An Approach to Quantum Gravity – Asymptotic Safety –

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Partly based on collaboration with Kevin Falls, Roberto Percacci and Lesław Rachwał.

Contents

1 Introduction

We are interested in how to formulate Quantum gravity within the framework of local field theory.

Why we need quantum gravity:

- Black hole singularity
- Big-Bang singularity

where the Einstein gravity is not applicable.

One possible approach to make sense of the quantum effects in gravity within the framework of field theory is the asymptotic safety.

2 Asymptotic safety

Integrating out all fluctuations of the fields with momenta larger than k.
$\Rightarrow$ effective average action Γ_k (Note: Γ_0 is the effective action.)

- The effective average action (EAA) itself is divergent, because all high energy modes are integrated.
- Dependence of the EEA on k, which gives the Wilsonian RG flow.

Important fact

RGE is free from any divergence and can be used to define sensible quantum effective action for gravity.

How:

FRGE gives flow of the effective action in the theory space defined by suitable bases $\mathcal{O}_i$.

$$\Gamma_k = \sum_i g_i(k)\mathcal{O}_i \qquad \Rightarrow \frac{d\Gamma_k}{dt} = \sum_i \beta_i \mathcal{O}_i \qquad \beta_i = \frac{dg_i}{dt}$$

We set initial conditions at some point and then flow to $k \to \infty$.

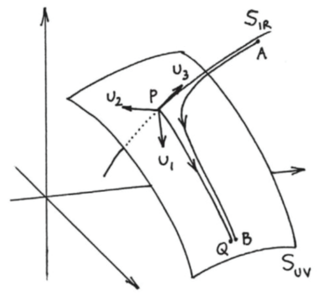

Figure 1: RG flow

- **If all couplings go to finite FPs at UV, physical quantities are well defined, giving the UV finite theory ⇒ Asymptotic safety**

The theories on the same trajectory belong to the same universality class. ⇒ The trajectories with the same FP make a surface, called critical surface, and its dimension is given by the number of relevant operators.

In the ideal case, we also require that the number of relevant operators (only which are retained) are finite. ⇒ Predictability

There is accumulating evidence (up to 70th order in R) that there are always nontrivial fixed points.

$\Rightarrow$ **Asymptotic safety program may be the right direction.**

In pertubation at Gaussian fixed point, the relevant operators have dimensions equal or less than four and are precisely renormalizable interactions.

2.1 Theory of scalar curvature

Start with the Einstein-Hilbert action

$$S(g) = Z_N \int d^d x \sqrt{g} (2\Lambda - g^{\mu\nu} R_{\mu\nu}(g))$$

with $Z_N = 1/(16\pi G)$, Λ: cosmological const.,G: Nweton const.

One can derive the beta functions

$$\begin{aligned} \frac{d\tilde{\Lambda}}{dt} &= -2\tilde{\Lambda} + \tilde{G} \frac{A_1 + 2B_1\tilde{\Lambda} + \tilde{G}(A_1 B_2 - A_2 B_1)}{2(1 + B_2 \tilde{G})}, \\ \frac{d\tilde{G}}{dt} &= (d-2)\tilde{G} + \frac{B_1 \tilde{G}^2}{1 + B_2 \tilde{G}} \end{aligned}$$

In 4 dims., this gives, in addition to Gaussian FP, nontrivial FP

$$\tilde{G} = 0.707321, \qquad \tilde{\Lambda} = 0.193201$$

But this is not enough. We would like to know how large we should take our "theory space."

This is just to identify nonperturbatively renormalizable theory.

The Important problem

What is the dimension of critical surface? $\cdots$ "Nonperturbatively Renormalizable theory"

The question: How to identify the dimension of the critical surface?

Though it is difficult to know the whole structure, we can study it close to the FP. Let

$$y_i = \tilde{g}_i - \tilde{g}_{i*}$$

and the flow equation near the FP is

$$\frac{dy_i}{dt} = M_{ij} y_j, \qquad M_{ij} = \left.\frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}\right|_{\tilde{g}_*} : \textbf{stability matrix}$$

Positive eigenvalues of the stability matrix correspond to the relevant direction!

The question is how many relevant operators exist!!

Current status: we can go on to add more terms in scalar curvature.

This system is now studied up to order R^{70}, and it is found that the critical surface is only 3-dimensional within the power series of scalar curvature.

K. G. Falls, D. F. Litim and J. Schröder, Phys. Rev. D 99 (2019) 126015 [arXiv:1810.08550 [gr-qc]].

One can even derive FRGE for arbitrary function $f(R)$ and make systematic expansion in R.

M. Demmel, F. Saueressig and O. Zanusso, JHEP 1508 (2015) 113 [arXiv:1504.07656 [hep-th]].

The problem is also studied by other choice of metric parametrization with practically the same result.

N. Ohta, R. Percacci and G. P. Vacca, Phys. Rev. D 92 (2015) 061501 [arXiv:1507.00968 [hep-th]];

N. Ohta, R. Percacci and G. P. Vacca, Eur. Phys. J. C 76 (2016) 46 [arXiv:1511.09393 [hep-th]].

Using the automorphism ambiguity, one can also have exact solution $f(R) = \Lambda + Z_N R + cR^2$.

The parametrization dependence is also studied:

N. Ohta, R. Percacci and A. D. Pereira, JHEP 1606 (2016) 115 [arXiv:1605.00454 [hep-th]];

Eur. Phys. J. C 77 (2017) 611 [arXiv:1610.07991 [hep-th]];

G. P. De Brito, N. Ohta, A. D. Pereira, A. A. Tomaz and M. Yamada, Phys. Rev. D 98 (2018) 026027 [arXiv:1805.09656 [hep-th]].

2.2 Dimension of the critical surface

What about other tensor structure?
Perturbative renormalizability suggests that $R_{\mu\nu}^2$ is also needed.

Namely the number of relevant directions is 4?
$\Lambda, R, R^2, R_{\mu\nu}^2 \cdots$ **on dimensional grounds.**

A surprise: D. Benedetti, P. F. Machado and F. Saueressig, Nucl. Phys. B 824 (2010) 168 [arXiv:0902.4630 [hep-th]]
studies the problem on Einstein space, keeping R^2 and extra term ${R_{\mu\nu\rho\lambda}}^2$.
$\Rightarrow$ Claims that there are only 3 relevant operators, the same as $f(R)$ theory, in contrast to perturbation theory which requires $R_{\mu\nu}^2$.

But more study on general backgrounds is needed to conclude this!

The reason is that in 4D, ${R_{\mu\nu\rho\lambda}}^2$ can be transformed into ${R_{\mu\nu}}^2$ and R^2 by GB theorem, and ${R_{\mu\nu}}^2$ is reduced to R^2 on Einstein space.
$\Rightarrow$ cannot distinguish $R_{\mu\nu\rho\lambda^2}, R_{\mu\nu}^2$ and R^2.

Most study found only Gaussian FPs for the higher curvature coefficients; $R_{\mu\nu}^2 \cdots$ known as asymptotically free theory

- K. Falls, C. R. King, D. F. Litim, K. Nikolakopoulos and C. Rahmede, Phys. Rev. D 97 (2018) 086006 [arXiv:1801.00162 [hep-th]].

$$f(R_{\mu\nu}^2) + R\, g(R_{\mu\nu}^2)$$

does not consider R^2 and $R_{\mu\nu}^2$ together (like $f(R_{\mu\nu}^2, R)$).

3 Our results on the dimension of the critical surface

We study the problem on arbitrary backgrounds in arbitrary dimensions for quadratic curvatures on general backgrounds.

(K. Falls, NO and R. Percacci, in preparation; following the work on general backgrounds, NO and R. Percacci, Class. Quant. Grav. 31 (2014) 015024 [arXiv:1308.3398 [hep-th]].)

$$S = \int d^d x \sqrt{-g} \Big[- Z_N (R - 2\Lambda) + \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 + \tau \nabla^2 R \Big],$$

where $Z_N = \frac{1}{16\pi G}$ with G being the d-dimensional gravitational constant, Λ is the cosmological constant,

$$C^2 = R^2_{\mu\nu\alpha\beta} - \frac{4}{d-2} R^2_{\mu\nu} + \frac{2}{(d-1)(d-2)} R^2,$$

the square of the Weyl tensor and the Gauss–Bonnet combination

$$E = R^2_{\mu\nu\alpha\beta} - 4 R^2_{\mu\nu} + R^2.$$

We choose the standard higher-derivative gauge fixing so as to to eliminate non-Laplacian terms ($\nabla^\mu \nabla^2 \nabla^\nu$ etc.) from the leading four-derivative part of the graviton kinetic operator $\Rightarrow \Delta^2 = \nabla^4$.

Our choice of the gauge-fixing and ghost is

$$\mathcal{L}_{GF+FP} = \sqrt{\bar{g}}\left[-\frac{1}{2\lambda}\chi_\mu Y^{\mu\nu}\chi_\nu + i\bar{c}_\mu \Delta^{(gh)\mu}{}_\nu c^\nu + \frac{1}{2\lambda} b_\mu Y^{\mu\nu} b_\nu\right],$$
$$\chi_\mu = \bar{\nabla}^\lambda h_{\lambda\mu} + b\bar{\nabla}_\mu h$$

where b_μ is a new anti-commuting hermitian ghost

$$\begin{aligned}\Delta^{(gh)}_{\mu\nu} &\equiv g_{\mu\nu}\bar{\nabla}^2 - \sigma_{gh}\bar{\nabla}_\mu\bar{\nabla}_\nu + \bar{R}_{\mu\nu},\\ Y_{\mu\nu} &\equiv \bar{g}_{\mu\nu}\bar{\nabla}^2 - \sigma_Y\bar{\nabla}_\mu\bar{\nabla}_\nu - R_{\mu\nu},\\ &\quad \sigma_{gh} = -\frac{1-2\omega}{2(1+\omega)}\ ; \qquad \sigma_Y = \frac{1-2\omega}{3}\ .\end{aligned}$$

The quadratic terms in the action can be written in the form

$$\mathcal{L}^{(2)} = h^{\mu\nu}\mathcal{K}_{\mu\nu,\alpha\beta}h^{\alpha\beta},$$

where

$$\mathcal{K} = K\mathcal{O} \equiv K(\Delta^2 + V_{\rho\lambda}\bar{\nabla}^\rho\bar{\nabla}^\lambda + U).$$

where $\Delta = -\bar{\nabla}^2$, and K, V, U are tensors depending on the curvature and metrics.

We have studied the problem in two schemes:

3.1 First scheme

Here we use the regulator

$$R_k(\Delta^2) = K(k^4 - \Delta^2)\theta(k^4 - \Delta^2)$$

for matter part.

$$T_g = \frac{1}{2}\mathbf{Tr}\frac{\partial_t[KR_k(\Delta^2)]}{K[\mathcal{O} + R_k(\Delta^2)]} = \frac{1}{2}\mathbf{Tr}\frac{\partial_t R_k(\Delta^2) + \eta_K R_k(\Delta^2)}{\mathcal{O} + R_k(\Delta^2)},$$

with $\eta_K = K^{-1}\frac{dK}{dt}$**. We evaluate these by expanding these terms to quadratic order in the curvature:**

$$T_g = \frac{1}{2}\mathbf{Tr}\left[\frac{\partial_t R_k(\Delta) + \eta_K R_k(\Delta)}{P_k(\Delta)}\left(1 - \frac{V_{\mu\nu}}{P_k(\Delta)}\bar\nabla^\mu\bar\nabla^\nu - \frac{U}{P_k(\Delta)} + \frac{\bar V_{\mu\nu}}{P_k(\Delta)}\bar\nabla^\mu\bar\nabla^\nu\frac{\bar V_{\alpha\beta}}{P_k(\Delta)}\bar\nabla^\alpha\bar\nabla^\beta + \dots\right)\right].$$

using off-diagonal heat kernel expansion.

For the ghost, the operators are of the form

$$\Delta\delta_\mu^\nu + \sigma\bar\nabla_\mu\bar\nabla^\nu + B_\mu^\nu,$$

σ **is a constant,** $B_\mu^\nu = s\bar{R}_\mu^\nu$**,** $s = -1$ **for** Δ_{gh} **and** $s = 1$ **for** Y**. Both operators are of non-minimal form. We have to use off-diagonal heat kernel expansion.**

At the fixed points, the anomalous dimensions vanish, and we find the beta functions are

$$\beta_\lambda = -\frac{133}{(4\pi)^2 10}\lambda^2, \qquad \beta_\omega = -\frac{200\omega^2 + 1098\omega + 25}{(4\pi)^2 60}\lambda, \qquad \beta_\rho = -\frac{49}{180\pi^2}\rho^2$$

We find that there are only Gaussian FPs for curvature squared terms! (new in including anomalous dimensions.)

We also get beta functions for Λ **and** G**, which have fixed points**

$$G = 2.388, \qquad \Lambda = 0.389.$$

This is consistent with known result.

cf. K. Groh, S. Rechenberger, F. Saueressig and O. Zanusso, "Higher Derivative Gravity from the Universal Renormalization Group Machine," PoS EPS -HEP2011 (2011) 124 [arXiv:1111.1743 [hep-th]].

However, we should keep $O(Z_N)$ **terms, which contributes to the beta functions of** λ **and then we expect that we have NGFP!** $\Rightarrow$ **Second scheme.**

3.2 Second scheme

We use the regulator

$$R_k(\Delta^2) = K(k^4 - \Delta^2)\theta(k^4 - \Delta^2) + K_N(k^2 - \Delta)\theta(k^2 - \Delta)$$

with anomalous dimensions (beyond one-loop). We expect that the independent operator ${R_{\mu\nu}}^2$ would play a role in this approach!

We find several nontrivial fixed points in the theory on arbitrary backgrounds.

G_*	λ_*	ξ_*	Λ_*	eigenvalues of stability matrix
1.4386	33.05	−98.776	0.3259	−2.781, −2.708, −0.5579, 1.363
4.1615	−350.96	1789.6	1.301	−10.38, −3.164, −1.782, 1.485
21.876	−4463.1	29667.3	−0.2990	−27.56, −0.7452, 43.98, 63.47
3.254	−288	279	0.8637	−7.590, −1.819, 1.188, 4.392
6.9031	−914.89	771.94	−0.2379	−1.114, 6.612, 11.30, 55.93

Table 1: fixed points and eigenvalues of the stability matrix (preliminary). The first two have 3 relevant directions.

The above result is a strong evidence that the dimension of critical surface is probably 3 if we include other tensors.

4 Quantum theory

4.1 Application to black holes

We would like to get some physical predictions from this approach!

Naively one identifies the cutoff k to the energy scale of physical process.

A. Bonanno and M. Reuter, Phys. Rev. D 62 (2000) 043008 [hep-th/0002196].

Using this ansatz, one adopts the scale-dependent Newton constant

$$G(k) = \frac{G_0}{1+\omega G_0 k^2}$$

and relate k to geodesic distance to obtain

$$G(r) = \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 [r + \gamma G_0 M]}$$

One then find that

- **Schwarzschild black hole becomes similar to Reissner-Nordström solution**
- **The singularity is made milder.** $\Leftarrow f(r) = 1 - \frac{2MG}{r}$

However, the cutoff k is unphysical parameter and the effective average action itself (to be distinguished from effective action) is not physical.

J. F. Donoghue, arXiv:1911.02967 [hep-th].

Quantum effects

Quantum effective action for gravity interacting with matter.

Most of the focus is the UV behavior of the flow; if the theory is well defined at UV.

We would like to have physical effective action directly related scattering amplitudes.

Typically this involves nonlocal interactions coming from integrating over quantum fluctuations!

There is only few work on this subject.

A. Codello, R. Percacci, L. Rachwal and A. Tonero, Eur. Phys. J. C 76 (2016) 226 [arXiv:1505.03119 [hep-th]].

We will find these nonlocal terms are uniquely determined once the matter contents are given. **(NO and Lesław Rachwał, in preparation)**

4.2 Effective action – basics –

The FRGE is given by

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k, \qquad (t = \ln k)$$

Our Laplacian: $-\nabla^2 \mathbf{1} + \mathbf{U}$, **U is a non-derivative term.**

We will take the trace of

$$h_k(\Delta, \omega) = \frac{\partial_t R_k(\Delta)}{\Delta + \omega + R_k(\Delta)}.$$

$$\partial_t \Gamma_k = \frac{1}{2} \int_0^\infty ds\, \tilde{h}_k(s,\omega)\, \mathrm{Tr}\, e^{-s\Delta}, \quad (\tilde{h}_k(s,\omega): \textbf{ the Laplace transform})$$

The **nonlocal** **heat kernel expansion: (not the usual local one)**

$$\mathrm{Tr}(e^{-s\Delta}) = \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g}\, \mathrm{g} \Big\{ \mathbf{1} + s\Big(\frac{R}{6}\mathbf{1} - \mathbf{U}\Big) + s^2 \Big[\mathbf{1} R_{\mu\nu} f_{\mathrm{Ric}}(-s\nabla^2) R^{\mu\nu}$$
$$+ \mathbf{1} R f_{\mathrm{R}}(-s\nabla^2) R + R f_{\mathrm{RU}}(-s\nabla^2)\mathbf{U} + \mathbf{U} f_{\mathrm{U}}(-s\nabla^2)\mathbf{U} + \Omega_{\mu\nu} f_{\Omega}(-s\nabla^2)\Omega^{\mu\nu} \Big] \Big\},$$

where $\Omega_{\mu\nu} \equiv [\nabla_\mu . \nabla_\nu]$ and f's are the structure functions

$$f_{\rm Ric}(x) = \frac{1}{6x} + \frac{1}{x^2}[f(x)-1], \quad f_{\rm R}(x) = \frac{1}{32}f(x) + \frac{1}{8x}f(x) - \frac{7}{48x} - \frac{1}{8x^2}[f(x)-1],$$
$$f_{\rm RU}(x) = -\frac{1}{4}f(x) - \frac{1}{2x}[f(x)-1], \quad f_{\rm U}(x) = \frac{1}{2}f(x), \quad f_\Omega(x) = -\frac{1}{2x}[f(x)-1],$$

where $f(x) = \int_0^1 d\xi e^{-x\xi(1-\xi)}$ ($x=0$ gives the local term).

Our master formula

$$\partial_t \Gamma_k = \frac{1}{2}\frac{1}{(4\pi)^{d/2}} \int d^dx \sqrt{g}\, \mathrm{g}\Big\{ \mathbf{1} Q_{\frac{d}{2}}[h_k] + \Big(\frac{R}{6} - \mathbf{U}\Big) Q_{\frac{d}{2}-1}[h_k] + R_{\mu\nu}\, g_{\rm Ric}\, R^{\mu\nu}\mathbf{1}$$
$$+ R\, g_{\rm R} R\mathbf{1} + R\, g_{\rm RU}\mathbf{U} + \mathbf{U} g_{\rm U}\mathbf{U} + \Omega_{\mu\nu} g_\Omega \Omega^{\mu\nu} + \cdots \Big\}.$$

$$g_A = g_A(z,\omega,k) = \int_0^\infty ds\, \tilde{h}_k(s,\omega) f_A(sz) s^{-d/2+2}, \quad (A = \textbf{Ric etc.},\ z = \Box)$$

Using optimized cutoff

$$R_k(z) = (k^2 - z)\theta(k^2 - z),$$

we obtain

$$g_{\rm Ric} = \frac{1}{30}\frac{1}{1+\tilde\omega}\left[1-\left(1-\frac{4k^2}{z}\right)^{\frac{5}{2}}\theta(z-4k^2)\right],$$

$$g_{\rm R} = \frac{1}{1+\tilde\omega}\left[\frac{1}{60}-\frac{1}{6}\left(1-\frac{4k^2}{z}\right)^{\frac{1}{2}}\theta(z-4k^2)+\frac{1}{24}\left(1-\frac{4k^2}{z}\right)^{\frac{3}{2}}\theta(z-4k^2)+\frac{1}{240}\left(1-\frac{4k^2}{z}\right)^{\frac{5}{2}}\theta(z-4k^2)\right],$$

$$g_{\rm RU} = \frac{1}{1+\tilde\omega}\left[-\frac{1}{3}+\frac{1}{2}\left(1-\frac{4k^2}{z}\right)^{\frac{1}{2}}\theta(z-4k^2)-\frac{1}{6}\left(1-\frac{4k^2}{z}\right)^{\frac{3}{2}}\theta(z-4k^2)\right],$$

$$g_{\rm U} = \frac{1}{1+\tilde\omega}\left[1-\left(1-\frac{4k^2}{z}\right)^{\frac{1}{2}}\theta(z-4k^2)\right],$$

$$g_{\Omega} = \frac{1}{6(1+\tilde\omega)}\left[1-\left(1-\frac{4k^2}{z}\right)^{\frac{3}{2}}\theta(z-4k^2)\right].$$

We have to integrate the flow equation down to $k=0$. These are all written as

$$g_a(z) = A_a + \left(-A_a+\frac{B_a}{z}+\frac{C_a}{z^2}\right)\sqrt{1-\frac{4}{z}}\theta(z-4)$$

and integration of this yields logarithmic nonlocal terms involving

$$\frac{A_a}{64\pi^2}\log\left(\frac{\Box}{\mu^2}\right)$$

What is important is that these nonlocal terms are uniquely determined.

Other local terms are subject to renormalization and are not unique.

Earlier perturbative work (Barvinsky et al.) assumes weak field:

$R \ll \nabla^2 R, F_{\mu\nu} \ll \nabla^2 F_{\mu\nu}$

but here we obtain the nonlocal effective action without such assumption.

Our task

1. **First calculate the Hessians for the quantum fields**
2. **Identify U and $\Omega_{\mu\nu}$ and calculate the traces of these and their squares.**
3. **Substitute these into the master formula, and integrate it down to $k = 0$.**
4. **Find out what terms we get.**

What is the physical implications of the result?

4.3 Details – may be skipped –

1st Step $\cdots$ Get Hessians!

Graviton

$$S_g = -\frac{1}{\kappa^2}\int d^dx\sqrt{g}\left[R(g)+\frac{1}{2}\chi_\mu\chi^\mu+\bar{C}_\mu(-\Box\delta^\mu_\nu-R^\mu_\nu)C^\nu\right],$$

with gauge fixing function

$$\chi_\mu=\bar{\nabla}^\nu h_{\mu\nu}-\frac{1}{2}\bar{\nabla}_\mu h.$$

Abelian gauge field

$$S_V=\int d^4x\sqrt{g}\left[\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}+\frac{1}{2}(\bar{\nabla}_\mu A^\mu)^2\right],$$

Dirac field

$$S_f=\int d^dx\sqrt{g}\,\frac{1}{2}\left(\bar{\psi}\gamma^\mu D_\mu\psi-D_\mu\bar{\psi}\gamma^\mu\psi+2m\bar{\psi}\psi\right),$$

with covariant derivatives

$$D_\mu\psi=(\partial_\mu-ieA_\mu)\psi+\frac{1}{2}\omega_{\mu ab}J^{ab}\psi,\quad D_\mu\bar{\psi}=(\partial_\mu+ieA_\mu)\bar{\psi}-\frac{1}{2}\omega_{\mu ab}\bar{\psi}J^{ab},$$

Charged scalar

$$S_s = \int d^d x \sqrt{g} \left[g^{\mu\nu} (D^S_\mu \phi^*)(D^S_\mu \phi) + V(|\phi|^2) \right],$$

where the covariant derivative on the scalar is

$$D_\mu \phi = (\nabla_\mu - i e_S A_\mu)\phi, \qquad D_\mu \phi^* = (\nabla_\mu + i e_S A_\mu)\phi^*.$$

The quantum fluctuations are defined by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}; \quad A_\mu \to \bar{A}_\mu + A_\mu; \quad \psi \to \psi + \chi; \quad \phi = \bar{\phi} + \varphi$$

The bosonic part of total Hessian

$$\frac{1}{2} \int d^d x \sqrt{g} (h^{\mu\nu}, A^\mu, \varphi^*, \varphi) H_T \begin{pmatrix} h^{\alpha\beta} \\ A^\alpha \\ \varphi \\ \varphi^* \end{pmatrix},$$

where

$$H_T = K_T(-\bar{D}^2) + 2V_T^\delta \bar{D}_\delta + U_T; \quad 4 \times 4 \textbf{ complicated matrix operator}$$

K_T **involves de Witt metric for graviton and diagonal metric for others.**

Extract an overall factor of K **and write the Hessian as**

$$H_T = K\Delta \equiv K(-\bar{D}^2 + 2Y^\delta \bar{D}_\delta + W).$$

Some details on fermionic contributions

Our action

$$S_f = \int d^d x \sqrt{g}\, \frac{1}{2} \left(\bar{\psi}\gamma^\mu D_\mu \psi - D_\mu \bar{\psi}\gamma^\mu \psi + 2m\bar{\psi}\psi \right)$$

Fluctuations of the vierbein in terms of $h_{\mu\nu}$:

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \quad \Rightarrow \quad e^a_\mu = \bar{e}^a_\mu + \frac{1}{2} h^a_\mu - \frac{1}{8} h_{\mu\rho} h^{\rho a} + \cdots ,$$

Tetrad postulate $\omega_\mu{}^a{}_b = e^a_\rho \Gamma^\rho_{\mu\sigma} e^\sigma_b + e^a_\rho \partial_\mu e^\rho_b, \quad \Rightarrow \quad \omega_\mu{}^{ab} = \bar{\omega}_\mu{}^{ab} + \omega_\mu{}^{ab\,(1)} + \omega_\mu{}^{ab\,(2)},$

where

$$\begin{aligned} \omega_\mu{}^{ab\,(1)} &= \frac{1}{2} \left(\bar{\nabla}^\beta h^\alpha_\mu - \bar{\nabla}^\alpha h^\beta_\mu \right) \bar{e}^a_\alpha \bar{e}^b_\beta, \\ \omega_\mu{}^{ab\,(2)} &= \frac{1}{8} \left(4h^{\alpha\rho} \bar{\nabla}_\rho h^\beta_\mu - 4h^{\alpha\rho} \bar{\nabla}^\beta h_{\mu\rho} - h^{\alpha\rho} \bar{\nabla}_\mu h^{\beta\rho} + h^{\beta\rho} \bar{\nabla}_\mu h^{\alpha\rho} \right) \bar{e}^a_\alpha \bar{e}^b_\beta. \end{aligned}$$

Making approximation to drop nonlocal terms, we have

$$\Gamma^{Dirac} = -\mathrm{Tr} \log \left[\bar{\not{D}} + m \right] = -\frac{1}{2} \mathrm{Tr} \log \left[-\bar{D}^2_\mu + \frac{ie}{2} F_{\mu\nu} \gamma^{\mu\nu} + \frac{1}{4} \bar{R}^2 + m^2 \right].$$

There are additional bosonic contributions which must be incorporated in the previous contributions from gravitons etc.

2nd Step $\cdots$ **Identify $\boldsymbol{U}$ and $\Omega \Rightarrow$ Their traces!**

Eliminate the first order term in $Y^\delta \bar{D}_\delta$ by writing

$$\Delta = -\tilde{D}_\mu^2 + \tilde{W} \quad (\tilde{D}_\mu = \bar{D}_\mu - Y_\mu), \quad \tilde{W} = W - \bar{D}_\delta Y^\delta + Y_\delta Y^\delta.$$

$$\tilde{\Omega}_{\rho\sigma} = \left[\tilde{D}_\rho, \tilde{D}_\sigma\right] = \Omega_{\rho\sigma} - 2\bar{D}_{[\rho} Y_{\sigma]} + 2Y_{[\rho} Y_{\sigma]}$$

$\tilde{W}$ **corresponds to U and $\tilde{\Omega}$ to Ω.**

After straightfoward calculation, we get

$$\mathrm{g}(\tilde{W}) = 7\bar{R} - 10V - 4(\bar{D}_\mu \bar{\phi})(\bar{D}^\mu \bar{\phi}^*) - \frac{3}{2} F_{\rho\lambda}^2 + 4e_S^2 |\bar{\phi}|^2 + 2V' + 2|\bar{\phi}|^2 V'' + \text{ **fermionic terms** }$$

$$\mathrm{g}(\tilde{W}^2) = -5\bar{R}_{\mu\nu}^2 + 5\bar{R}^2 - 12\bar{R}V + 10V^2 - 5\bar{F}_{\mu\nu}\bar{F}^{\mu\rho}\bar{R}^\nu{}_\rho + \frac{3}{2}\bar{F}_{\mu\nu}\bar{F}_{\rho\lambda}(\bar{R}^{\mu\nu\rho\lambda} - \bar{R}^{\mu\lambda\nu\rho}) + \cdots$$

$$\mathrm{g}(\tilde{\Omega}^2) = -5(\bar{\nabla}^\rho \bar{F}_{\mu\nu})^2 + \bar{\nabla}^\rho \bar{F}^{\mu\nu} \bar{\nabla}_\nu \bar{F}_{\mu\rho} + (\bar{\nabla}^\mu \bar{F}_{\mu\nu})^2 + 3ie_S(\bar{\phi}^* \bar{D}^\mu \bar{\phi} - \bar{\phi}\bar{D}^\mu \bar{\phi}^*)\nabla^\nu \bar{F}_{\mu\nu} + \cdots$$

3rd Step $\cdots$ Plugge the traces into the FRGE and integrate it!

Results: terms like

$$\bar{R}_{\mu\nu}\log\left(\frac{\Box}{\mu^2}\right)\bar{R}^{\mu\nu}, \quad \bar{R}\log\left(\frac{\Box}{\mu^2}\right)\bar{R}, \quad \bar{F}_{\mu\nu}\log\left(\frac{\Box}{\mu^2}\right)\bar{F}^{\mu\nu},$$

$$\bar{\nabla}_\rho\bar{F}_{\mu\nu}\log\left(\frac{\Box}{\mu^2}\right)\bar{\nabla}^\rho\bar{F}^{\mu\nu}, \quad \bar{F}_{\mu\nu}\bar{F}_{\rho\lambda}\log\left(\frac{\Box}{\mu^2}\right)(\bar{R}^{\mu\nu\rho\lambda}-\bar{R}^{\mu\lambda\nu\rho}), \quad \text{terms involving scalar}$$

are generated with unique coefficients. It is interesting to see what physical predictions or explanations of exisitng data they make.

4.4 Application – an example

In our universe, magnetic fields are observed on various scales such as in galaxies and galaxy clusters.

There is some evidence for the presence of magnetic fields in intergalactic voids.

There is some literature arguing that Weyl invariance in the electromagnetic interactions is violated by quantum effects, and electromagnetic fields generated in the early universe!

Our effective action may well give a mechanism for generating magnetic fields. . . . yet to be examined!

Announcement: Next ERG conference (ERG2020) will be at Yukawa Institute for Theoretical Physics, Kyoto University.

Period is 7 – 11 September 2020. Be prepared!

Thank you for your attention!