Hydrodynamic fluctuations and fluctuation theorem in heavy-ion collisions

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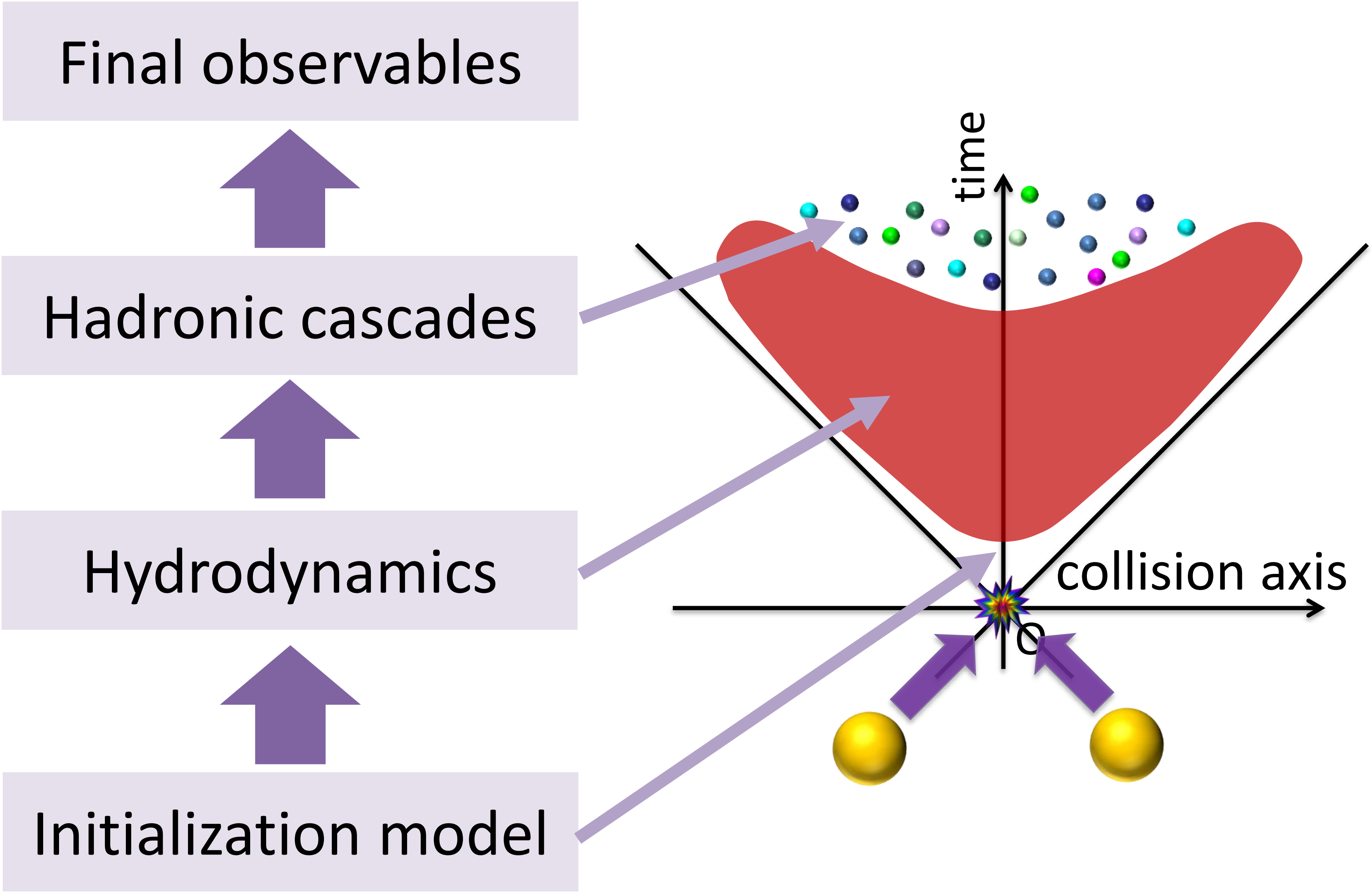
Dynamical models for heavy-ion collisions



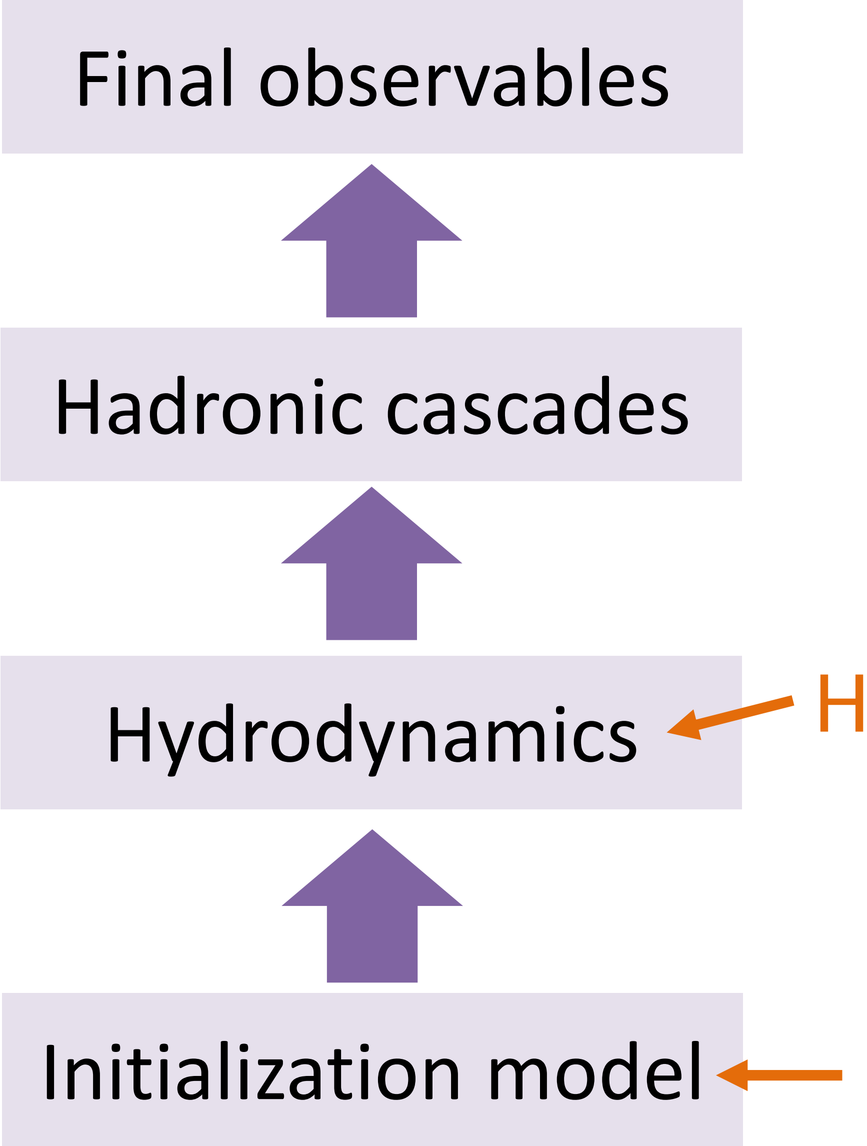
Final observables

Hadronic cascades

Hydrodynamics

Initialization model

Fluctuations/Correlations



Final observables

Hadronic cascades

Hydrodynamics

Initialization model

Non-flow contributions:
jets/mini-jets,
core-corona,
etc.

Hydrodynamic fluctuations:
thermal fluctuations of
hydrodynamics

Initial state fluctuations

Fluctuating hydrodynamics

= viscous hydrodynamics with thermal fluctuations

Stochastic partial differential equations (SPDE)

Conservation law $\partial_\mu T^{\mu\nu} = 0.$

Constitutive eqs.

hydrodynamic fluctuations

$$\pi^{\mu\nu} + \tau_\pi \Delta^{\mu\nu}{}_{\alpha\beta} \mathrm{D}\pi^{\alpha\beta} = 2\eta\partial^{\langle\mu}u^{\rangle\nu} + \cdots + \xi_\pi^{\mu\nu},$$

$$[1 + \tau_\Pi \mathrm{D}]\Pi = -\zeta\partial_\mu u^\mu + \cdots + \xi_\Pi.$$

Fluctuation-dissipation relation (FDR)

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x')\rangle = 4T\eta\Delta^{\mu\nu\alpha\beta}\delta^{(4)}(x - x'),$$

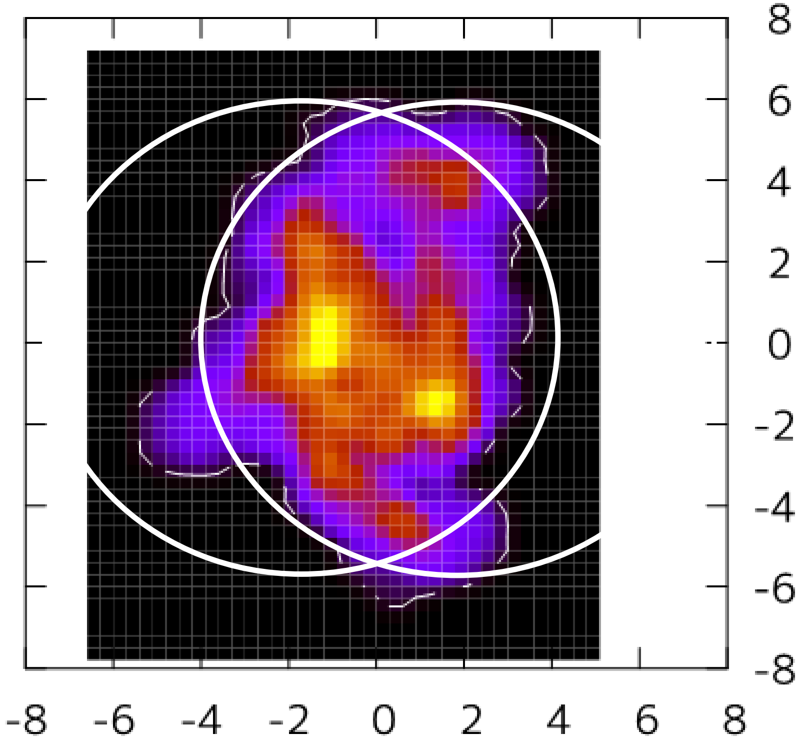
$$\langle \xi(x)\xi(x')\rangle = 2T\zeta\delta^{(4)}(x - x'),$$

FDR MODIFICATION

based on K. Murase, arXiv:1904.11217

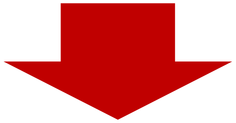
Inhomogeneous and non-static matter

Matter created in nuclear collisions
is not ***static*** and ***homogeneous***



Au+Au 200GeV MC-KLN
(*x*-*y* plane)

Usual FDR relies on
the linear-response
of ***global equilibrium***
to small perturbations



How is FDR modified in
inhomogeneous and
non-static matter?

Constitutive equation (CE)

Constitutive eq. in linear response regime

- ***Integral form***: $\Gamma = \Pi, \pi^{\mu\nu}, \nu_i^\mu$

$$\Gamma(x) = \int d^4x' \underbrace{G(x-x')}_{\text{memory function}} \underbrace{\kappa F(x')}_{\text{1st order term}}$$

- ***Differential form***:

$$L(\mathrm{D}, \nabla^\mu)\Gamma(x) = M(\nabla^\mu) \underbrace{\kappa F(x)}_{\text{1st order}}$$

Γ + (higher order terms)

finite-order in derivatives

e.g. Bulk pressure

$$L = 1 + \tau_\Pi \mathrm{D},$$

$$M = 1$$

$$\Downarrow$$

$$\tau_\Pi D\Pi + \Pi = -\zeta\theta$$

Integral form FDR

$$\Gamma(x) = \int d^4x' G(x,x')\kappa(x')F(x') + \boxed{\delta\Gamma(x)}.$$

Typical form of FDR for integral form noise in *inhomogeneous and non-static backgrounds*

$$\langle \delta\Gamma(x)\delta\Gamma(x')^{\mathrm{T}}\rangle\Theta(x^0 - x'^0) = \underline{G(x,x')}\kappa(x')T(x'),$$

$\neq G(x-x')$

No translational symmetry

See, e.g., D. N. Zubarev, Nonequilibrium Statistical Thermodynamics (Plenum, New York, 1974),
A. Hosoya, M.-a. Sakagami, and M. Takao, Annals Phys. 154, 229 (1984).

Equivalent to

$$\langle \delta\Gamma(x)\delta\Gamma(x')^{\mathrm{T}}\rangle = \underline{G(x,x')}\kappa(x')T(x') + T(x)\kappa(x)\underline{G(x',x)^{\mathrm{T}}}$$

Differential form FDR

Linear-response differential CE with noise

$$L(\mathrm{D}, \nabla^{\mu}; x)\Gamma(x) = M(\nabla^{\mu}; x)\kappa F(x) \boxed{+ \xi(x)}.$$

Relations to (*G*, *δ*Γ) in integral form CE

$$L(\mathrm{D}, \nabla^{\mu}; x)G(x, x') = M(\nabla^{\mu}; x)\delta^{(4)}(x - x'),$$

$$\xi(x) = L(\mathrm{D}, \nabla^{\mu}; x)\delta\Gamma(x).$$

+ Integral form FDR

→ Differential form FDR

$$\begin{aligned}\langle \xi(x)\xi(x')^{\mathrm{T}} \rangle = {} & M(\nabla^{\mu}; x)\delta^{(4)}(x - x')\kappa(x')T(x')L(\overleftarrow{\mathrm{D}}', \overleftarrow{\nabla}'^{\mu}; x')^{\mathrm{T}} \\ & + L(\mathrm{D}, \nabla^{\mu}; x)T(x)\kappa(x)\delta^{(4)}(x - x')M(\overleftarrow{\nabla}'^{\mu}; x')^{\mathrm{T}}.\end{aligned}$$

Simplified IS case

Differential form FDR for simplified Israel-Stewart theory

$$\langle \xi_\Pi(x)\xi_\Pi(x')\rangle = \left(2 + \tau_\Pi \mathrm{D}\ln\frac{T\zeta}{\tau_\Pi} - \tau_\Pi\theta\right) T\zeta\delta^{(4)}(x-x'),$$

$$\langle \xi_\pi^{\mu\nu}(x)\xi_\pi^{\alpha\beta}(x')\rangle = 2\left[\left(2 + \tau_\pi \mathrm{D}\ln\frac{T\eta}{\tau_\pi} - \tau_\pi\theta\right)\Delta^{\mu\nu\alpha\beta} + \tau_\pi\mathcal{D}\Delta^{\mu\nu\alpha\beta}\right] T\eta\delta^{(4)}(x-x'),$$

$$\begin{aligned}\langle \xi_i^\mu(x)\xi_j^\alpha(x')\rangle &= -2T\kappa_{ij}\Delta^{\mu\alpha}\delta^{(4)}(x-x') \\ &\quad - \Delta^{\mu\alpha}[K_{ij}^{\mathrm{A}}(x)\mathcal{D} - K_{ij}^{\mathrm{A}}(x')\mathcal{D}']\delta^{(4)}(x-x') \\ &\quad + \sum_{k=1}^{n}\left\{-\Delta^{\mu\alpha}\left[\tau_{ik}\mathrm{D}T\kappa_{kj} - (\mathrm{D}\tau_{ik})T\kappa_{kj} - \tau_{ik}\theta T\kappa_{kj}\right]^{\mathrm{S}} - K_{ij}^{\mathrm{S}}\mathcal{D}\Delta^{\mu\alpha}\right\}\delta^{(4)}(x-x'),\end{aligned}$$

where $K_{ij}^{\mathrm{S/A}}(x) = \sum_{k=1}^{n} T(x)(\tau_{ik}(x)\kappa_{kj}(x) \pm \tau_{jk}(x)\kappa_{ki}(x))/2$, and $[\circ_{ij}]^{\mathrm{S}} = (\circ_{ij} + \circ_{ji})/2$.

$$\begin{aligned}\mathcal{D}\Delta^{\mu\nu\alpha\beta} &= \Delta^{\mu\nu}{}_{\kappa\lambda}\Delta^{\alpha\beta}{}_{\gamma\delta}\mathrm{D}\Delta^{\kappa\lambda\gamma\delta}, \\ \mathcal{D}\Delta^{\mu\alpha} &= \Delta^{\mu}{}_{\kappa}\Delta^{\alpha}{}_{\gamma}\mathrm{D}\Delta^{\kappa\gamma}.\end{aligned}$$

complicated expression due to the tensor structure…

Essential structure

$$\langle \xi(x)\xi(x')\rangle = \left(2 \boxed{+ \underline{\tau_R}\mathrm{D}\ln\frac{T\kappa}{\tau_R} - \underline{\tau_R}\theta}\right) T\kappa\delta^{(4)}(x-x').$$

new modification terms ∝ relaxation time

FLUCTUATION THEOREM IN BJORKEN EXPANSION

based on T. Hirano, R. Kurita, KM, Nucl. Phys. A984 (2019) 44-67

Fluctuation theorem (FT)

D. J. Evans, E. G. D. Cohen, G. P. Morriss, Phys. Rev. Lett. **71**, 2401–2404 (1993)

Relations for probability density **Pr(δ*S*)** of entropy production **δ*S***

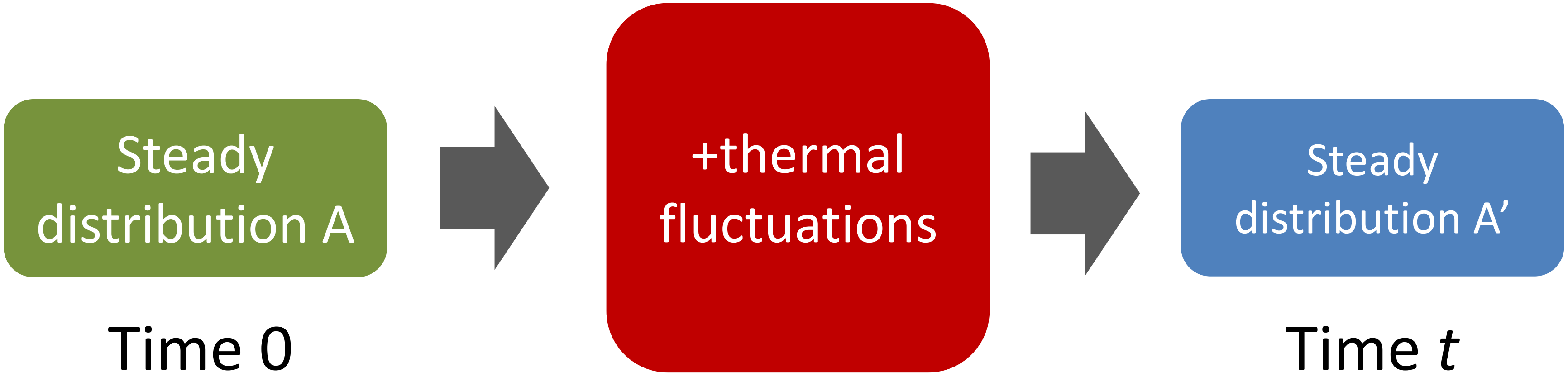
known in non-equilibrium statistical mechanics:

$$\ln \frac{\Pr(\delta S = \alpha)}{\Pr^{\dagger}(\delta S^{\dagger} = -\alpha)} = \alpha$$

Note: definitions of δS, $\delta S^{\dagger}$, $\Pr(\delta S)$, $\Pr^{\dagger}(\delta S^{\dagger})$
depends on systems and processes.
→ many variations of FT

Fluctuation theorem (FT)

(Example) Steady-state FT (**SSFT**)



$$\ln \frac{\Pr(\bar{\sigma} = \alpha)}{\Pr(\bar{\sigma} = -\alpha)} = \alpha t + O(\tau_R / t)$$

- $\sigma = \delta S / t$: entropy production rate per unit time
- τ_R: relaxation time scale of the system

FT in high-energy nuclear collisions

FT: generalization of FDR near the equilibrium

→ Relations to FDR in fluctuating hydrodynamics?

→ Entropy distribution through multiplicity?

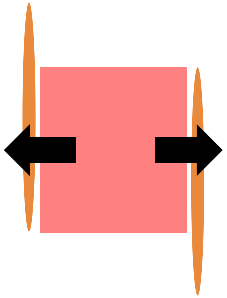
➡ Qualitative understanding under idealized assumptions

Simplification

① **Bjorken flow**: (0+1)-dim evolution

② **Linear fluctuations**: no non-linear fluctuations

③ **Navier-Stokes limit**: negligible τ_R



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Hydrodynamic equations

(0+1)-dim Bjorken flow (assumption ①)

2nd order fluctuating hydro

Conservation

τ: proper time

$$\frac{\mathrm{d}e}{\mathrm{d}\tau} = -\frac{e+p}{\tau}\left(1 - \frac{\pi - \Pi}{sT}\right)$$

Constitutive eqs.

$\pi = \pi^{00} - \pi^{33}$

$$\left(\tau_\pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\pi = \frac{4\eta}{3\tau} + \underline{\xi_\pi},$$

$$\left(\tau_\Pi \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\Pi = -\frac{\zeta}{\tau} + \underline{\xi_\Pi}.$$

Hydrodynamic fluctuations

Gaussian noise

FDR (without modifications)

Cell

$V(\tau) = \tau \Delta\eta_s \Delta x \Delta y$

$$\langle \xi_\pi(\tau)\xi_\pi(\tau')\rangle = \frac{8\eta T}{3\tau\Delta\eta_s\Delta x\Delta y}\delta(\tau - \tau'),$$

$$\langle \xi_\Pi(\tau)\xi_\Pi(\tau')\rangle = \frac{2\zeta T}{\tau\Delta\eta_s\Delta x\Delta y}\delta(\tau - \tau').$$

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Entropy production rate

Step 1. Def. Entropy production in a cell

$$\bar{\sigma} := \frac{s(\tau)V(\tau) - s(\tau_{\mathrm{i}})V(\tau_{\mathrm{i}})}{\tau - \tau_{\mathrm{i}}}.$$

s: Equilibrium entropy
V(τ): volume of the cell
τ_{i}: initial time

Step 2. Result of time evolution

$$\bar{\sigma} = \frac{1}{\tau - \tau_{\mathrm{i}}} \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' \frac{\pi(\tau') - \Pi(\tau')}{T(\tau')} \Delta\eta_s \Delta x \Delta y.$$

where

$$\pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\pi}(\tau, \tau') \frac{4\eta}{3\tau'} + \delta\pi(\tau), \qquad \delta\pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\pi}(\tau, \tau') \xi_{\pi}(\tau'),$$

$$\Pi(\tau) = -\int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\Pi}(\tau, \tau') \frac{\zeta}{\tau'} + \delta\Pi(\tau), \qquad \delta\Pi(\tau) = \int_{\tau_{\mathrm{i}}}^{\tau} \mathrm{d}\tau' G_{\Pi}(\tau, \tau') \xi_{\Pi}(\tau').$$

$$G_{\pi/\Pi}(\tau_2, \tau_1) := \exp\left(-\int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{\tau_{\pi/\Pi}(\tau)} \right) \frac{1}{\tau_{\pi/\Pi}(\tau_1)}$$

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Entropy production rate

Step 3. Distribution of entropy production

Linear fluctuations (assumption ②) → Gaussian distributions

Navier-Stokes limit (assumption ③) → Simplified expr.

average
$$\langle \bar{\sigma} \rangle = \frac{\Delta \eta_s \Delta x \Delta y}{\tau - \tau_{\mathrm{i}}} \boxed{\int_{\tau_{\mathrm{i}}}^{\tau} \frac{\mathrm{d}\tau'}{T_0(\tau')} \left(\frac{4\eta}{3\tau'} + \frac{\zeta}{\tau'} \right)},$$

variance
$$\begin{aligned} a^2 &= \langle \bar{\sigma}^2 \rangle - \langle \bar{\sigma} \rangle^2 \\ &= \frac{2\Delta \eta_s \Delta x \Delta y}{(\tau - \tau_{\mathrm{i}})^2} \boxed{\int_{\tau_{\mathrm{i}}}^{\tau} \frac{\mathrm{d}\tau'}{T_0(\tau')} \left(\frac{4\eta}{3\tau'} + \frac{\zeta}{\tau'} \right)} \end{aligned}$$

→ $$\frac{2\langle \bar{\sigma} \rangle}{a^2} = \tau - \tau_{\mathrm{i}} \quad \Leftrightarrow \quad \ln \frac{\Pr(\bar{\sigma} = \alpha)}{\Pr(\bar{\sigma} = -\alpha)} = \alpha t + O(\tau_R / t)$$

✓ Expression equivalent to SSFT in expanding system

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Effects of non-linearity/relaxation

Simplification

① **Bjorken flow**

② ~~Linear fluctuations~~

③ ~~Navier-Stokes limit~~

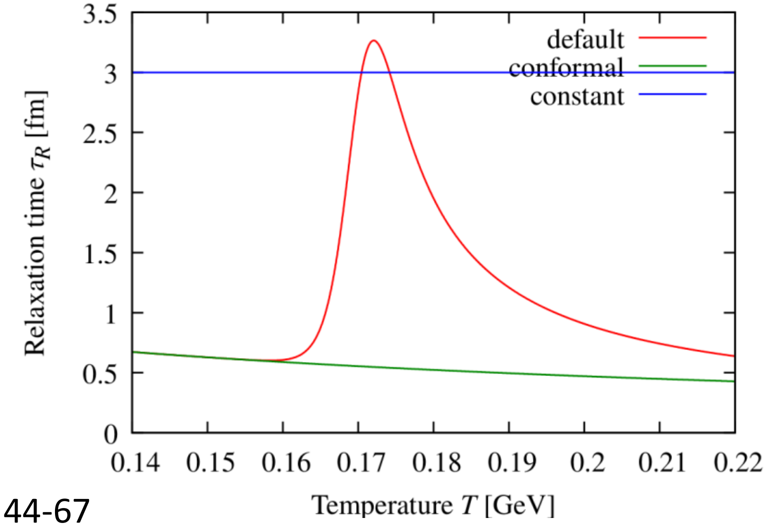
Effects of *non-linear fluctuations* and *relaxation time* by 0+1 dim numerical simulations

3 models of relaxation time

1. default $\tau_\pi = \tau_\Pi = \dfrac{3\eta}{2p}$

2. conformal $\tau_\pi = \tau_\Pi = 3/2\pi T$

3. constant $\tau_\pi = \tau_\Pi = 0.3\mathrm{fm}$



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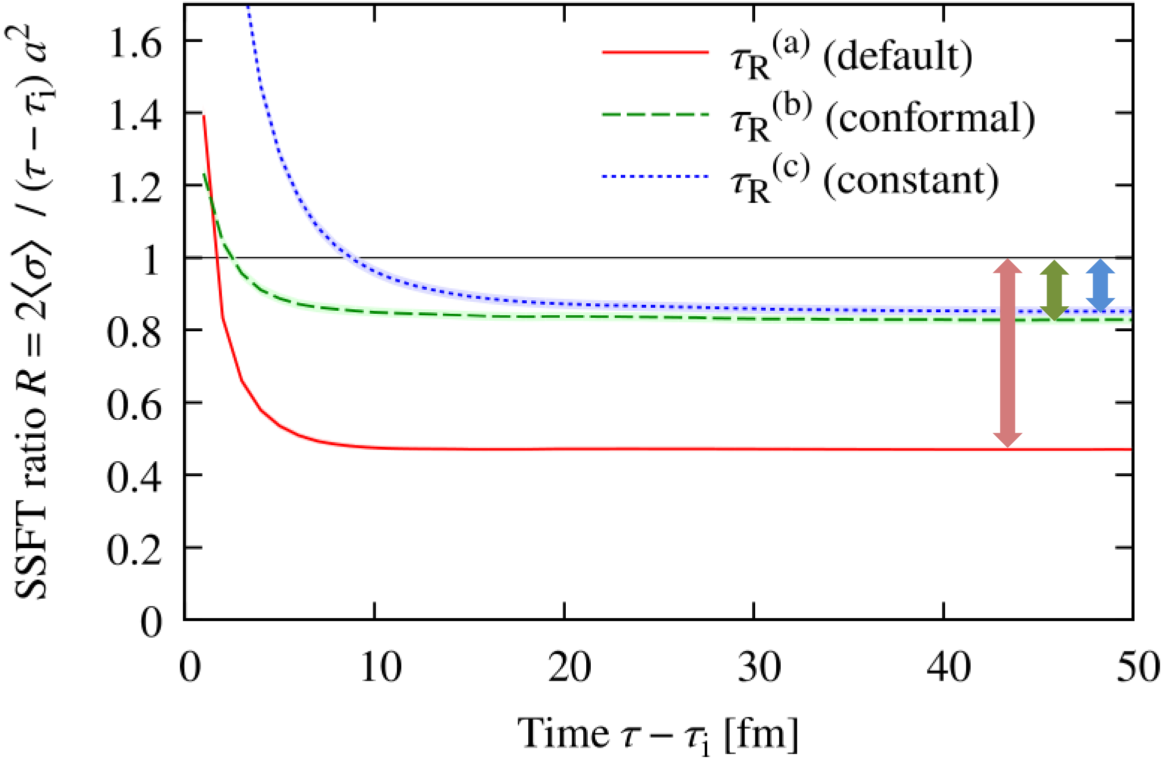
SSFT Ratio *R*

SSFT $\frac{2\langle\bar{\sigma}\rangle}{a^2} = \tau - \tau_\mathrm{i}$ → **Ratio** $R = \frac{2\langle\bar{\sigma}\rangle}{a^2 \cdot (\tau - \tau_\mathrm{i})}$

Time evolution of *R*



R ≠ 1

SSFT breaking

Significant
in default model

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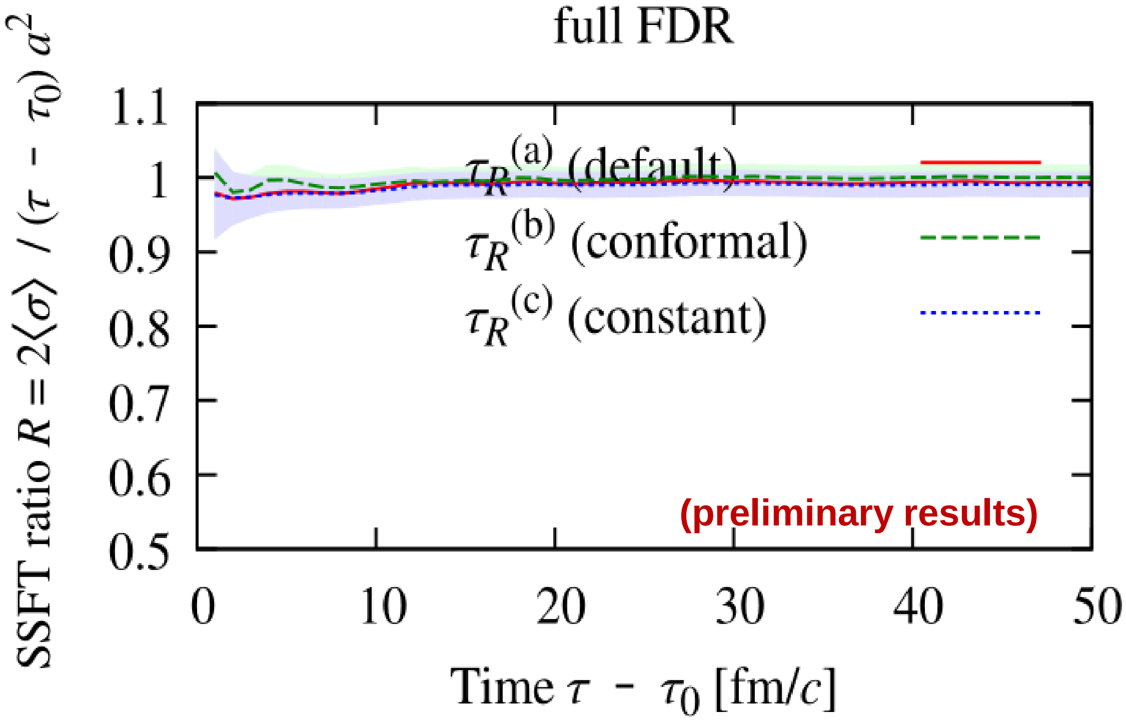
With FDR modifications

SSFT breaking by relaxation times was actually artifact.

With the FDR modification, SSFT recovers:

$$\langle \xi(x)\xi(x')\rangle = \left(2 \boxed{+ \tau_R \mathrm{D} \ln \frac{T\kappa}{\tau_R} - \tau_R \theta}\right) T\kappa\delta^{(4)}(x-x').$$

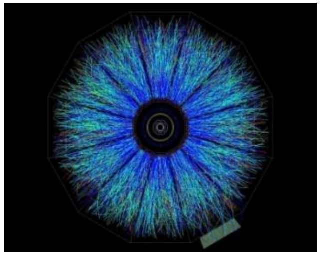


Note: there is still small breaking due to non-linear fluctuations

✓ FDR modification is needed to maintain correct entropy distribution obeying FT

Multiplicity fluctuations

In high-energy nuclear collisions?



Au+Au $\sqrt{s_{NN}}$ = 200 GeV
STAR Collaboration (RHIC)

Upper bound in entropy fluctuations

$$\frac{\Delta S(\tau)}{\langle S(\tau)\rangle} = \frac{a(\tau-\tau_i)}{S_i+\langle\bar{\sigma}\rangle(\tau-\tau_i)}$$

$$= \frac{\sqrt{2\langle\bar{\sigma}\rangle(\tau-\tau_i)}}{S_i+\langle\bar{\sigma}\rangle(\tau-\tau_i)}$$

$$\leq \frac{1}{\sqrt{2S_\mathrm{i}}}$$

S: entropy
S_i: initial entropy

∵ SSFT $\frac{2\langle\bar{\sigma}\rangle}{a^2} = \tau-\tau_\mathrm{i}$

∵ inequality $\frac{\sqrt{2y}}{x+y} \leq \frac{1}{\sqrt{2x}}$

→ Upper bound of multiplicity fluctuations

$$\frac{(\Delta_\mathrm{ev}N)^2 - \underbrace{\langle N\rangle_\mathrm{ev}}_{\text{Poisson statistics}}}{\langle N\rangle_\mathrm{ev}^2} \leq \underbrace{\frac{(\Delta_\mathrm{ev}S_\mathrm{tot,i})^2}{\langle S_\mathrm{tot,i}\rangle_\mathrm{ev}^2}}_{\text{Initial state fluctuations}} + \underbrace{\frac{1}{2\langle S_\mathrm{tot,i}\rangle_\mathrm{ev}}}_{\text{SSFT upper bound}}$$

LHS: observables
RHS: initial state
→constraining initial state independently of intermediate dynamics?

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SUMMARY

Summary

- ***Fluctuation-dissipation relation (FDR)*** *for second-order hydro*, which determines the power of hydrodynamic fluctuations, has ***modifications proportional to τ_R*** in inhomogeneous and non-static systems
- ***Steady-state Fluctuation theorem (SSFT)*** holds for relativistic hydrodynamics with FDR modifications (for linear fluctuations)
- FDR modifications should be incorporated into dynamical models with relaxation times