Linked Cluster Expansion for $SU(3)$ Spin Model with a Sign Problem

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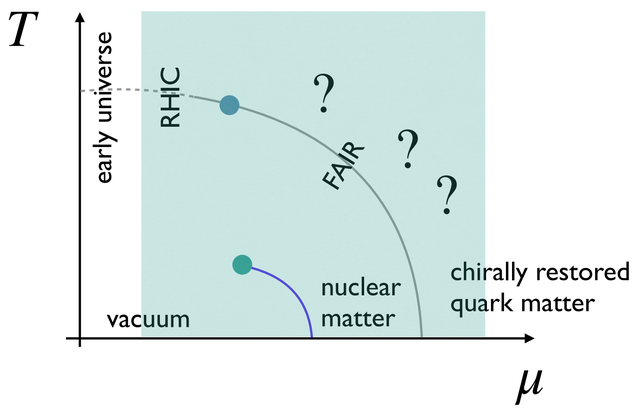
Introduction



Recently, a lot of attention has been paid to whether a critical point exists on the phase diagram of strongly interacting matter (quark matter). The main tool for extracting quantitative results is based on Monte Carlo algorithms. Unfortunately, at finite baryon densities all Monte Carlo algorithms suffer from the sign problem.

We revisit a well-known method in statistical mechanics called Series Expansion Method, which provides an alternative way to overcome the sign problem.

An $SU(3)$ spin model is used as a first step to deal with QCD at finite chemical potential.

$SU(3)$ spin model

The $SU(3)$ spin model is a three dimensional effective theory of QCD at strong coupling limit and large quark mass limit. The partition function of the model reads

$$Z = \int D[U_0] \exp\left(\tau \sum_{\langle \vec{x},\vec{y}\rangle} (L_{\vec{x}} L^*_{\vec{y}} + L^*_{\vec{x}} L_{\vec{y}}) + \kappa \sum_{\vec{x}} (e^{a\mu} L_{\vec{x}} + e^{-a\mu} L^*_{\vec{x}})\right),$$

where $L_{\vec{x}}$'s are Polyakov loops, $\kappa \sim m_q^{-2}$. The first term presents the nearest neighbour interaction of Polyakov loops, while the second term couples the system to quarks at finite chemical potential μ.

A particular observable that was used in our computation is the susceptibility of equilibrium Polyakov loop L, which diverges near critical points:

$$L = \frac{1}{V} \sum_{\vec{x}} (L_{\vec{x}} + L^*_{\vec{x}}),$$

$$\chi_L = \frac{1}{V} \frac{\partial^2 \log Z[J]}{\partial J^2}$$

Conclusions

- Analytic calculations for the $SU(3)$ spin model via LCE are done for the first time.
- Our results agree with those from simulation with flux representation algorithm within errors.
- To include the parameter μ it is needed to compute higher orders.
- Our next goal is to apply the series expansion method to improved effective theories.

Linked Cluster Expansion (LCE)

Similar to the generating functional for connected diagrams in QFT, we define the grand canonical potential as

$$W[J, \nu] = \log Z[J, \nu]$$

where $Z[J, \nu]$ is the partition function coupled to an external source J. A linked cluster expansion (LCE) of the grand canonical potential is then defined as the Taylor expansion with respect to the coupling v(x,y) around the free theory,

$$W[J, \nu] = \left(\exp\left(\nu(\vec{x}, \vec{y}) \frac{\delta}{\delta \hat{\nu}(\vec{x}, \vec{y})}\right) W[J, \hat{\nu}]\right)\Big|_{\hat{\nu}=0}$$

There is a very simple expression of W through **graph representation**,

$$\mathcal{W}[\nu] = \bullet + \tfrac{1}{2} \bullet\!\!-\!\!\bullet + \tfrac{1}{2} \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet + \tfrac{1}{4} \bullet\!\!=\!\!\bullet + \ldots$$

In each term of the graph representation, the number of bonds corresponds to the order of the coupling ν. In additions, to complete the computation one also needs to calculate integrals at each vertices of graphs, e.g.

$$I_{mn} = \int dU_0 L^m L^{*n}$$

Series Analysis

The Polyakov loop susceptibility computed up to order $K = 13$ is a series of the interaction coupling τ while the coefficients are functions of κ and μ,

$$\chi_L = \sum_{n=0}^{K} c_n(\kappa, \mu) \tau^n.$$

These polynomials are obviously not divergent near critical points. Thus, the (generalised) Pade' approximant is needed for extracting information about critical points and critical exponents. Example: for one coupling series ($\kappa = 0$), the Polyakov loop susceptibility near τ_c behaves as

$$\chi_L \sim \frac{1}{(\tau - \tau_c)^\gamma} \quad \rightarrow \quad \frac{d}{d\tau} \log \chi_L \sim \frac{-\gamma}{\tau - \tau_c}$$

$$\sim [X/Y] = \frac{a_0 + a_1\tau + \cdots + a_X \tau^X}{1 + b_1\tau + \cdots + b_Y \tau^Y}$$

References

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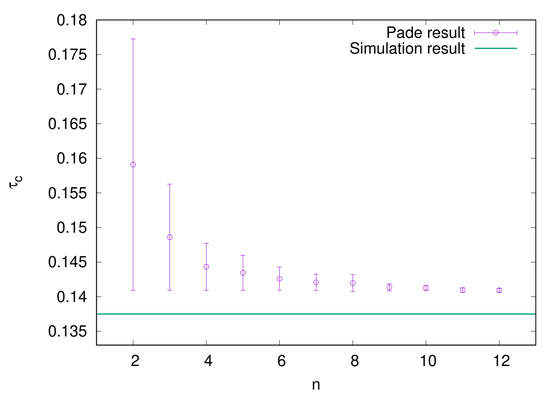
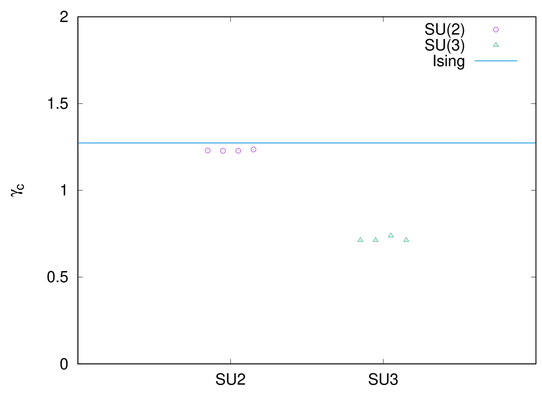
Results

Estimation of τ_c and critical exponents for $SU(2)$, $SU(3)$. $\gamma_c^{SU(3)} \approx 0.712$ which is unknown universality class ~ first order phase transition





Unbiased estimations of critical points (κ_c, τ_c) at $\mu = 0$ from our series