The Schwinger model in the canonical formulation

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Motivation for the canonical formulation

- Consider the grand-canonical partition function at finite μ:

$$Z_{\mathrm{GC}}(\mu) = \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t(\mu)$$

- The sign problem at finite density is a manifestation of huge cancellations between different states:
  - all states are present for any μ and T
  - some states need to cancel out at different μ and T

- In the canonical formulation:

$$Z_{\mathrm{C}}(N_f) = \mathrm{Tr}_{N_f}\left[e^{-\mathcal{H}/T}\right] = \mathrm{Tr}\prod_t \mathcal{T}_t^{(N_f)}$$

  - dimension of Fock space tremendously reduced
  - less cancellations necessary:
  - e.g. $Z_{\mathrm{C}}^{\mathrm{QCD}}(N_Q) = 0$ for $N_Q \neq 0 \bmod N_c$

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  - less cancellations necessary:
  - e.g. "Silver Blaze" phenomenon realised automatically

Motivation for canonical formulation of QCD

Canonical transfer matrices can be obtained explicitly!

- based on the dimensional reduction of the QCD fermion determinant [Alexandru, Wenger '10; Nagata, Nakamura '10]

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Outline:

- Overview
- Definition of the transfer matrices in canonical formulation
- Relation to fermion loop and worldline formulations
- Hubbard model and Super Yang-Mills QM
- Schwinger model

Overview

- Identification of transfer matrices:
  - Dimensional reduction in QCD [Alexandru, UW '10]
  - SUSY QM and SUSY Yang-Mills QM [Baumgartner, Steinhauer, UW '12-'15]
    - solution of the sign problem
    - connection with fermion loop formulation
  - QCD in the heavy-dense limit
    - absence of the sign problem at strong coupling
    - solution of the sign problem in the 3-state Potts model [Alexandru, Bergner, Schaich, UW '18]
  - Hubbard model [Burri, UW '19]
    - HS field can be integrated out analytically
  - $N_f = 1, 2$ Schwinger model [Bühlmann, UW '19]

General construction

- For a generic Hamiltonian $\mathcal{H}$ with $\mu \equiv \{\mu_\sigma\}$ one has

$$\begin{aligned} Z_{\mathrm{GC}}(\mu) &= \mathrm{Tr}\left[e^{-\mathcal{H}(\mu)/T}\right] \\ &= \sum_{\{N_\sigma\}} e^{-\sum_\sigma N_\sigma \mu_\sigma / T} \cdot Z_C(\{N_\sigma\}) \end{aligned}$$

where $Z_C(\{N_\sigma\}) = \mathrm{Tr} \prod_t \mathcal{T}_t^{(\{N_\sigma\})}$.

- Trotter decomposition and coherent state representation yields

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}\phi e^{-S_b[\phi]} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-S[\psi^\dagger, \psi, \phi; \mu]}$$

with Euclidean action S_b and fermion matrix M

$$S[\psi^\dagger, \psi, \phi; \mu] = \sum_\sigma \psi_\sigma^\dagger M[\phi; \mu] \psi_\sigma \,.$$

Fermion matrix and dimensional reduction

- The fermion matrix $M[\phi;\mu_\sigma]$ has the generic structure

$$M = \begin{pmatrix} B_0 & e^{-\mu_\sigma} C'_0 & 0 & \dots & \pm e^{\mu_\sigma} C_{N_t-1} \\ e^{\mu_\sigma} C_0 & B_1 & e^{-\mu_\sigma} C'_1 & & 0 \\ 0 & e^{\mu_\sigma} C_1 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \\ & & & B_{N_t-2} & e^{-\mu_\sigma} C'_{N_t-2} \\ \pm e^{-\mu_\sigma} C'_{N_t-1} & 0 & & e^{\mu_\sigma} C_{N_t-2} & B_{N_t-1} \end{pmatrix}$$

  for which the determinant can be reduced to

$$\det M[\phi;\mu_\sigma] = \prod_t \det \tilde{B}_t \cdot \det\left(1 \mp e^{N_t\mu_\sigma}\mathcal{T}[\phi]\right)$$

  where $\mathcal{T}[\phi] = \mathcal{T}_{N_t-1} \cdot \ldots \cdot \mathcal{T}_0$.

- $M[\phi;\mu_\sigma]$ is $(L_s \cdot N_t) \times (L_s \cdot N_t)$, while $\mathcal{T}[\phi]$ is $L_s \times L_s$.

Fermion matrix and canonical determinants

- Fugacity expansion

$$\det M[\phi;\mu_\sigma] = \sum_{N_\sigma} e^{-N_\sigma \mu_\sigma / T} \cdot \det{}_{N_\sigma} M[\phi]$$

  yields the canonical determinants

$$\det{}_{N_\sigma} M[\phi] = \sum_{J} \det \mathcal{T}^{JJ}[\phi] = \mathrm{Tr}\left[\prod_t \mathcal{T}_t^{(N_\sigma)}\right].$$

  where $\det \mathcal{T}^{JJ}$ is the principal minor of order N_σ.

- States are labeled by index sets $J \subset \{1, \ldots, L_s\}$, $|J| = N_\sigma$
  - number of states grows exponentially with L_s at half-filling

$$N_{\text{states}} = \begin{pmatrix} L_s \\ N_\sigma \end{pmatrix} = N_{\text{principal minors}}$$

  - sum can be evaluated stochastically with MC

Transfer matrices

- Use Cauchy-Binet formula

$$\det(A \cdot B)^{IK} = \sum_J \det A^{IJ} \cdot \det B^{JK}$$

to factorize into product of transfer matrices

- Transfer matrices in sector N_σ are hence given by

$$(\mathcal{T}_t^{(N_\sigma)})_{IK} = \det \tilde{B}_t \cdot \det \left[\mathcal{T}_t\right]^{IK}$$

with $\mathrm{Tr}\left[\prod_t \mathcal{T}_t^{(N_\sigma)}\right] = (\mathcal{T}_{N_t-1}^{(N_\sigma)})_{IJ} \cdot (\mathcal{T}_{N_t-2}^{(N_\sigma)})_{JK} \cdot \ldots \cdot (\mathcal{T}_0^{(N_\sigma)})_{LI}$.

- Finally, we have

$$Z_C(\{N_\sigma\}) = \int \mathcal{D}\phi \, e^{-S_b[\phi]} \prod_t \det \tilde{B}_t \cdot \sum_{\{J_t^\sigma\}} \prod_t \left(\prod_\sigma \det \left[\mathcal{T}_t^\sigma\right]^{J_{t-1}^\sigma J_t^\sigma} \right)$$

where $|J_t^\sigma| = N_\sigma$ and $J_{N_t}^\sigma = J_0^\sigma$.

Example: Hubbard model

- Consider the Hamiltonian for the Hubbard model

$$\mathcal{H}(\mu) = -\sum_{\langle x,y\rangle,\sigma} t_\sigma\, \hat{c}^\dagger_{x,\sigma}\hat{c}_{y,\sigma} + \sum_{x,\sigma}\mu_\sigma N_{x,\sigma} + U\sum_x N_{x,\uparrow}N_{x,\downarrow}$$

 with particle number $N_{x,\sigma} = \hat{c}^\dagger_{x,\sigma}\hat{c}_{x,\sigma}$.

- After Trotter decomposition and Hubbard-Stratonovich transformation we have

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}\psi^\dagger\mathcal{D}\psi\mathcal{D}\phi\,\rho[\phi]e^{-\sum_\sigma S[\psi^\dagger_\sigma,\psi_\sigma,\phi;\mu_\sigma]}$$

 with $S[\psi^\dagger_\sigma,\psi_\sigma,\phi;\mu_\sigma] = \psi^\dagger_\sigma M[\phi;\mu_\sigma]\psi_\sigma$, and hence

$$= \int \mathcal{D}\phi\,\rho[\phi]\prod_\sigma \det M[\phi;\mu_\sigma]\,.$$

Example: Hubbard model

- The fermion matrix has the structure

$$M[\phi;\mu_\sigma] = \begin{pmatrix} B & 0 & \dots & \pm e^{\mu_\sigma} C(\phi_{N_t-1}) \\ -e^{\mu_\sigma} C(\phi_0) & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -e^{\mu_\sigma} C(\phi_{N_t-2}) & B \end{pmatrix}$$

for which the determinant can be reduced to

$$\det M[\phi;\mu_\sigma] = \det B^{N_t} \cdot \det\left(1 \mp e^{N_t\mu_\sigma}\mathcal{T}[\phi]\right)$$

where $\mathcal{T}[\phi] = B^{-1}C(\phi_{N_t-1}) \cdot \ldots \cdot B^{-1}C(\phi_0)$.

- Fugacity expansion yields the canonical determinants

$$\det M_{N_\sigma}[\phi] = \sum_J \det \mathcal{T}^{JJ}[\phi] = \mathrm{Tr}\left[\prod_t \mathcal{T}_t^{(N_\sigma)}\right].$$

where $\det\mathcal{T}^{JJ}$ is the principal minor of order N_σ.

Example: Hubbard model

- Transfer matrices are hence given by

$$
\begin{aligned}
(\mathcal{T}_t)_{IK} &= \det B \cdot \det\left[B^{-1} \cdot C(\phi_t)\right]^{\not{I}\not{K}} \\
&= \det B \cdot \det(B^{-1})^{\not{I}\not{J}} \cdot \det C(\phi_t)^{\not{J}\not{K}}
\end{aligned}
$$

- Moreover, using the complementary cofactor we get

$$
\det B \cdot \det(B^{-1})^{\not{J}\not{I}} = (-1)^{p(I,J)} \det B^{IJ}
$$

where $p(I,J) = \sum_i (I_i + J_i)$ and HS field can be integrated out,

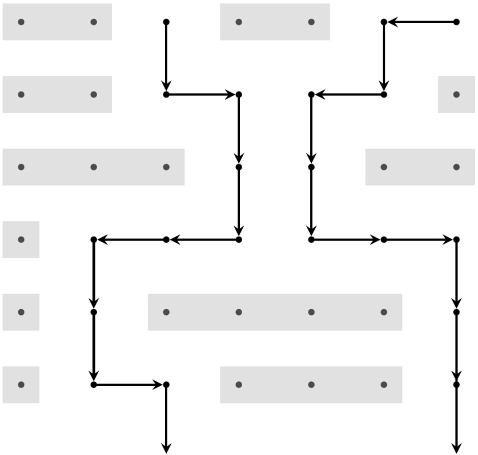
$$
\det C(\phi_t)^{\not{J}\not{K}} = \delta_{JK} \prod_{x \notin J} \phi_{x,t} \implies \prod_{x} w_{x,t} \equiv W\left(\{J_t^\sigma\}\right) .
$$

- Finally, only sum over discrete index sets is left:

$$
Z_C(\{N_\sigma\}) = \sum_{\{J_t^\sigma\}} \prod_t \left(\prod_\sigma \det B^{J_{t-1}^\sigma J_t^\sigma} \right) W\left(\{J_t^\sigma\}\right), \quad |J_t^\sigma| = N_\sigma
$$

Example: Hubbard model

$$Z_C(\{N_\sigma\}) = \sum_{\{J_t^\sigma\}} \prod_t \left(\prod_\sigma \det B^{J_{t-1}^\sigma J_t^\sigma} \right) W\left(\{J_t^\sigma\}\right)$$



index sets J_t:

$\{3,6\}$

$\{4,5\}$

$\{4,5\}$

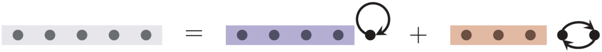
$\{2,7\}$

$\{2,7\}$

$\{3,7\}$

Example: Hubbard model

- In $d = 1$ dimension the 'fermion bags' $\det B^{IJ}$ can be calculated analytically:



and one can prove that

$$\det B^{IJ} \geq 0 \quad \text{for open b.c.}$$

$\Rightarrow$ there is no sign problem

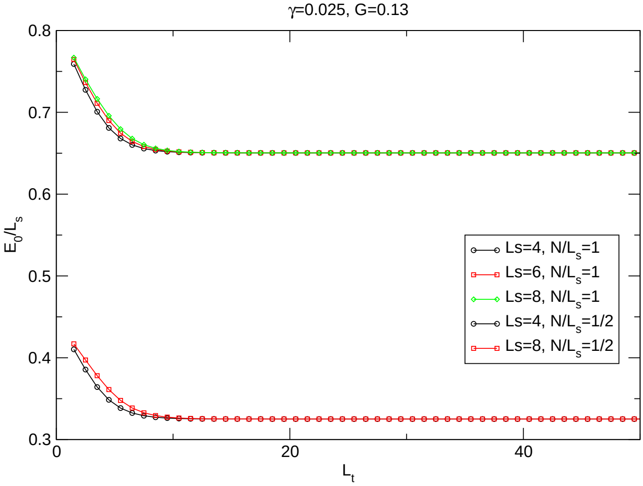
- For periodic b.c. there is no sign problem either, because

$$Z_C^{\text{pbc}}(L_s \to \infty) = Z_C^{\text{obc}}(L_s \to \infty)$$

Example: Hubbard model

- Since our formulation is factorized in time, we have

$$E_0 = \lim_{L_t\to\infty} \frac{Z_C(L_t)}{Z_C(L_{t+1})} = \left\langle \prod_\sigma \left(\frac{\det B^{J^\sigma_{t-1} J^\sigma_{t+1}}}{\det B^{J^\sigma_{t-1} J^\sigma_t} \det B^{J^\sigma_t J^\sigma_{t+1}}} \right) \frac{1}{W(\{J^\sigma_t\})} \right\rangle_{Z_C(L_{t+1})}$$



Grand canonical gauge theories

- Consider gauge theory, e.g. Schwinger model or QCD:

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}U\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\, e^{-S_g[U]-S_f[\overline{\psi},\psi,U;\mu]}$$

  where

$$S_g[U] = \beta \sum_P \left[1 - \frac{1}{2}\left(U_P + U_P^\dagger\right)\right],$$
$$S_f[\overline{\psi},\psi,U;\mu] = \overline{\psi} M[U;\mu]\psi\,.$$

  - for QCD: $d = 4$, $U \in SU(N_c)$
  - for the Schwinger model: $d = 2$, $U \in U(1)$

- Integrating out the Grassmann fields for N_f flavours yields

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}U\, e^{-S_g[U]} \left(\det M[U;\mu]\right)^{N_f}\,.$$

Dimensional reduction of gauge theories

- Consider the Wilson fermion matrix for a single quark with chemical potential μ:

$$M_{\pm}(\mu) = \begin{pmatrix} B_0 & P_+A_0^+ & & & \pm P_-A_{L_t-1}^- \\ P_-A_0^- & B_1 & P_+A_1^+ & & \\ & P_-A_1^- & B_2 & \ddots & \\ & & \ddots & \ddots & \\ & & & & P_+A_{L_t-2}^+ \\ \pm P_+A_{L_t-1}^+ & & & P_- & B_{L_t-1} \end{pmatrix}$$

  - B_t are (spatial) Wilson Dirac operators on time-slice t,
  - Dirac projectors $P_{\pm} = \frac{1}{2}(\mathbb{I} \mp \Gamma_4)$,
  - temporal hoppings are

$$A_t^+ = e^{+\mu} \cdot \mathbb{I}_{d\times d} \otimes \mathcal{U}_t = \left(A_t^-\right)^{-1}$$

  - all blocks are $(d \cdot N_c \cdot L_s^3 \times d \cdot N_c \cdot L_s^3)$-matrices

Dimensional reduction of gauge theories

- Reduced Wilson fermion determinant is given by

$$\det M_{p,a}(\mu) = \prod_t \det Q_t^+ \cdot \det\left[\mathbb{I} \pm e^{+\mu L_t}\mathcal{T}\right]$$

where $\mathcal{T}$ is a product of transfer matrices given by

$$\mathcal{T} = \prod_t \mathcal{U}_{t-1}^+ \cdot (Q_t^-)^{-1} \cdot Q_t^+ \cdot \mathcal{U}_t^-$$

with

$$Q_t^\pm = B_t P_\pm + P_\mp, \qquad \mathcal{U}_t^\pm = \mathcal{U}_t P_\pm + P_\mp$$

- Fugacity expansion yields with $N_Q^{\max} = d \cdot N_c \cdot L_s^3$

$$\det M_a(\mu) = \sum_{N_Q=-N_Q^{\max}}^{N_Q^{\max}} e^{\mu N_Q/T} \cdot \det M_{N_Q}$$

Canonical formulation of gauge theories

Canonical transfer matrices of gauge theories

$$\det M_{N_Q} = \prod_t \det Q_t^+ \cdot \sum_A \det \mathcal{T}^{\backslash A \backslash A} = \mathrm{Tr} \prod_t \mathcal{T}_t^{(N_Q)}$$

- sum is over all index sets $A \in \{1, 2, \ldots, 2N_Q^{\max}\}$ of size N_Q,
- i.e. the trace over the minor matrix of rank N_Q of $\mathcal{T}$

- Provides a complete temporal factorization of the fermion determinant.

Relation between quark and baryon number in QCD

- Consider $\mathbb{Z}(N_c)$-transformation by $z_k = e^{2\pi i \cdot k/N_c} \in \mathbb{Z}(N_c)$:

$$U_4(x) \rightarrow U_4(x)' = (1 + \delta_{x_4,t} \cdot (z_k - 1)) \cdot U_4(x)$$

- Hence, $\mathcal{U}_{x_4}$ transforms as $\mathcal{U}_{x_4} \rightarrow \mathcal{U}'_{x_4} = z_k \cdot \mathcal{U}_{x_4}$, while for all others $\mathcal{U}'_{t \neq x_4} = \mathcal{U}_{t \neq x_4}$.
- As a consequence we have

$$\begin{aligned} \det M_{N_Q} \rightarrow \det M'_{N_Q} &= \prod_t \det Q_t^+ \cdot \sum_A \det(z_k \cdot \mathcal{T})^{|A|} \\ &= z_k^{-N_Q} \cdot \det M_{N_Q} \end{aligned}$$

and summing over z_k therefore yields

$$\det M_{N_Q} = 0 \qquad \text{for } N_Q \neq 0 \bmod N_c$$

  - reduces cancellations by factor of N_c

Gauss' law in the $N_f = 1$ Schwinger model

- Consider $U(1)$-transformation by $e^{i\alpha} \in U(1)$:

$$e^{i\phi_2(x)} \to e^{i\phi_2(x)\prime} = \left(1 + \delta_{x_2,t} \cdot (e^{i\alpha} - 1)\right) \cdot e^{i\phi_2(x)}$$

- Hence, $\mathcal{U}_{x_2}$ transforms as $\mathcal{U}_{x_2} \to \mathcal{U}'_{x_2} = e^{i\alpha} \cdot \mathcal{U}_{x_2}$, while for all others $\mathcal{U}'_{t \neq x_2} = \mathcal{U}_{t \neq x_2}$.
- As a consequence we have

$$\begin{aligned} \det M_{N_Q} \to \det M'_{N_Q} &= \prod_t \det Q_t^+ \cdot \sum_A \det(e^{i\alpha} \cdot \mathcal{T})^{\cancel{A}\cancel{A}} \\ &= e^{-i\alpha N_Q} \cdot \det M_{N_Q} \end{aligned}$$

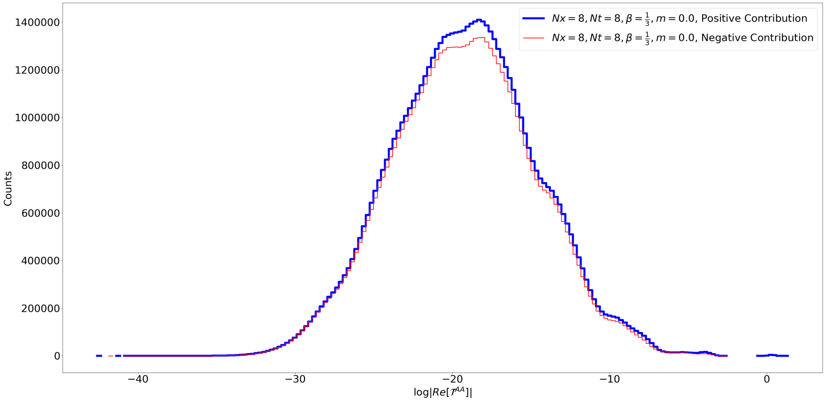
and integrating over α therefore yields

$$\det M_{N_Q} = 0 \qquad \text{for } N_Q \neq 0$$

  - only zero charge sector is allowed!

$N_f = 1$ Schwinger model in $d = 2$

- Distribution of principal minors:



$N_f = 1$ Schwinger model in $d = 2$

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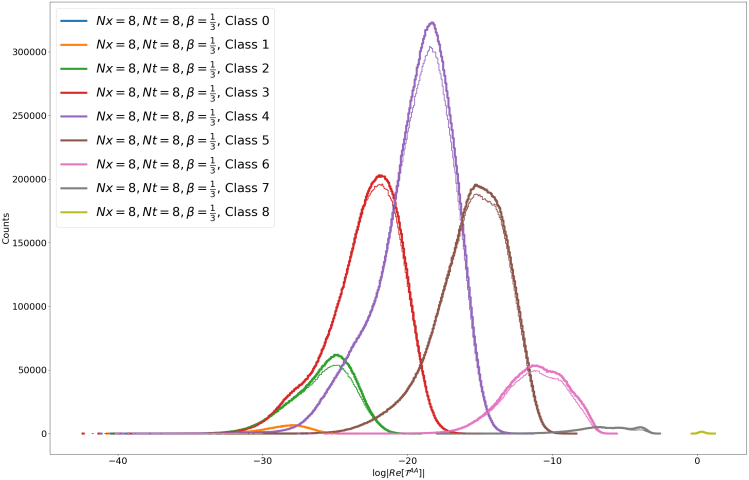
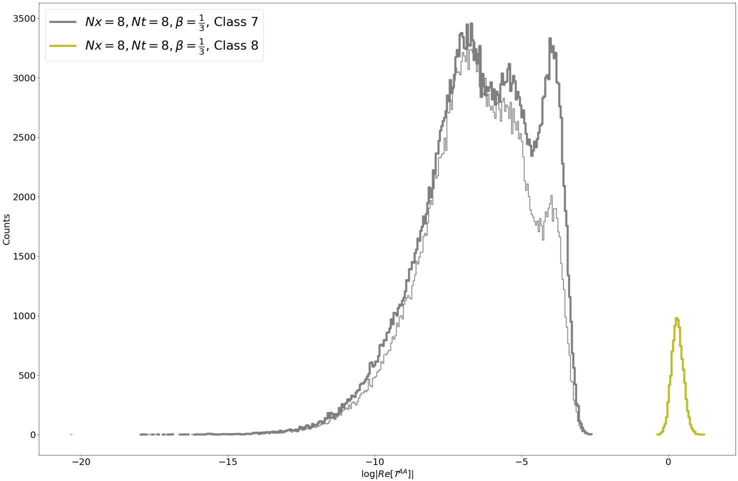


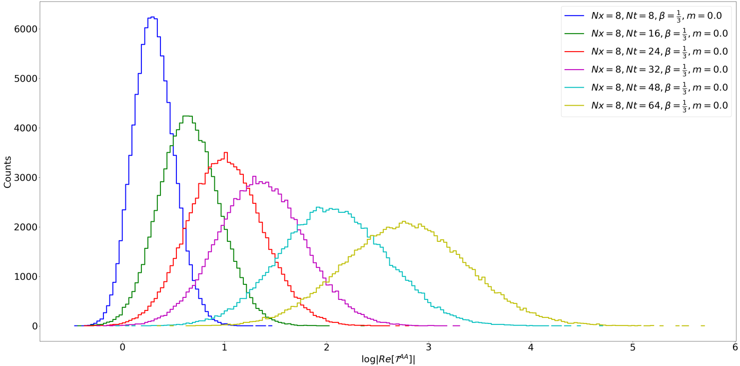
$N_f = 1$ Schwinger model in $d = 2$

- Distribution of maximal principal minors:



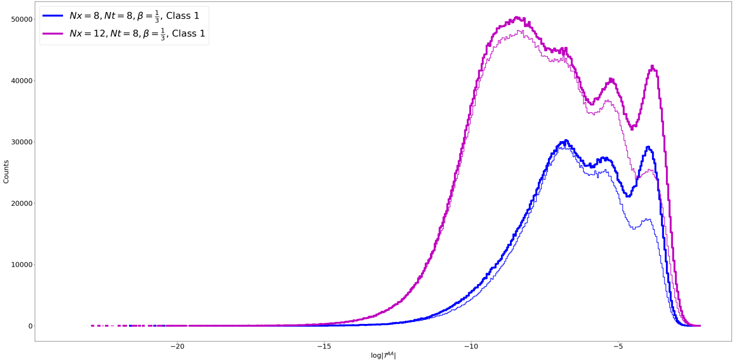
$N_f = 1$ Schwinger model in $d = 2$

- Distribution of maximal principal minors (ground state):



$N_f = 1$ Schwinger model in $d = 2$

- Distribution of next-to-maximal principal minors (exc. states):



$N_f = 2$ Schwinger model in $d = 2$

- Physics in the 2-flavour model is more interesting,
  - denote the fermion flavours by u and d.
- Isospin chemical potential generates multi-meson states.
- Number of u- and d-fermions must be equal:

$$\text{charge } Q = n_u + n_d = 0 \quad \Leftrightarrow \quad \text{Gauss' law},$$
$$\text{isospin } I = (n_u - n_d)/2 \quad \text{arbitrary}$$

- Corresponding canonical partition functions (with $n_u = -n_d$):

$$Z_{n_u,n_d} = \int \mathcal{D}\phi \, e^{-S_g[\phi]} \det{}_{n_u} M_u[\phi] \det{}_{n_d} M_d[\phi] \, .$$

- Vacuum sector is described by $Z_{0,0}$.

Calculating the pion energy

- The flavour-triplet meson (pion) $|\overline{\psi}\gamma_5\tau^a\psi\rangle$ has quantum numbers

$$Q = 0 \quad \text{fermion number}$$
$$I = 1 \quad \text{isospin}$$

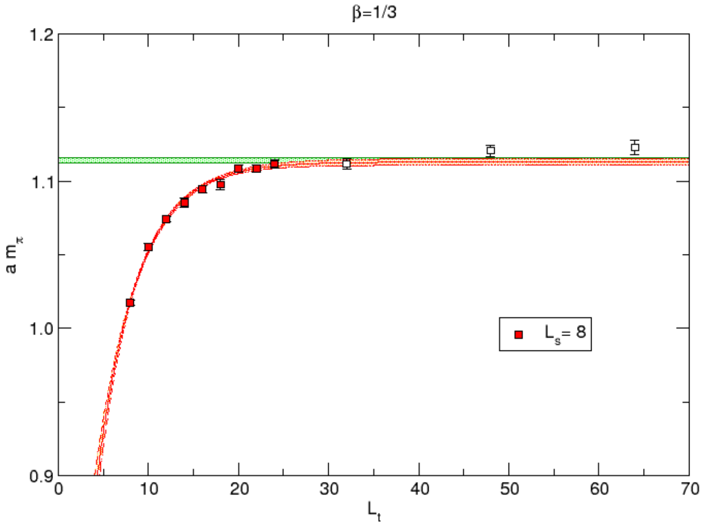
  and is the groundstate of the system with $n_u = +1, n_d = -1$:

$$Z_{+1,-1} = \int \mathcal{D}\phi \, e^{-S_g[\phi]} \, \mathrm{det}_{+1} M_u[\phi] \, \mathrm{det}_{-1} M_d[\phi] \, .$$

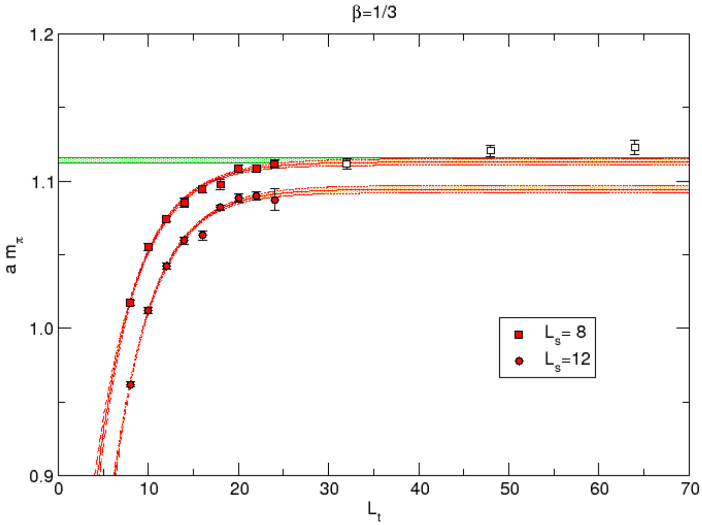
- The free energy difference to the vacuum at $T \to 0$ defines the pion mass:

$$m_\pi(L) = - \lim_{L_t \to \infty} \frac{1}{L_t} \log \frac{Z_{+1,-1}(L_t)}{Z_{0,0}(L_t)} \equiv \mu_1(L)$$

Calculating the pion energy



Calculating the pion energy



Calculating the 2-pion energy

- The flavour-triplet 2-meson (pion) state $|\pi\pi\rangle$ has quantum numbers

$$Q = 0 \quad \text{fermion number}$$
$$I = 2 \quad \text{isospin}$$

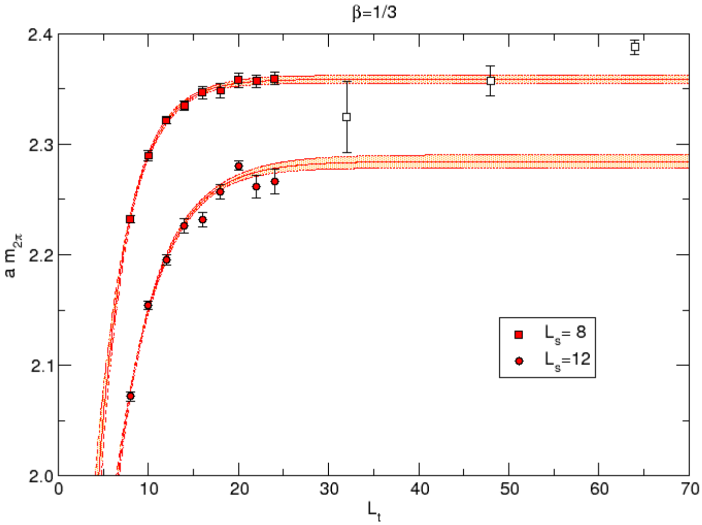
  and is the groundstate of the system with $n_u = +2, n_d = -2$:

$$Z_{+2,-2} = \int \mathcal{D}\phi \, e^{-S_g[\phi]} \det{}_{+2} M_u[\phi] \det{}_{-2} M_d[\phi] \, .$$

- The free energy difference to the vacuum at $T \to 0$ defines the energy of the 2-pion system:

$$E_{2\pi}(L) = - \lim_{L_t \to \infty} \frac{1}{L_t} \log \frac{Z_{+2,-2}(L_t)}{Z_{0,0}(L_t)} \equiv \mu_1(L) + \mu_2(L)$$

Calculating the 2-pion energy



Scattering phase shifts

- $m_\pi(L)$ and $E_{2\pi}(L)$ can be described by 3 parameters:

$$m_\pi(L) = m_\infty + Ae^{-m_\infty L}/\sqrt{L}$$
$$E_{2\pi}(L) = 2\sqrt{m_\pi(L)^2 + p^2}$$

where p is determined through the scattering phase shift

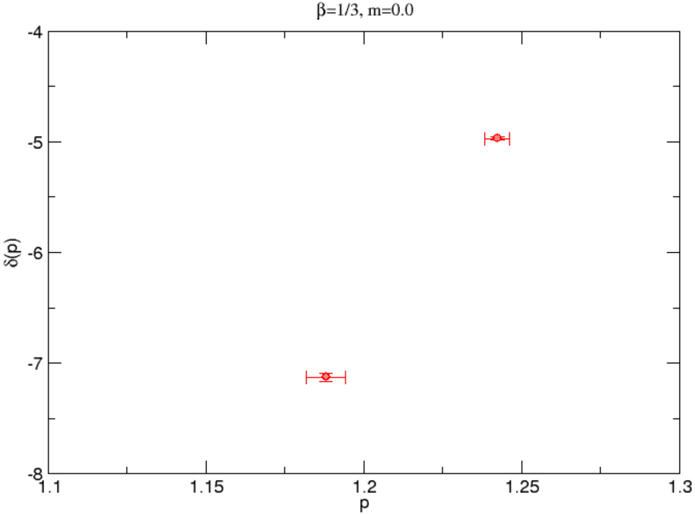
$$\delta(p) = -\frac{pL}{2}\,, \quad \text{or rather} \quad \delta(p(L)) = -\frac{p(L)L}{2} \equiv \delta(L)\,.$$

- From this one can predict the 3-pion energy

$$E_{3\pi}(L) = \sum_{j=1}^{3} \sqrt{m_\pi(L)^2 + p_j^3} \equiv \sum_{i=1}^{3} \mu_i(L)$$

with $p_2 = p_3 = -p_1/2 = -2\delta(L)/L$.

Scattering phase shifts



Summary

- Canonical formulation of field theories:
  - transfer matrices can be obtained explicitly
  - close connection to fermion loop or worldline formulations
  - fermionic degrees of freedom are local occupation numbers $n_x = 0, 1$ (encoded in index sets)

Formalism and techniques are generically applicable:

- sometimes solves (or avoids) the fermion sign problem,
- improved estimators for fermionic correlation functions,
- integrating out (auxiliary) fields in some cases possible:
  - $\Rightarrow$ projection to baryon or zero charge sectors
  - $\Rightarrow$ the HS field in the Hubbard model