Tensor network study of two dimensional complex ϕ^4 theory at finite density

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Complex ϕ^4 theory (relativistic bose gas) at finite density

$$S = \int d^2x \left\{ |\partial_\nu \phi|^2 + \left(m^2 - \mu^2\right) |\phi|^2 + \mu \left(\phi^* \partial_2 \phi - \partial_2 \phi^* \phi\right) + \lambda |\phi|^4 \right\}.$$

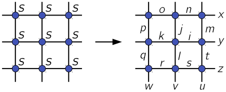
- ϕ: complex scalar field; $\phi = (\phi_1, \phi_2)$
- m, λ, μ: mass, coupling, chemical potential

- Typical model suffering from the severe sign problem
- Many previous studies, *e.g.* [Endres 2007; Aarts 2009; Gattringer and Kloiber 2013]
  $\Rightarrow$ Good practice table
- We analyze this model using TRG, a sign problem free method.

Tensor renormalization group (TRG) [Levin and Nave 2007]

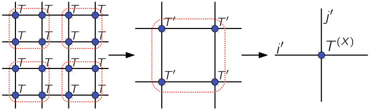
1. Tensor network representation of Z

$$Z = \sum_{\{s\}} e^{-\beta H[s]} \rightarrow \sum_{\dots,i,j,k,l,\dots} \cdots T_{ijkl} T_{ymit} T_{xunm} \cdots$$



2. Coarse-graining of the tensor network

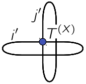
$$TT\cdots T \underbrace{\rightarrow \cdots \xrightarrow{\text{Coarse-graining}} \cdots \rightarrow}_{X \text{ times}} T^{(X)}$$



3. Contraction of the "effective tensor" $T^{(X)}$

$$Z \approx \sum_{i',j'} T^{(X)}_{i'j'i'j'}$$

$$Z \approx \sum_{i',j'}$$



- √ Deterministic data compression
- √ Completely free of the sign problem
- √ Systematic error is controlled by one parameter (the size of $T^{(X)}$).

Examples of TRG study for relativistic field theories

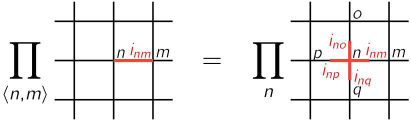
- Bosons:
  - 2D real ϕ^4 model [Shimizu 2012; Kadoh, Kuramashi, Nakamura, RS, Takeda, Yoshimura 2019]
  - 2D CP(1) model [Kawauchi and Takeda 2016]
- (Gauged) fermions:
  - 2D Schwinger model w/ θ term [Shimizu and Kuramashi 2014a, 2014b, 2018]
  - 2D Thirring model w/ chemical potential [Takeda and Yoshimura 2015]
  - 3D free fermions [RS, Takeda, Yoshimura 2017; Yoshimura et al. 2017]
- Pure gauge theories:
  - 2D/3D U(1) and SU(2) theory [Liu et al. 2013]
  - 3D Z_2 gauge theory [Kuramashi and Yoshimura 2018]
- Supersymmetric theory
  - 2D $\mathcal{N} = 1$ Wess–Zumino model [Kadoh, Kuramashi, Nakamura, RS, Takeda, Yoshimura 2018]

Ex) partition function of the 2D Ising model

$$Z = \sum_{\{s\}} e^{-\beta H[s]} = \sum_{\{s\}} \prod_{\langle n,m \rangle} e^{\beta s_n s_m},$$
$$H[s] = -\sum_{\langle n,m \rangle} s_n s_m$$

How to construct a tensor network representation of Z:

1. Expand the Boltzmann factor
   - High T expansion: $e^{\beta s_n s_m} = \cosh\beta \sum_{i_{nm}=0}^{1} (s_n s_m \tanh\beta)^{i_{nm}}$
2. Reread $\prod_{\langle n,m \rangle}$ as $\prod_n$, and trace out the old d.o.f. (spin s)



3. Consider the remaining integer d.o.f. as tensor indices

Partition function of the 2D complex ϕ^4 model

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S},$$

$$S = \sum_n \left[\left(4 + m^2\right) |\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\nu=1}^{2} \left(e^{\mu\delta_{\nu,2}} \phi_n^* \phi_{n+\hat{\nu}} + e^{-\mu\delta_{\nu,2}} \phi_{n+\hat{\nu}}^* \phi_n \right) \right]$$

- ϕ: complex scalar field; $\phi = (\phi_1, \phi_2)$
- m, λ, μ: mass, coupling, chemical potential
- n, $\hat{\nu}$: lattice coordinate, unit vector along ν-direction

How to construct a tensor network representation of Z:

1. Expand the Boltzmann factor
   - *cf.* Ising model: high T expansion
2. Integrate out the original d.o.f. (the scalar field ϕ)
3. Consider the remaining integer d.o.f. as tensor indices

TN rep. of scalar theory

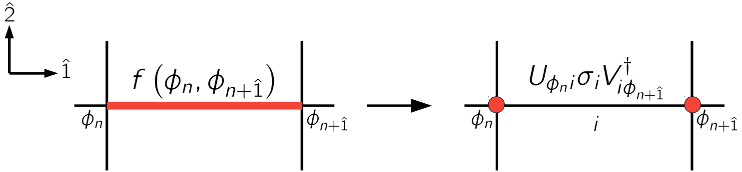
- Since S takes nearest-neighbor form,

$$e^{-S} = \prod_n \prod_\nu f_\nu\left(\phi_n, \phi_{n+\hat{\nu}}\right),$$

$$f_\nu\left(\phi_1, \phi_2\right) = \exp\left\{ -\frac{1}{4}\left(4+m^2\right)\left(|\phi_1|^2 + |\phi_2|^2\right) - \frac{\lambda}{4}\left(|\phi_1|^4 + |\phi_2|^4\right) + e^{\mu\delta_{\nu,2}}\phi_1^*\phi_2 + e^{-\mu\delta_{\nu,2}}\phi_1\phi_2^* \right\}.$$

- If f can be decomposed using discrete d.o.f,

$$f\left(\phi_n, \phi_{n+\hat{\mu}}\right) = \sum_i U_{\phi_n i}\sigma_i V^\dagger_{i\phi_{n+\hat{\mu}}}$$



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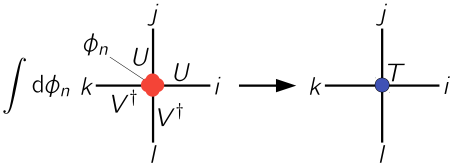
- If f can be decomposed using discrete d.o.f,

$$T_{ijkl} = \int \mathrm{d}\phi_n \sqrt{\sigma_i\sigma_j\sigma_k\sigma_l}\, U_{\phi_n i} U_{\phi_n j} V^\dagger_{k\phi_n} V^\dagger_{l\phi_n}$$



Gauss–Hermite quadrature rule

$$\int_{-\infty}^{\infty} \mathrm{d}\phi e^{-\phi^2} g(\phi) \approx \sum_{\alpha=1}^{K} w_\alpha g(x_\alpha)$$

- $g(\phi)$: (well-behaved) arbitrary function [1]
- K: degree of the Hermite polynomial
- x_α: α-th root of the Hermite polynomial
- w_α: α-th weight of the GH quadrature

[1] If g is a polynomial function of order $2K-1$ or less, the Gaussian quadrature rule yields the exact solution.

Spectral decomposition of local Boltzmann weight

Replace integral of ϕ_n using GH quadrature: $\phi_n = (\phi_{n,1}, \phi_{n,2}) \to (x_{\alpha_1}, x_{\alpha_2})$
(Interpolate f using the roots of the Hermite polynomial)

$$\int d\phi_{n,1} d\phi_{n,2} \prod_{\nu=1}^{2} f_\nu\left((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (\phi_{n,1}, \phi_{n,2})\right) \cdot f_\nu\left((\phi_{n,1}, \phi_{n,2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2})\right)$$

$$\approx \sum_{\alpha_1,\alpha_2=1}^{K} w_{\alpha_1} w_{\alpha_2} e^{x_{\alpha_1}^2 + x_{\alpha_2}^2} \prod_{\nu=1}^{2} f_\nu\left((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (x_{\alpha_1}, x_{\alpha_2})\right) \cdot f_\nu\left((x_{\alpha_1}, x_{\alpha_2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2})\right),$$

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$$\begin{aligned}\int d\phi_{n,1} d\phi_{n,2} \prod_{\nu=1}^{2} & f_\nu \left((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (\phi_{n,1}, \phi_{n,2}) \right) \\ & \cdot f_\nu \left((\phi_{n,1}, \phi_{n,2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2}) \right) \\ \approx \sum_{\alpha_1,\alpha_2=1}^{K} w_{\alpha_1} w_{\alpha_2} e^{x_{\alpha_1}^2 + x_{\alpha_2}^2} \prod_{\nu=1}^{2} & f_\nu \left((\phi_{n-\hat{\nu},1}, \phi_{n-\hat{\nu},2}), (x_{\alpha_1}, x_{\alpha_2}) \right) \\ & \cdot f_\nu \left((x_{\alpha_1}, x_{\alpha_2}), (\phi_{n+\hat{\nu},1}, \phi_{n+\hat{\nu},2}) \right),\end{aligned}$$

$$f_\nu (\phi_n, \phi_{n+\hat{\nu}}) \to f_\nu \left((x_{\alpha_1}, x_{\alpha_2}), (x_{\beta_1}, x_{\beta_2}) \right) = \sum_{m=1}^{K^2} U_{(x_{\alpha_1}, x_{\alpha_2})m} \sigma_m V^\dagger_{m(x_{\beta_1}, x_{\beta_2})}$$

- f becomes a matrix and can be numerically decomposed by singular values!

Tensor network representation

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S} \approx \sum_{i,j,k,l,\ldots} T_{ijkl} T_{mnio} \cdots$$

- Z is written as $\int$ of field ϕ → $\sum$ of tensor indices $i, j, k, \ldots$!
- TN rep. of Green's functions can be similarly constructed.
- This is just another representation of original quantity, and it is hard to fully contract the tensor indices.
  → coarse-graining of tensor network

Coarse-graining of the tensor network

Eckart–Young theorem [Eckart and Young 1936] states that the SVD is the best low rank approximation of a matrix [2].

$$T_{ijkl} = M_{(jk)(li)} = \sum_{m=1}^{D^2} U_{(jk)m}\sigma_m V^\dagger_{m(li)}$$

$$\approx \sum_{m=1}^{D_{\mathrm{cut}}} U_{(jk)m}\sigma_m V^\dagger_{m(li)} \qquad \leftarrow \text{the best approx!}$$

- $1 \leq i, j, k, l \leq D$
- $D_{\mathrm{cut}} < D^2$
- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{D^2} \geq 0$

[2] The Frobenius norm of the difference between the original and the approximated matrices is minimized.

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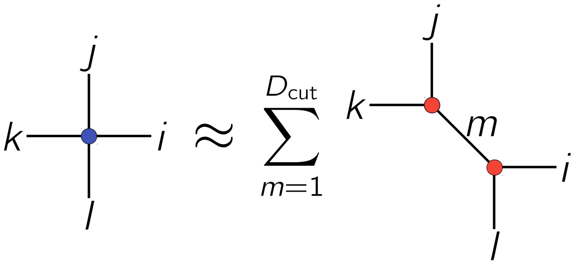

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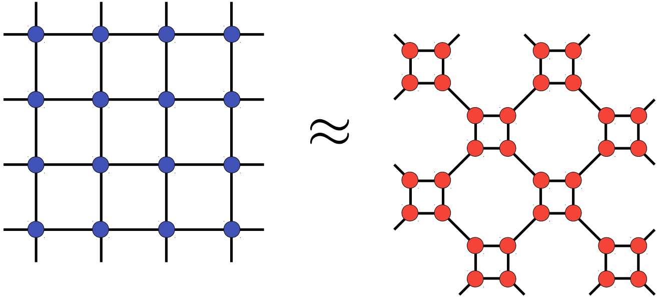

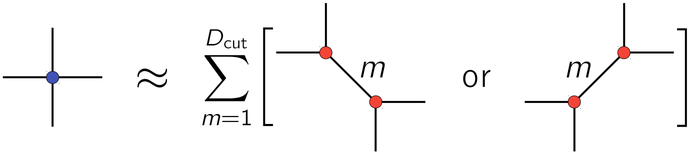

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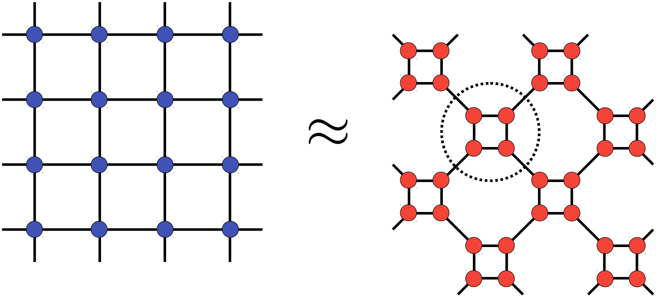

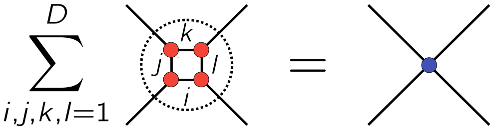

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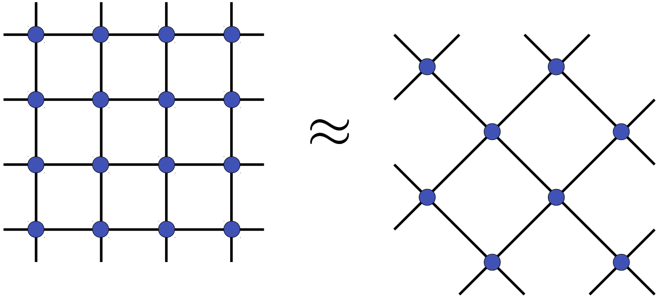

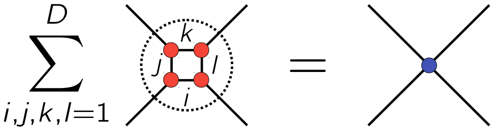

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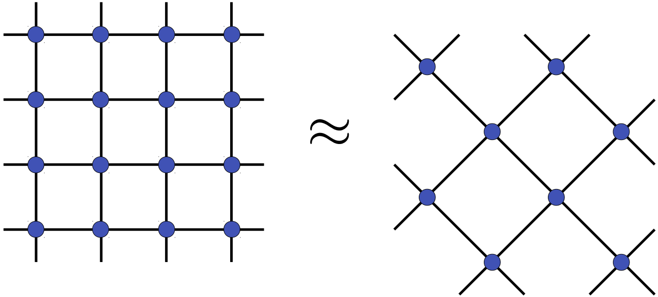
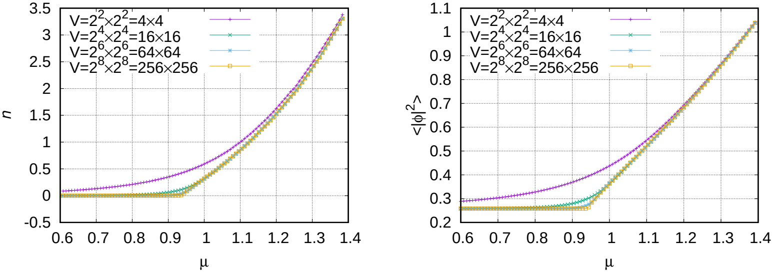


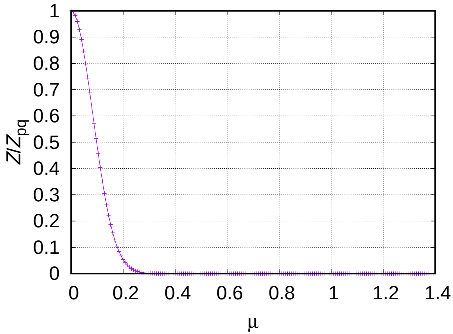
- # of lattice sites is reduced by 1/2 through a single cycle.
- TN is uniform. ⇒ One needs to just repeat local procedures.
- Total cost $\propto \log_2(\text{Vol.})$.

Silver Blaze phenomenon



- left) particle number density,
  right) expectation value of squared absolute value of field
- $m^2 = 0.01,\ \lambda = 1,\ K^2 = 4096,\ D = 64.$

Average of phase factor $\langle e^{i\theta} \rangle$



- $Z_{\mathrm{pq}} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-\mathrm{Re}(S)}$.
- $m^2 = 0.01,\ \lambda = 1,\ K^2 = 4096,\ D = 64,\ V = 8 \times 128$.
- In the large μ region, the ratio deviates from 1, and the sign problem is severe.

naive TRG vs. sign problem free formulation of Z

Using

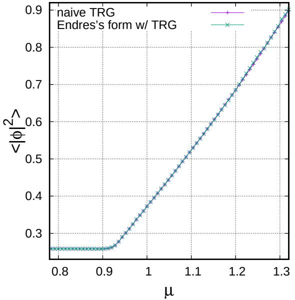
- polar coordinate expression: $\phi_n = (\phi_{n,1}, \phi_{n,2}) \to (r_n \cos\theta_n, r_n \sin\theta_n)$,
- character expansion: $e^{x\cos z} = \sum_{l=-\infty}^{\infty} I_l(x)\, e^{ilz}$ for $x \in \mathbb{R}$, $z \in \mathbb{C}$,

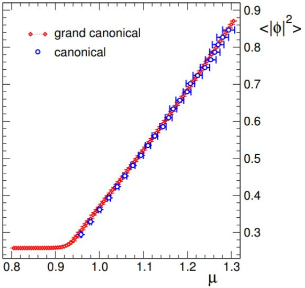
Z can be expressed in a sign problem free version:

$$Z = \left(\prod_n \sum_{l_{n,1}, l_{n,2}=-\infty}^{\infty}\right)\left(\prod_n \int_0^\infty \mathrm{d}r_n\right)\prod_n 2\pi r_n \prod_{\nu=1}^{2} e^{-\frac{1}{4}(4+m^2)(r_n^2+r_{n+\hat{\nu}}^2)-\frac{\lambda}{4}(r_n^4+r_{n+\hat{\nu}}^4)} \cdot I_{l_{n,\nu}}(2r_n r_{n+\hat{\nu}})\, e^{l_{n,\nu}\mu\delta_{\nu,2}}\delta_{(l_{n,1}+l_{n,2}-l_{n-\hat{1},1}-l_{n-\hat{2},2}),0}$$

- I_l: modified Bessel function of first kind (non-negative)
- [Endres 2007] (*cf.* loop formulation, world line representation).
- Efficient MC evaluation using worm algorithm [Orasch and Gattringer 2018]

naive TRG vs. sign problem free formulation of Z





- left) naive TRG and Endres's form w/ TRG, $V = 8 \times 128$.
  right) Endres's form w/ worm, $V = 10 \times 100$ (adapted from [Orasch and Gattringer 2018]).
- $m^2 = 0.01$, $\lambda = 1$.
- $D = 64$, truncation order of CE $= 128$ (in left panel).
- Good agreement in the large μ region (severe sign problem region)

Summary and outlook

- TN formulation of the 2D complex ϕ^4 theory
- Silver Blaze phenomenon is observed
- Result by naive TRG is consistent with those of sign problem free formulation

In future

- Phase diagram
- Properties of Silver Blaze transition
- Complicated models (*e.g.* SUSY and chiral gauge theories)
  - 2D $\mathcal{N}=1$ WZ model [Kadoh, Kuramashi, Nakamura, RS, Takeda, Yoshimura 2018]
  - 3D will be reasonable in the viewpoint of both accuracy and computational complexity
  - 4D application is still limited to Ising model [Akiyama et al. 2019] (but hopeful result!)

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