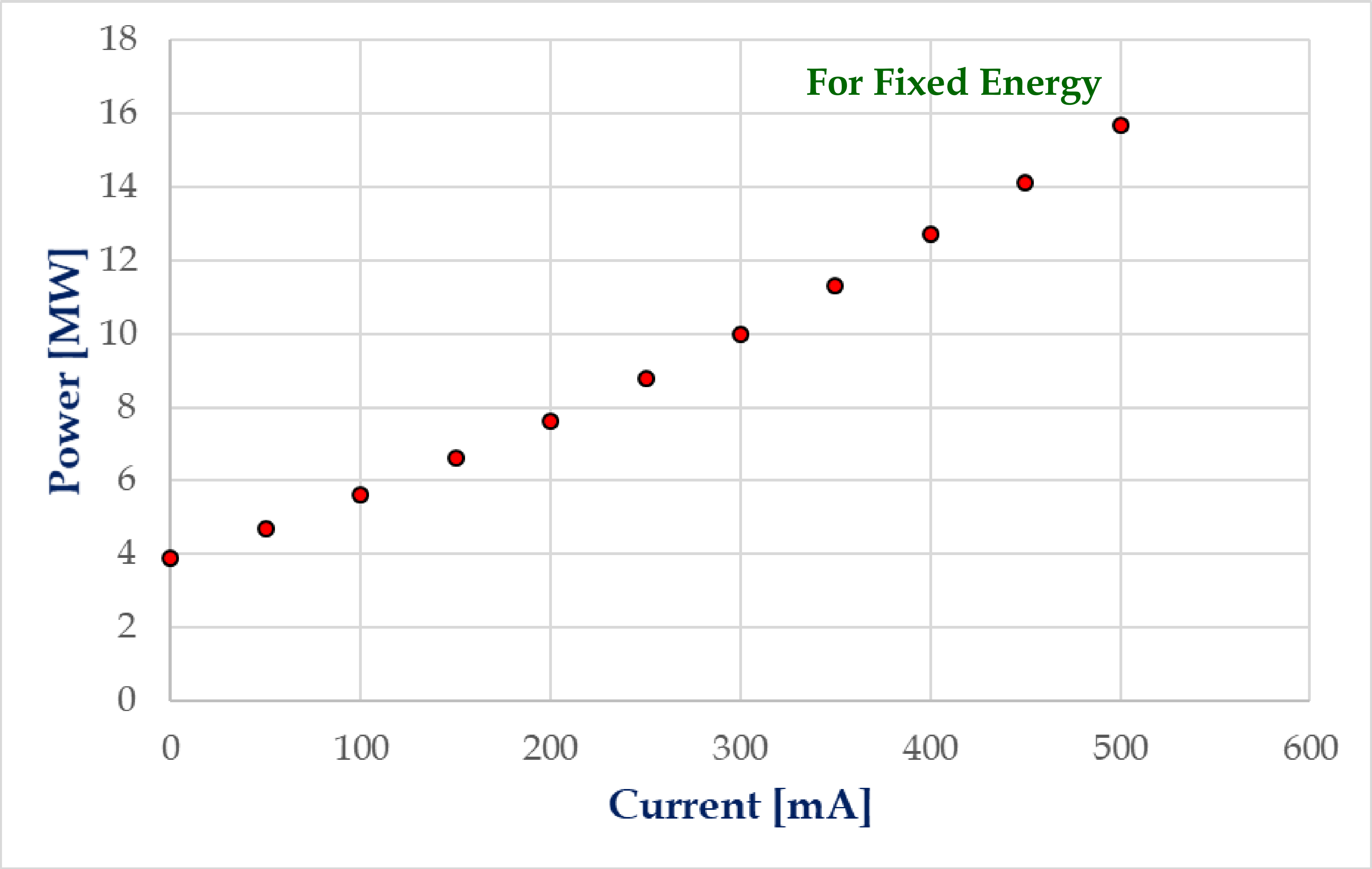
- As discussed above, in presence of bunches the power available for acceleration will go down and therefore we expect reduction in energy.

The energy gain with beam loading is given by

$$\boldsymbol{V = \frac{2\sqrt{\beta P_0 (ZT^2) l} - i\left(ZT^2\right) l}{1+\beta} \left\{1 - exp\left(\frac{\omega t}{2Q_l}\right)\right\}}$$

Where β is coupling coefficient and ***i*** is the peak current of the bunch.


For Fixed Power
Energy [MeV]
Current [mA]
25
20
15
10
5
0
0
100
200
300
400
500
600

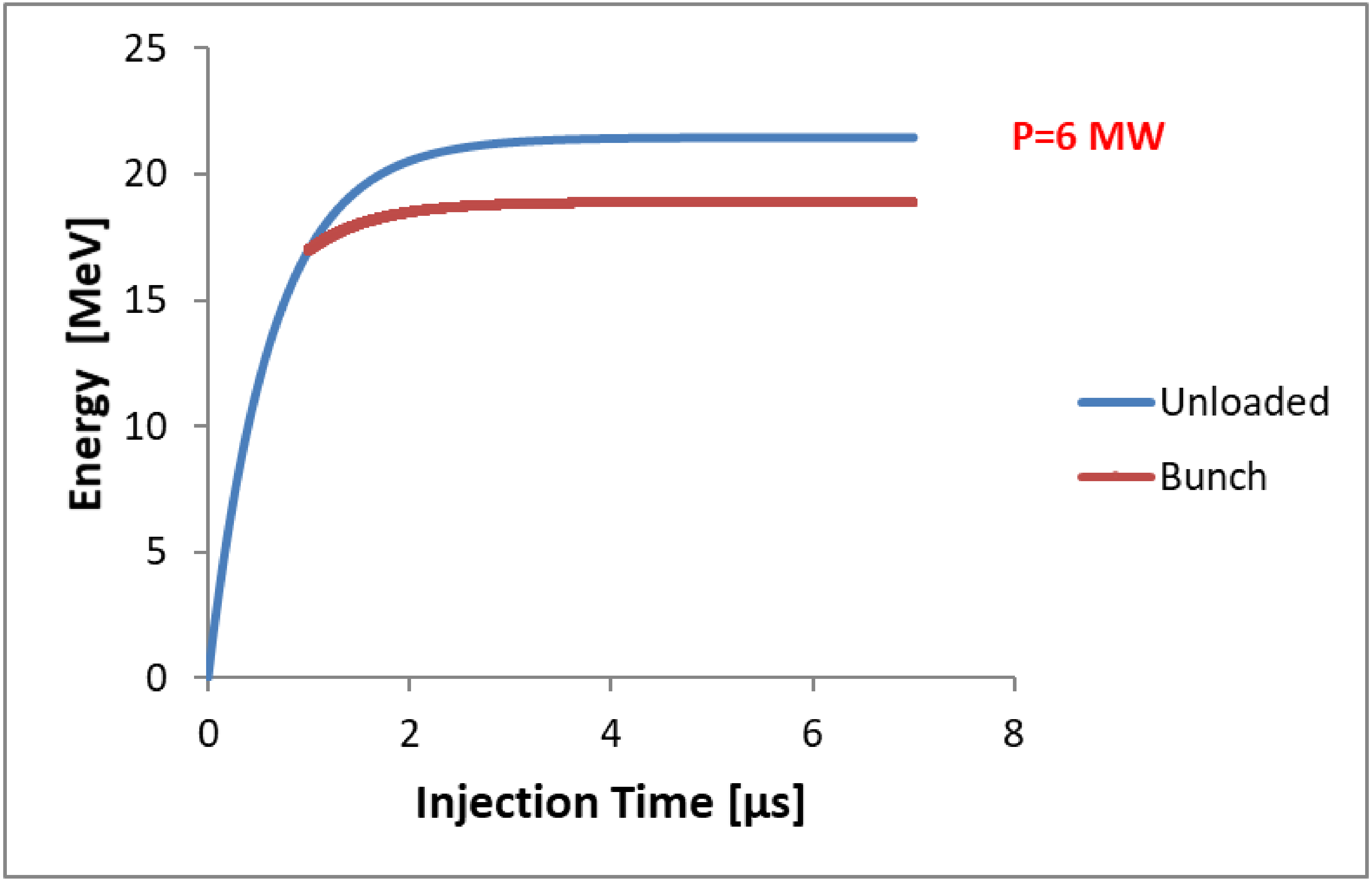

For Fixed Energy
Power [MW]
Current [mA]
0
2
4
6
8
10
12
14
16
18
0
100
200
300
400
500
600

Beam Loading

The transient beam loading can be calculated as:

$$V_{RFG} = \frac{2\sqrt{\beta P_0 Z T^2 l}}{(1+\beta)}(1 - e^{-t/T_f}) - V_{b0}(\frac{1 - e^{-(t-t_{inj})/T_f}}{(1 - e^{-t_b/T_f})} + \frac{1}{2})$$

Where T_f is filling time and β is the coupling factor

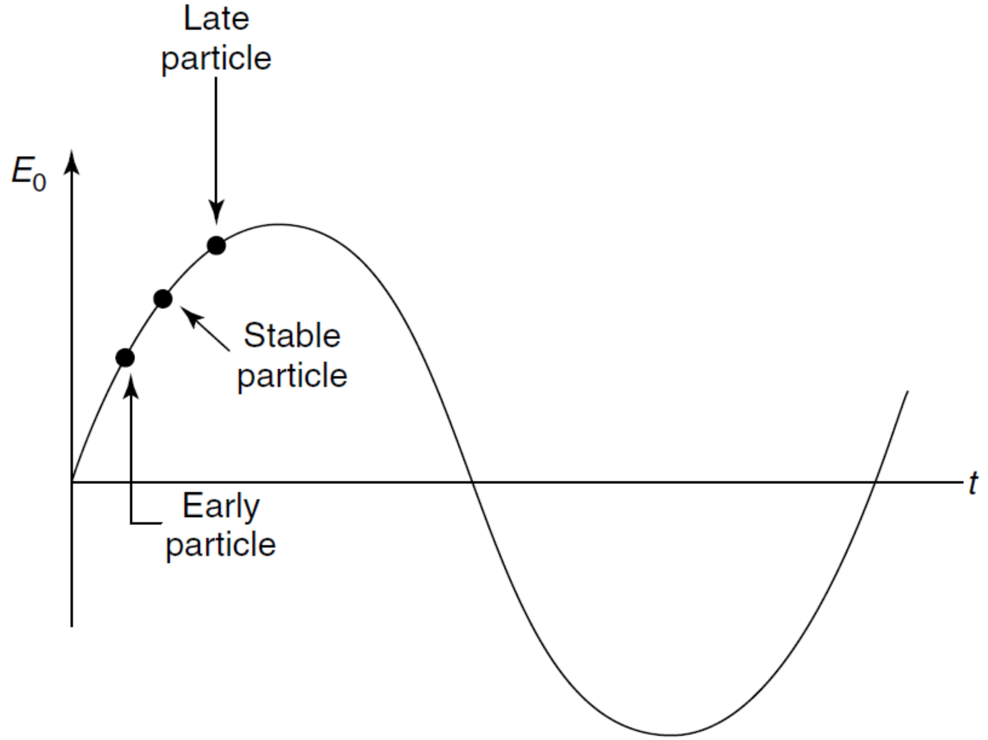


25
20
15
10
5
0
0
2
4
6
8
Energy [MeV]
Injection Time [μs]
P=6 MW
Unloaded
Bunch

LONGITUDINAL DYNAMICS

Longitudinal beam dynamics

Consider multicell accelerator with a particle in n^th^ cell with synchronous phase ϕ_{sn} , energy W_{sn} and velocity β_{sn} . For any other particle the parameters will be ϕ_n , W_n and β_n



Longitudinal beam dynamics

Then the following equations govern the longitudinal motion:

$$\frac{d}{dz}(\varphi - \varphi_s) = -\frac{k\Delta W}{\beta_s^3\gamma_s^3 m_0 c^2}$$

$$\frac{d\Delta W}{dz} = eE(cos\varphi - cos\varphi_s)$$

$$\frac{d}{dz}\left(\beta_s^3\gamma_s^3\frac{d}{dz}(\varphi - \varphi_s)\right) = -\frac{keE(cos\varphi - cos\varphi_s)}{m_0 c^2}$$

For small amplitude we can have $\varphi - \varphi_s = \Delta\varphi$ and then the equation (2) becomes :

$$\frac{dW}{dz} = -eEsin\varphi_s\,\Delta\varphi$$

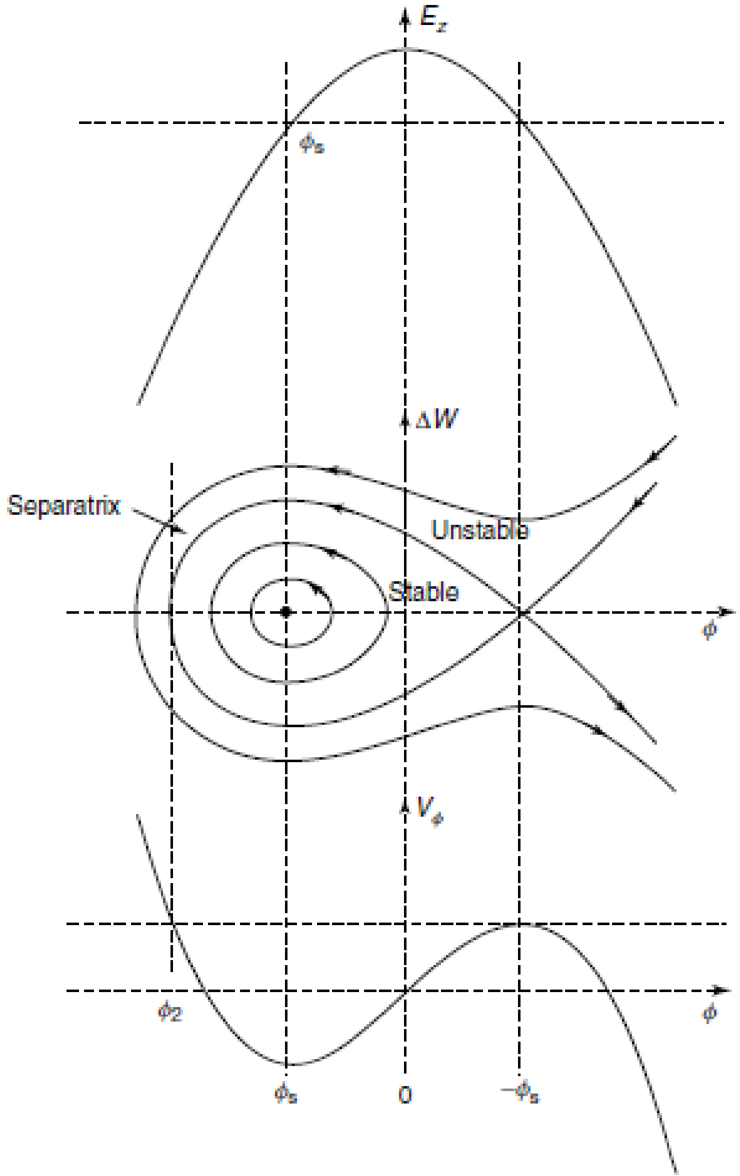
Frequency of small amplitude

Further if the coefficients $\beta_s^3, \gamma_s^3, Esin\varphi$ were constants then it can be shown that the particles will show simple harmonic oscillations of $\Delta\varphi$ and ΔW with wavelength as:

$$\frac{\lambda_\varphi}{2\pi} = \sqrt{\frac{\boldsymbol{\beta_s^3\gamma_s^3 m_0 c^2}}{\boldsymbol{-ekEsin\varphi}}}$$

The phase difference is translated in spatial term.

The frequency of the oscillations decrease as γ increases.



Accelerating field as function of phase. Note synchronous phase $\boldsymbol{\varphi_s}$ is negative

Longitudinal phase space trajectory and limiting values

Corresponding potential

Range of stability

To check the range of stability, we have to look for large amplitude variations. We multiply the equation (3) by ${}^{d\Delta\varphi}/_{dz}$:

$$\boldsymbol{\beta_s^3\gamma_s^3\Delta\varphi'\Delta\varphi'' = -\frac{keE(cos\varphi - cos\varphi_s)}{m_0c^2}\Delta\varphi'}$$

Integrating with respect to z and rearranging,

$$\boldsymbol{\frac{k}{2\beta_s^3\gamma_s^3m_0c^2}(\Delta W)^2 + eE(sin\varphi - \varphi cos\varphi_s + C) = 0}$$

For each value of constant C, we can have a trajectory. The stationary point $\boldsymbol{\varphi_s}$ determines the limiting amplitude. The special trajectory is obtained when

$$\boldsymbol{C = sin\varphi_s - \varphi_s cos\varphi_s}$$

This bounding trajectory is called as “Separatrix”

Adiabatic Damping

We have already seen equations defining $\boldsymbol{\Delta\varphi}$ and $\boldsymbol{\Delta W}$. The ratio of the two quantities can be seen as:

$$\frac{\Delta W}{\Delta\varphi} = \sqrt{\frac{eE(-sin\varphi_s)\beta_s^3\gamma_s^3 m_0 c^2}{k}}$$

Let us get back to the phase space diagram.

Consider a bunch of particle following the stable trajectory. For small amplitude oscillations, they trace a smooth oval. As the acceleration begins, the parameters change and the trajectories start getting complicated.

Adiabatic Damping

Let particle enter with parameters marked as 1.

$$\frac{\Delta W_1}{\Delta \varphi_1} = \sqrt{\frac{eE_1(-sin\varphi_{s,1})\beta_{s,1}^3\gamma_{s,1}^3 m_0 c^2}{k}}$$

If the changes are slow (sufficient to be adiabatic) then at the end parameters will be as marked with subscript 2

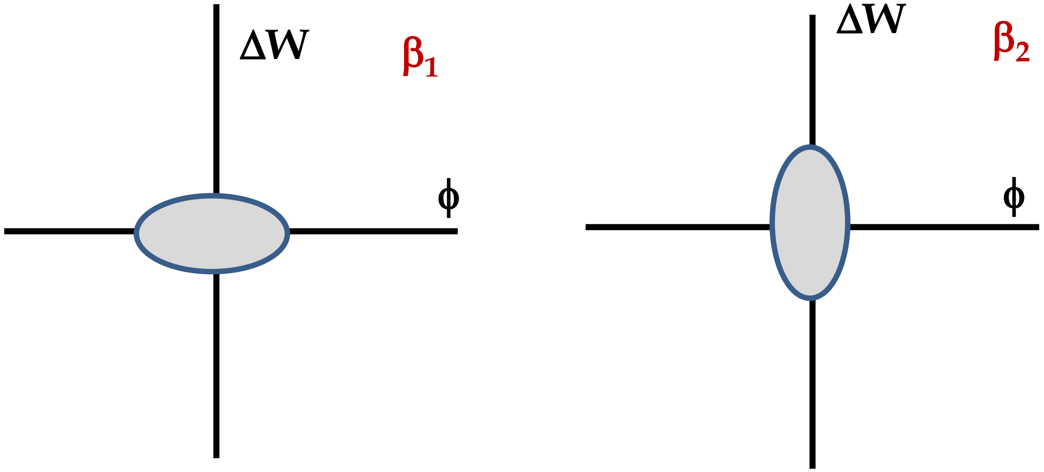
$$\frac{\Delta W_2}{\Delta \varphi_2} = \sqrt{\frac{eE_2(-sin\varphi_{s,2})\beta_{s,2}^3\gamma_{s,2}^3 m_0 c^2}{k}}$$

Since the area of phase space and the particle density in the phase space in neighbourhood of any given particle are constants in motion. Hence the ratio of phase spread can be shown as per the following equation.

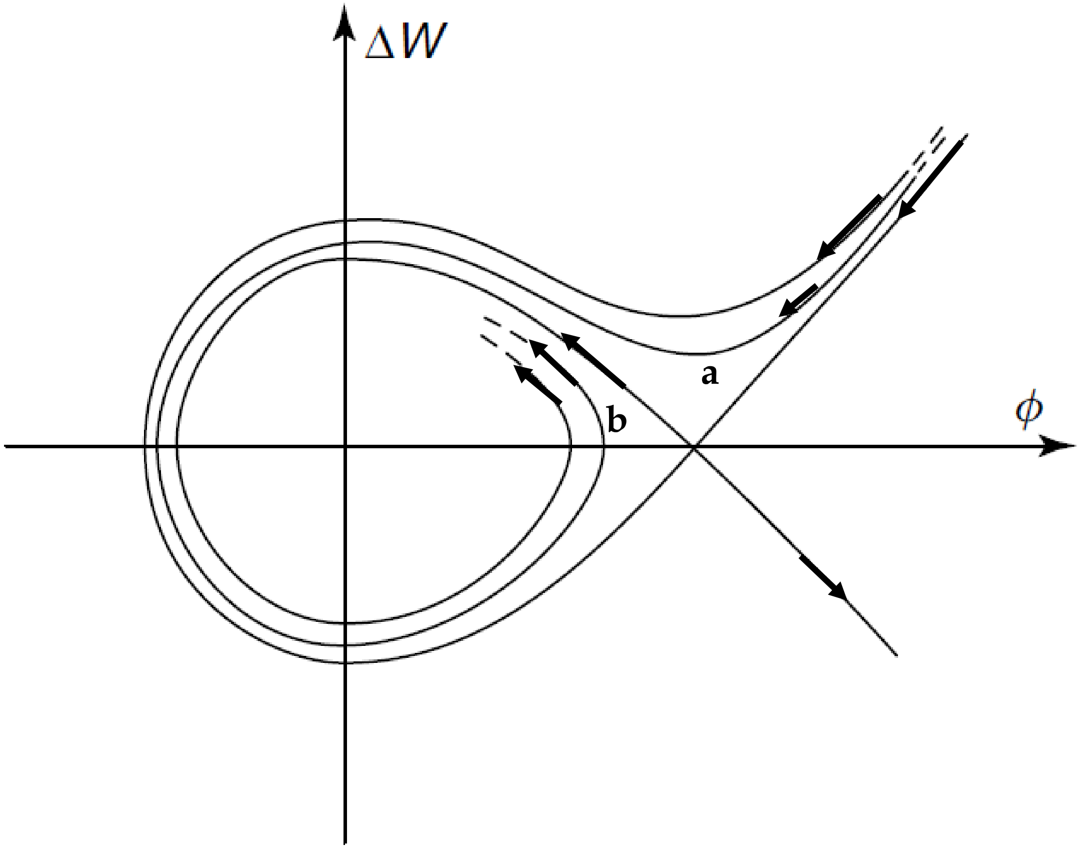
Adiabatic Damping

$$\frac{\Delta\varphi_2}{\Delta\varphi_1} = \left(\frac{E_1(-sin\varphi_{s,1})\beta_{s,1}^3\gamma_{s,1}^3}{E_2(-sin\varphi_{s,2})\beta_{s,2}^3\gamma_{s,2}^3}\right)^{1/4}$$

Clearly, as the energy amplitude is increased, the phase spread is decreased.



Adiabatic Damping



Comments

- We covered the evolution of linac from single cell to multicell cavities
- We discussed important pointers to take care while making linac design.
- Review of TW and SW linac was done.
- It is important for linac designer to establish simple techniques and make an accelerator which can give stable operation over extended period of time.
- SC linac technology has brought high current accelerators in great demand…so onus is on NC linac designers to evolve high current accelerators at much low cost.

References

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- RF Linear Accelerators, Thomas Wangler, Wiley 2008
- Linac Lectures 1 & 2, T. Higo ILC School 2008 and Indo Japan Accelerator School 2012.
- SLAC Report 2448, P. Wilson